Analysis I (English)
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Analysis 1 - Exercise Set 1

1. Real Numbers.

   (a) Explain the difference between a rational and an irrational number.

   (b) Classify the following numbers as rational, irrational, natural, integer. (A number may belong to more than one set).

      (i) -2

      (ii) $4\frac{1}{3}$

      (iii) $\sqrt{10}$

      (iv) 0

      (v) π

   (c) Show that $\sqrt{7}$ is an irrational number. (*Hint: assume that you can write* $\sqrt{7} = \frac{a}{b}$ *where* a *and* b *are integers where their greatest common divisor is* $\gcd(a, b) = 1$. *Then try to find a contradiction.*)

Solution:

(a) A rational number can be expressed as the ratio of two integers. An irrational number is any real number that is not rational.

(b) (i) rational, integer

   (ii) rational

   (iii) irrational

   (iv) rational, integer, natural

   (v) irrational

(c) Let $\sqrt{7} = \frac{a}{b}$ such that $\gcd(a, b) = 1$. We square both sides to get

$$7 = \frac{a^2}{b^2}$$

Now we multiply both sides with b^2 to get

$$a^2 = 7b^2$$

Clearly 7 is a factor of right hand side so 7 must be a factor of left hand side too. But since 7 is a factor of a^2, a is an integer and 7 is a prime number, we deduce that 7 is also a factor of a. Hence we can find an integer c such that $a = 7c$. By replacing this in the above equation we get

$$a^2 = 7^2c^2 = 7b^2 \implies b^2 = 7c^2$$

Using an argument similar to before we can deduct that 7 is also a factor of b. This is a contradiction since we assumed that $\gcd(a, b) = 1$ but 7 divides both a and b.

2. Trigonometry.

   Show that the following identities hold:

   (a) $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$

   (b) $\sin x + \sin y = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$

   (c) $\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$

   (d) $\cos x + \cos y = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$

   (e) $\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$

   (Hints: For (a) use the identities: $(a^3+b^3) = (a+b)(a^2-ab+b^2)$, $(a+b)^2 = a^2+2ab+b^2$, $\sin^2 x + \cos^2 x = 1$. *For (b)-(e) use the identities*

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

   and compute $\sin(\alpha+\beta) + \sin(\alpha-\beta)$, $\sin(\alpha+\beta) - \sin(\alpha-\beta)$ *etc. then try to find* x *and* y *in terms of* α *and* β*)*

Solution:

(a)

$$\begin{aligned}
\sin^6 x + \cos^6 x &= \underbrace{(\sin^2 x + \cos^2 x)}_{=1}(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) \\
&= (\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) \\
&= \underbrace{(\sin^2 x + \cos^2 x)}_{=1}^2 - 2\sin^2 x\cos^2 x - \sin^2 x\cos^2 x \\
&= 1 - 3\sin^2 x\cos^2 x
\end{aligned}$$

(b) Using the hint we get

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

So it is enough to let $x = \alpha+\beta$ and $y = \alpha-\beta$. Solving the system of equations yields $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$.

(c) We have

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

So we take $x = \alpha+\beta$ and $y = \alpha-\beta$. Solving the system of equations yields $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$.

(d) We have

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

So we take $x = \alpha+\beta$ and $y = \alpha-\beta$. Solving the system of equations yields $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$.

(e) We have

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$

So we take $x = \alpha+\beta$ and $y = \alpha-\beta$. Solving the system of equations yields $\alpha = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$.

3. Trigonomentry.

Simplify the following trigonometric expressions to obtain algebraic expressions (i.e., only involving sums, ratios, roots, etc.).
Example: if $-1 \leq x \leq 1$, we have $\cos(\arcsin x) = \sqrt{1-x^2}$.

(a) $\sin(\arcsin x)$, where $-1 \leq x \leq 1$

(b) $\sin(\arccos x)$, where $-1 \leq x \leq 1$

(c) $\tan(\arccos x)$, where $-1 \leq x \leq 1$

Solution: In each case, the restriction on the values of x guarantees that x is in the domain of the corresponding inverse trigonometric function.

(a)
$$\sin(\arcsin x) = (\sin \circ \arcsin)(x) = id(x) = x$$
where id is the identity mapping, i.e., $id(x) = x$ for all x.

(b) the function arccos takes values in $[0, \pi]$, and in this interval we have the relation
$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - x^2}$$
(side question: what does fail outside this interval?), so we have
$$\sin(\arccos x) = \sqrt{1 - \cos^2(\arccos x)} = \sqrt{1 - x^2}$$

(c) Arguing as above, we get
$$\tan(\arccos x) = \frac{\sin(\arccos x)}{\cos(\arccos x)} = \frac{\sqrt{1-x^2}}{x}$$

4. Arithmetic manipulations.

Prove the following identities.

(a)
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad n \geq 1.$$
(*Hint: try to add the elements of the two finite sequences* $(1, 2, \ldots, n)$ *and* $(n, n-1, n-2, \ldots, 1)$ *term by term*)

(b) Give an alternative proof, to the one given in the first lecture, for the equality
$$a + a^2 + a^3 + \cdots + a^n = a \cdot \frac{1-a^n}{1-a}, \quad a \neq 1,\ n > 1.$$
(*Hint: use the identity*
$$(b^n - a^n) = (b-a)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}),$$
and replace b with 1)

(c)* For any $n \in \mathbb{N}$,
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$
(*Hint: start by computing* $k^3 - (k-1)^3$ *for a natural number* k)

Solution:

(a) We observe that if we add elements of the two sequences $(1, 2, \ldots, n)$ and $(n, n-1, n-2, \ldots, 1)$ we get the sequence $(n+1, n+1, n+1, \ldots, n+1)$, whose sum is $n(n+1)$. As we counted each term of the original sequence twice, we conclude that,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

(b) Using the identity and replacing b with 1 we obtain

$$(1 - a^n) = (1 - a)(a^{n-1} + a^{n-2} + \cdots + a + 1)$$

We divide both sides by $(1 - a)$, which is not 0 as $a \neq 1$, to get

$$\frac{(1 - a^n)}{(1 - a)} = a^{n-1} + a^{n-2} + \cdots + a + 1$$

Finally we multiply both sides by a to get

$$a + a^2 + a^3 + \cdots + a^n = a \cdot \frac{1 - a^n}{1 - a}$$

(c) We compute

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1.$$

Hence,

$$k^2 = \frac{k^3 - (k-1)^3}{3} + k - \frac{1}{3}.$$

Thus,

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} \left(\frac{i^3 - (i-1)^3}{3} + i - \frac{1}{3} \right) = \sum_{i=1}^{n} \frac{i^3 - (i-1)^3}{3} + \sum_{i=1}^{n} i - \sum_{i=1}^{n} \frac{1}{3}. \quad (1)$$

Now, the first sum is a telescopic sum (i.e., there are cancellations)

$$\sum_{i=1}^{n} \frac{i^3 - (i-1)^3}{3} = \frac{n^3}{3} - \frac{(n-1)^3}{3} + \frac{(n-1)^3}{3} - \frac{(n-2)^3}{3} + \frac{(n-2)^3}{3} - \ldots \quad (2)$$

$$- \frac{3^3}{3} + \frac{3^3}{3} - \frac{2^3}{3} + \frac{2^3}{3} - \frac{1^3}{3} + \frac{1^3}{3} - 0^3 = \frac{n^3}{3}.$$

The second sum has been computed in part (a), that is

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}, \quad (3)$$

while the last sum is the sum of n copies of $\frac{1}{3}$

$$\sum_{i=1}^{n} \frac{1}{3} = \frac{n}{3}. \quad (4)$$

Putting equations (1)-(4) together, we get that

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + 3n - 2n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(2n+1)(n+1)}{6}.$$

Alternative proof, using induction (check the book at page 469 if you do not know what induction is; you can also read the text in italics below).
We prove this by induction. That the conclusions holds for $n = 1$ is a simple direct computation by substituting n with 1 in the formula.
Now, suppose that the formula that we are requested to prove holds for some natural number $n = k$. That is,

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Then,

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

where the second equality follows from our inductive hypothesis and the third equality can be verified by expanding each term.

Hence, the formula holds by induction.

Proof by induction: We have shown that the formula holds whenever $n = 1$. We later show that if the formula holds for some natural number $n = k$, then the formula also holds whenever $n = k+1$. Consequently, since the formula holds whenever $n = 1$, it must also holds for $n = 1 + 1 = 2$. But this means that it must also hold for $n = 2 + 1 = 3$, and therefore $n = 3 + 1 = 4, n = 4 + 1 = 5, n = 5 + 1 = 6, \cdots$ and so on.

5. Equations.

   Solve the following equations:

   (a) $\frac{2x}{x+1} = \frac{2x-1}{x}$;
   (b) $x^4 - 3x^2 + 2 = 0$;
   (c) $3|x - 4| = 10$.

Solution:

(a) By the denominators, we have the conditions $x \neq -1$ and $x \neq 0$. Now, we multiply both sides by $x(x + 1)$, and we get

$$\begin{aligned} 2x^2 &= (2x-1)(x+1) \\ 2x^2 &= 2x^2 + 2x - x - 1 \\ 0 &= -x - 1 \\ x &= 1, \end{aligned}$$

which is an acceptable solution.

(b) First, we regard the polynomial as a polynomial in $t = x^2$. Then, we realized that we can factor $t^2 - 3t + 2 = (t-1)(t-2)$. So, the equation becomes $(x^2 - 1)(x^2 - 2) = 0$. Since we have a product and we want it to be 0, it will be 0 when at least one of the factors is 0. So, first we study $x^2 - 1 = 0$, which has solutions $x = \pm 1$. Then, we study $x^2 - 2 = 0$, which has solutions $x = \pm\sqrt{2}$. Thus, the equation has solutions $1, -1, \sqrt{2}, -\sqrt{2}$.

(c) We study the equation by cases. If $x - 4 \geq 0$, we have $|x-4| = x - 4$. Thus, subject to the condition $x \geq 4$, the equation becomes $3(x-4) = 10$. So, it is

$$\begin{aligned} 3(x-4) &= 10 \\ 3x - 12 &= 10 \\ 3x &= 22 \\ x &= \frac{22}{3}. \end{aligned}$$

Since $\frac{22}{3} \geq \frac{12}{3} = 4$, this solution is acceptable.

Now, we analyze the case $x < 4$, which allows to simplify the equation as $-3(x-4) = 10$. Thus, we have

$$\begin{aligned} -3x + 12 &= 10 \\ -3x &= -2 \\ x &= \frac{2}{3}. \end{aligned}$$

Since $\frac{2}{3} < 4$, also this solution is acceptable.

6. Inequalities.

Determine the solutions to the following inequalities.

(a) $x^2 < 2x + 8$;
(b) $x(x-1)(x-2) > 0$;
(c) $\frac{2x-3}{x+1} \leq 1$;
(d) $|x^2 - 1| \leq 1$.

Solution:

(a) We will bring all the summands to one side and try to factor the polynomial we get. So, the inequality is equivalent to $x^2 - 2x - 8 < 0$. Now, we factor $x^2 - 2x - 8 = (x+2)(x-4)$, and the roots are -2 and 4. If we think of the graph of $y = x^2 - 2x - 8$, it is a parabola, and $x = -2$ and $x = 4$ determine its x-intercepts. So, since we want the portion of the parabola whose y-coordinate is negative (i.e., $y = x^2 - 2x - 8 < 0$) and the parabola is concave up, the solution is $-2 < x < 4$.

(b) We study the sign of each factor, and then we use the sign rule to determine the sign of their product. The factor x is positive if $x > 0$ and negative if $x < 0$. The factor $x - 1$ is positive if $x > 1$ and negative if $x < -1$. Lastly, the factor $x - 2$ is positive if $x > 2$ and negative if $x < -2$. Since we want a strict inequality, we exclude the values $x = 0, 1, 2$ and only consider open intervals. On $(-\infty, 0)$ all three factors are negative, so their product is negative, as $- \cdot - \cdot - = -$. On $(0, 1)$, the factor x is positive and the other two are negative, so $+ \cdot - \cdot - = +$, and we have a solution. On $(1, 2)$ only the factor $x - 2$ is negative and the other two are positive, so we have $+ \cdot + \cdot - = -$. Lastly, on $(2, +\infty)$ all three are positive and we get a solution, as $+ \cdot + \cdot + = +$. So, the solution set is $(0, 1) \cup (2, +\infty)$.

(c) In order to solve this equation, we want to clear denominators. Yet, the sign of $x + 1$ will determine whether we have to flip the inequality or not. So, the first case is when $x + 1 > 0$, that is, $x > -1$. In this case, when we clear the denominator, we multiply by a positive number, and the direction of the inequality is preserved. So, if $x > -1$, we obtain $2x - 3 \leq x + 1$. Now, we solve this linear inequality, and we get $x \leq 4$. So, since we have the constraint $x > -1$, we obtain the solutions $-1 < x \leq 4$.

Now, we consider the case $x < -1$, which corresponds to having a negative denominator. So, when we clear the denominator, we multiply by a negative number, and the inequality becomes $2x - 3 \geq x + 1$, which has solution $x \geq 4$. Since $(-\infty, -1) \cap [4, +\infty) = \emptyset$, this case does not provide any valid solution.

So, the overall solution set is $-1 < x \leq 4$.

(d) In this case, we use the following fact: if $b \geq 0$, the inequality $|a| \leq b$ is equivalent to $-b \leq a \leq b$. So, in our case, we get $-1 \leq x^2 - 1 \leq 1$. Adding 1 all the way, we get $0 \leq x^2 \leq 2$. Since $x^2 \geq 0$ is always true, we are left with $x^2 \leq 2$. If we interpret is as the graph of a parabola that is concave up, we are looking at the portion that is below the line $y = 2$. So, we can solve $x^2 = 2$, and the solution will be the interval between the two roots of this equation. Thus, the overall solution is $-\sqrt{2} \leq x \leq \sqrt{2}$.

7. Functions.

Let

$$f(x) = \frac{1}{1 - \frac{2}{1+\frac{1}{1-x}}}$$

(a) Find x, such that $f(x) = 3$.

(b) Find the domain of f.

Solution:

(a) there are many easy alternative ways to solve this equation; one is the following

$$\begin{aligned}
\frac{1}{1 - \frac{2}{1+\frac{1}{1-x}}} = 3 &\iff 1 - \frac{2}{1 + \frac{1}{1-x}} = \frac{1}{3} \\
&\iff \frac{1}{1 + \frac{1}{1-x}} = \frac{1}{3} \\
&\iff 1 + \frac{1}{1-x} = 3 \\
&\iff \frac{1}{1-x} = 2 \\
&\iff x = \frac{1}{2}
\end{aligned}$$

(b) The domain is the entire $\mathbb{R}$ except for the values that make the denominator of the fractions $\frac{1}{1-x}$, $\frac{2}{1+\frac{1}{1-x}}$ and $\frac{1}{1-\frac{2}{1+\frac{1}{1-x}}}$ zero. These values are, $x = 1$, $x = 2$ and $x = 0$. So $D_f = \mathbb{R} \backslash \{0, 1, 2\}$.

8. Functions.

Do there exist functions f and g defined on $\mathbb{R}$ such that

$$f(x) + g(y) = xy$$

for all real numbers x and y? (*Hint: Try to evaluate for* $(x, y) = (0, 0),\ (x, y) = (1, 0),\ (x, y) = (0, 1),\ (x, y) = (1, 1)$)

Solution: The answer is no. To explain look at

$$\begin{aligned} f(0) + g(0) &= 0 \quad (1)\\ f(1) + g(0) &= 0 \quad (2)\\ f(0) + g(1) &= 0 \quad (3)\\ f(1) + g(1) &= 1 \quad (4) \end{aligned}$$

If we add equations (2) and (3) we get

$$f(1) + g(1) + f(0) + g(0) = 0$$

But from equation (4) we know that $f(1) + g(1) = 1$ and $f(0) + g(0) = 0$ from equation (1). So the left hand side adds up to 1 while the right hand side is zero. This is the contradiction with the hypothesis that we can have the form $f(x) + g(y) = xy$.

9. Functions.

Recall that a function $F\colon X \to Y$ is called injective if for every pair of elements a and b in X, $F(a) = F(b)$ implies that $a = b$; in other words, it is injective if distinct elements have distinct images.
Consider now three functions $f, g, h : \mathbb{R} \to \mathbb{R}$. For each of the following statements, say whether that is true or false. If you think it is true, then provide a proof of that, or, else, if false, provide a counterexample.

(a) $f \circ (g + h) = f \circ g + f \circ h$;

(b) $(f + g) \circ h = (f \circ h) + (g \circ h)$;

(c) $f \circ g = g \circ f$ if and only if $f = g$;

(d) if $f \circ f$ is injective then f is injective;

(e) if $f \circ g$ is injective then g is injective;

(f) if $f \circ g$ is injective then f is injective.

Solution:

(a) False. Take $f(x) = x^2$, $g(x) = x$ and $h(x) = x$ then $f \circ (g + h) = 4x^2$ and $f \circ g + f \circ h = 2x^2$.

(b) True.

(c) False. Take $f(x) = x$ and g to be the function identically 0, i.e., $g(x) = 0$ for all x. Then $f \circ g = g \circ f = 0$

(d) True. We want to show that f is injective. Assume we have x_1 and x_2 such that $f(x_1) = f(x_2)$, we have to show that $x_1 = x_2$. We have

$$f(x_1) = f(x_2) \Longrightarrow f(f(x_1)) = f(f(x_2)) \Longrightarrow (f \circ f)(x_1) = (f \circ f)(x_2) \Longrightarrow x_1 = x_2$$

The last implication is because $f \circ f$ is injective.

(e) True. Similar to the previous exercise.

(f) False. The following is a counterexample with easy sets, rather than $\mathbb{R}$. Let g be the inclusion of $\{a\}$ in $\{a, b\}$, and f a map from $\{a, b\}$ to a set with a single element $\{*\}$, so $f(a) = f(b) = *$, then the composition is injective but f is not injective. To have a counterexample on $\mathbb{R}$, take as g the exponential and $f(x) = x^2$.

10. Functions.

Let $f \colon \mathbb{N} \to \mathbb{N}$ and $g \colon \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$ and $g(n) = n + 1$, respectively.

(a) Compute $f \circ g$;

(b) compute $g \circ f$;

(c) compute g^m for every $m \in \mathbb{N}$.

Solution:

(a) We have $f \circ g(n) = f(g(n)) = f(n+1) = (n+1)^2 = n^2 + 2n + 1$ for every $n \in \mathbb{N}$.

(b) We have $g \circ f(n) = g(f(n)) = g(n^2) = n^2 + 1$ for every $n \in \mathbb{N}$.

(c) The function g adds 1 to the input. If we repeat it m times, g^m will add 1 m times to the original input. That is, $g^m(n) = n + m$.

11. * Functions.

Consider the following set of $n + 1$ points in $\mathbb{R}^2$:

$$S := \{(x_i, y_i) | i = 0, 1, \cdots, n\},$$

where $x_i \neq x_j$ for $i \neq j$.
Find a polynomial p of degree at most n such that $p(x_i) = y_i$ for any $i = 0, 1, \cdots, n$.
You may use the following fact: If p is a sum of polynomials of degree n, then p is a polynomial of degree at most n.
(*Hint: Try to first find degree n polynomials φ_i for $i = 0, 1, \cdots, n$ s.t. $\varphi_i(x_j) = \delta_{ij}$, where δ_{ij} is defined as follows:*

$$\delta_{ij} = \begin{cases} 0 & \textit{for } i \neq j, \\ 1 & \textit{for } i = j. \end{cases}$$

Using the polynomials φ_i, can you construct p?)

Solution: We start with the hint. The aim is to find a polynomial φ_i of degree n with roots at $x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ and $\varphi_i(x_i) = 1$. We know the n roots of φ_i. We can therefore conclude that φ_i must be of the following form

$$\varphi_i(x) = C(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n) = C \prod_{j=0, j\neq i}^{n} (x - x_j)$$

for some non-zero constant C. As long as $C \neq 0$ the roots of φ_i remain unchanged whatever we choose C to be. We therefore choose C s.t. $\varphi(x_i) = 1$. Hence,

$$1 = \varphi_i(x_i) = C \prod_{j=0, j\neq i}^{n} (x_i - x_j) \Rightarrow C = \frac{1}{\prod\limits_{j=0, j\neq i}^{n} (x_i - x_j)}$$

Hence,

$$\varphi_i(x) = \frac{\prod\limits_{j=0, j\neq i}^{n} (x - x_j)}{\prod\limits_{j=0, j\neq i}^{n} (x_i - x_j)}$$

Hence, the polynomial p of degree at most n that interpolates the points $\{(x_i, y_i) : i = 0, 1, \cdots, n\}$ is

$$p(x) = \sum_{j=0}^{n} y_j \varphi_j(x)$$

since by construction of φ_i

$$p(x_i) = \sum_{j=0}^{n} y_j \varphi_j(x_i) = \sum_{j=0}^{n} y_j \delta_{ij} = y_i \delta_{ii} = y_i$$

Note that p has degree at most n since it is a sum of degree n polynomials.

12. Sets.

For sets E, F and D prove the following:

(a) Commutativity: $E \cap F = F \cap E$ and $E \cup F = F \cup E$;

(b) Associativity: $D \cap (E \cap F) = (D \cap E) \cap F$ and $D \cup (E \cup F) = (D \cup E) \cup F$;

(c) Distributivity: $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$ and $D \cup (E \cap F) = (D \cup E) \cap (D \cup F)$;

(d) De Morgan laws: $(E \cap F)^c = E^c \cup F^c$ and $(E \cup F)^c = E^c \cap F^c$.

Solution: A standard approach to show that two sets A and B are equal is to first show $A \subseteq B$ and then $B \subseteq A$. These two inclusions together imply $A = B$. We are going to use this approach in each part of the problem.

(a) Let $x \in E \cap F$. As $E \cap F \subseteq F$, we have $x \in F$. As $E \cap F \subseteq E$, we have $x \in E$. Thus, we have $x \in F$ and $x \in E$; that is, $x \in F \cap E$. Since any element of $E \cap F$ is in $F \cap E$, we have $E \cap F \subseteq F \cap E$. The reversed inclusion (i.e., $F \cap E \subseteq E \cap F$) is proved analogously (just swap the roles). Thus, we conclude $E \cap F = F \cap E$.

Now, let $x \in E \cup F$. By definition of $E \cup F$, at least one among $x \in E$ and $x \in F$ holds true. If $x \in E$, we have $x \in F \cup E$, as $E \subseteq F \cup E$. If $x \in F$, we have $x \in F \cup E$, as $F \subseteq F \cup E$. Thus, in either case we have $x \in F \cup E$. Thus, it follows that $E \cup F \subseteq F \cup E$. Similarly, we can show $F \cup E \subseteq F \cup F$, which allows to conclude $E \cup F = F \cup E$.

(b) We show the associativity of intersection. The associativity of inclusion is proved with a similar argument.

Let $x \in D \cap (E \cap F)$. Thus, $x \in D$ and $x \in E \cap F$. As $x \in E \cap F$, we have $x \in E$ and $x \in F$. Now, as $x \in D$ and $x \in E$, we have $x \in D \cap E$. As $x \in D \cap E$ and $x \in F$, we conclude $x \in (D \cap E) \cap F$. Thus, we have $D \cap (E \cap F) \subseteq (D \cap E) \cap F$. The reversed inclusion is proved analogously and allows to conclude $D \cap (E \cap F) = (D \cap E) \cap F$.

(c) Let $x \in D \cap (E \cup F)$. Then $x \in D$ and in at least one of E and F. If $x \in E$, we have $x \in D \cap E$. If $x \in F$, we have $x \in D \cap F$. So, x is always in at least one among $D \cap E$ and $D \cap F$; that is, $x \in (D \cap E) \cup (D \cap F)$. This shows $D \cap (E \cup F) \subseteq (D \cap E) \cup (D \cap F)$.

Now, let $x \in (D \cap E) \cup (D \cap F)$. If $x \in D \cap E$, we have $x \in D$ and $x \in E \subseteq E \cup F$; thus, $x \in D \cap (E \cup F)$. Similarly, if $x \in D \cap F$, we have $x \in D$ and $x \in F \subseteq E \cup F$l this, $x \in D \cap (E \cup F)$. This shows the reversed inclusion and we conclude $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$.

Now, let $x \in (D \cup E) \cap (D \cup F)$. Then, $x \in D \cup E$ and $x \in D \cup F$. If $x \in D$, then $x \in D \subseteq D \cup (E \cap F)$. Now, assume $x \notin D$. Then, as $x \in D \cup E$, it follows $x \in E$. Similarly, as $x \in D \cup F$, it follows $x \in F$. Thus, $x \in E$ and $x \in F$. Hence, $x \in E \cap F \subseteq D \cup (E \cap F)$. This shows $(D \cup E) \cap (D \cup F) \subseteq D \cup (E \cap F)$.

Now, let $x \in D \cup (E \cap F)$. If $x \in D$, then $x \in D \subseteq D \cup E$ and $x \in D \subseteq D \cup F$; as $x \in D \cup E$ and $D \cup F$, we have $x \in (D \cup E) \cap (D \cap F)$. Now, assume $x \notin D$. As $x \in D \cup (E \cap F)$, it follows that $x \in E \cap F$. In particular, $x \in E \subseteq D \cup E$ and $x \in F \subseteq D \cup F$. So, $x \in (D \cup E) \cap (D \cup F)$. This proves the reversed inclusion and we get the sought equality.

(d) Let $x \in (E \cap F)^c$. If $x \in E^c$, then $x \in E^c \cup F^c$. So, assume $x \notin E^c$; that is, assume $x \in E$. Then, it has to be the case that $x \in F^c$: indeed, if $x \in F$ were true, we'd have $x \in E \cap F$, contradicting that $x \in (E \cap F)^c$. Thus, if $x \in (E \cap F)^c$, at least one among $x \in E^c$ and $x \in F^c$ holds; that is $x \in E^c \cup F^c$.

Now, assume that $x \in E^c \cup F^c$. At least one among $x \in E^c$ and $x \in F^c$ holds. If $x \in E^c$, then $x \notin E$, so $x \notin E \cap F$, as $E \cap F \subseteq E$. If $x \in F^c$, then $x \notin F$, so $x \notin E \cap F$, as $E \cap F \subseteq F$. In either case, we have $x \notin E \cap F$; that is, $x \in (E \cap F)^c$. This concludes the proof of $(E \cap F)^c = E^c \cup F^c$.

Let $x \in (E \cup F)^c$. Then, $x \notin E$: indeed, if $x \in E$, then $x \in E \subset E \cup F$, which is impossible, as $x \in (E \cup F)^c$. Similarly, $x \notin F$. Thus, $x \in E^c$ and $x \in F^c$. Thus, $x \in E^c \cap F^c$. Hence, $(E \cup F)^c \subseteq E^c \cap F^c$.

Now, assume $x \in E^c \cap F^c$. As $x \in E^c$, $x \notin E$. Similarly, $x \notin F$. So, x is neither in E nor in F. Thus, by definition of $E \cup F$, $x \notin E \cup F$. Hence, $x \in (E \cup F)^c$. So, we get the reversed inclusion.

13. Representations of numbers.

Prove that a real number is a rational number if and only if the decimal representation becomes periodic.

Solution: Let $q \in \mathbb{Q}$. Without loss of generality (in short, wlog) we may assume $q \in [0, 1)$ by removing the integral part.
Let $q = \frac{a}{b}$ where $0 \leq a < b$ and a, b are natural numbers. We define recursively two sequences of numbers, $(a_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$, by means of long division by b. We set $a_0 = 0$ and $r_0 = a$. Then, the recursion is as follows: $10r_{n-1} = a_n b + r_n$, where $0 \leq r_n < b$. That is, r_n is the remainder of the division of $10r_{n-1}$ by b (convince yourself of this by writing out a few terms!). By construction of $(a_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$, $\frac{a}{b} - (\frac{a_1}{10} + \cdots + \frac{a_n}{10^n}) = \frac{r_n}{b}10^{-n} < 10^{-n}$. Therefore, a_i must be the i-th digit in the decimal expansion of q, that is, $q = 0.a_1a_2a_3\cdots$. Since r_i is a natural number that is the remainder of a division by b, we have $r_i \in \{0, \cdots, b-1\}$. So, after $b+1$ steps of this, a remainder r_i must have been repeated more once, because r_i can only take at most b values. This means that the sequence of a_i must repeat at one point and the recurring sequence cannot be longer than b in length. This finishes the proof.

($\Leftarrow$) Let $a = z.x_1 \cdots x_m\overline{y_1 \cdots y_n}$ be the decimal expansion of a real number a. Then,

$$10^m a = zx_1 \cdots x_m.\overline{y_1 \cdots y_n}$$

and

$$10^{m+n}a = zx_1 \cdots x_m y_1 \cdots y_n.\overline{y_1 \cdots y_n}.$$

Thus,

$$(10^{m+n} - 10^m)a = zx_1 \cdots x_m y_1 \cdots y_n - zx_1 \cdots x_m \in \mathbb{Z}$$
$$\implies a = \frac{zx_1 \cdots x_m y_1 \cdots y_n - zx_1 \cdots x_m}{10^{m+n} - 10^m} \Rightarrow a \in \mathbb{Q}.$$