

Analysis 1 - Exercise Set 14

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State on which closed intervals the following function is integrable and compute the antiderivatives:

$$f(x) = \frac{1}{1-x^2}.$$

Solution:

The function is continuous and hence integrable on every closed interval contained in $\mathbb{R} \setminus \{\pm 1\}$. We compute the antiderivative formally as follows: this is a rational function, so we write

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{2} \frac{1-x+1+x}{(1-x)(1+x)} = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right)$$

So

$$\begin{aligned} F(x) &= \frac{1}{2} \left(\int \frac{1}{1+x} dx + \int \frac{1}{1-x} dx \right) \\ &= \frac{1}{2} (\log |1+x| - \log |1-x|) + c \\ &= \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + c. \end{aligned}$$

2. (a) Compute $\int_0^1 \frac{3x^2+3}{x^6+6x^4+9x^2+1} dx$. (*Hint: recall $\arctan(t)'$*)
 (b) Compute $\int_0^1 \frac{x^2+x-2}{x^3+2x^2+x+2} dx$.
 (c) Compute $\int_0^1 \frac{x^4+8x^3-4x^2-4x}{x^4-x^2-12} dx$.

Solution:

- (a) We know that $\arctan(t)' = \frac{1}{t^2+1}$. We observe that $x^6 + 6x^4 + 9x^2 = (x^3 + 3x)^2$ and that $(x^3 + 3x)' = 3x^2 + 3$. So

$$\int_0^1 \frac{3x^2+3}{x^6+6x^4+9x^2+1} dx = \int_0^1 \frac{3x^2+3}{(x^3+3x)^2+1} dx = \arctan(x^3+3x)|_0^1 = \arctan(4).$$

- (b) We observe that $\frac{x^2+x-2}{x^3+2x^2+x+2} = \frac{(x-1)(x+2)}{(x^2+1)(x+2)} = \frac{x-1}{x^2+1}$, so

$$\begin{aligned} \int_0^1 \frac{x^2+x-2}{x^3+2x^2+x+2} dx &= \frac{1}{2} \int_0^1 \frac{2x}{x^2+1} dx - \int_0^1 \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \log(x^2+1)|_0^1 - \arctan(x)|_0^1 = \frac{1}{2} \log(2) - \frac{\pi}{4} \end{aligned}$$

(c) We factor the denominator into irreducible factors:

$$x^4 - x^2 - 12 = (x^2 + 3)(x - 2)(x + 2).$$

We want to write

$$\frac{x^4 + 8x^3 - 4x^2 - 4x}{(x^2 + 3)(x - 2)(x + 2)} = \frac{Ax + B}{x^2 + 3} + \frac{C}{x - 2} + \frac{D}{x + 2} + E.$$

We compute

$$\begin{aligned} & \frac{Ax + B}{x^2 + 3} + \frac{C}{x - 2} + \frac{D}{x + 2} + E = \\ & \frac{(Ax + B)(x^2 - 4) + C(x^2 + 3)(x + 2) + D(x^2 + 3)(x - 2) + E(x^2 + 3)(x^2 - 4)}{(x^2 + 3)(x^2 - 4)} \\ & = \frac{Ex^4 + (A + C + D)x^3 + (B + 2C - 2D - E)x^2 - (4A - 3C - 3D)x - 4B + 6C - 6D - 12E}{(x^2 + 3)(x^2 - 4)} \end{aligned}$$

So we need to solve the linear system

$$\begin{cases} E = 1 \\ A + C + D = 8 \\ B + 2C - 2D - E = -4 \\ -4A + 3C + 3D = -4 \\ -4B + 6C - 6D - 12E = 0 \end{cases}$$

It has solution $A = 4, B = -3, C = 2, D = 2, E = 1$. So

$$\frac{x^4 + 8x^3 - 4x^2 - 4x}{x^4 - x^2 - 12} = \frac{4x - 3}{x^2 + 3} + \frac{2}{x - 2} + \frac{2}{x + 2} + 1$$

We compute the resulting integrals separately

$$\int_0^1 \frac{2}{x - 2} + \frac{2}{x + 2} + 1 dx = (2 \log |x^2 - 4| + x)|_0^1 = 2 \log \frac{3}{4} + 1$$

$$\begin{aligned} \int_0^1 \frac{4x - 3}{x^2 + 3} dx &= 2 \int_0^1 \frac{2x}{x^2 + 3} dx - 3 \int_0^1 \frac{1}{x^2 + 3} dx = (2 \log |x^2 + 3|)|_0^1 - \sqrt{3} \int_0^1 \frac{\frac{1}{\sqrt{3}}}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1} dx \\ &= 2 \log \frac{4}{3} - \sqrt{3} \arctan \left(\frac{x}{\sqrt{3}} \right) \Big|_0^1 = 2 \log \frac{4}{3} - \frac{\pi}{2\sqrt{3}} \end{aligned}$$

So

$$\int_0^1 \frac{x^4 + 8x^3 - 10x^2 - 2x}{x^4 - x^2 - 12} dx = 1 - \frac{\pi}{2\sqrt{3}}.$$

3. Calculate the following formal integrals

(a) $\int \frac{x(x^2 + x - 2)}{(x^2 - x + 2)(x - 2)^2} dx.$

(b) $\int \frac{3x + 4}{1 + x^2} dx$ (*Hint: recall $(\arctan(x))'$*).

Solution:

- (a) We observe that $x^2 - x + 2$ has no real solutions because the discriminant of the quadratic polynomial is $(-1)^2 - 4 \cdot 1 \cdot 2 = -7 < 0$. We want to write the rational function as

$$\frac{x(x^2 + x - 2)}{(x^2 - x + 2)(x - 2)^2} = \frac{Ax + B}{x^2 - x + 2} + \frac{C}{(x - 2)^2} + \frac{D}{x - 2} + E$$

for suitable $A, B, C, D, E, F \in \mathbb{R}$. So we compute

$$\begin{aligned} & \frac{Ax + B}{x^2 - x + 2} + \frac{C}{(x - 2)^2} + \frac{D}{x - 2} + E \\ = & \frac{(Ax + B)(x^2 - 4x + 4) + C(x^2 - x + 2) + D(x - 2)(x^2 - x + 2) + E(x^2 - 4x + 4)(x^2 - x + 2)}{(x^2 - x + 2)(x - 2)^2} \\ = & \frac{Ex^4 + (A + D - 5E)x^3 + (B - 4A + C - 3D + 10E)x^2}{(x^2 - x + 2)(x - 2)^2} \\ & + \frac{(4A - 4B - C + 4D - 12E)x + 4B + 2C - 4D + 8E}{(x^2 - x + 2)(x - 2)^2} \end{aligned}$$

Since $x(x^2 + x - 2) = x^3 + x^2 - 2x$, we have to take $E = 0$ and we need to solve the linear system of equations

$$\begin{cases} A + D = 1 \\ B - 4A + C - 3D = 1 \\ 4A - 4B - C + 4D = -2 \\ 4B + 2C - 4D = 0 \end{cases}$$

It has solution $A = -1, B = 1, C = 2, D = 2$, so

$$\frac{x(x^2 + x - 2)}{(x^2 - x + 2)(x - 2)^2} = \frac{1 - x}{x^2 - x + 2} + \frac{2}{(x - 2)^2} + \frac{2}{x - 2}.$$

We compute the three resulting integral separately.

$$\begin{aligned} \int \frac{2}{(x - 2)^2} dx &= -\frac{2}{x - 2} + C, \quad \int \frac{2}{x - 2} dx = 2 \log |x - 2| + C \\ \int \frac{1 - x}{(x^2 - x + 2)} dx &= -\frac{1}{2} \int \frac{2x - 1}{x^2 - x + 2} - \frac{1}{x^2 - x + 2} dx = -\frac{1}{2} \log |x^2 - x + 2| + \frac{1}{2} \int \frac{1}{(x - \frac{1}{2})^2 + (2 - \frac{1}{4})} dx \end{aligned}$$

the integral on the right is

$$\frac{1}{2} \int \frac{1}{(x - \frac{1}{2})^2 + \frac{7}{4}} dx = \frac{4}{7} \frac{\sqrt{7}}{4} \int \frac{\frac{2}{\sqrt{7}}}{(\frac{2}{\sqrt{7}}(x - \frac{1}{2}))^2 + 1} dx = \frac{1}{\sqrt{7}} \arctan \left(\frac{2x - 1}{\sqrt{7}} \right) + C$$

So

$$\int \frac{x(x^2 + x - 2)}{(x^2 - x + 2)(x - 2)^2} dx = -\frac{2}{x - 2} - 2 \log |x - 2| + \frac{1}{2} \log |x^2 - x + 2| + \frac{1}{\sqrt{7}} \arctan \left(\frac{2x - 1}{\sqrt{7}} \right) + C.$$

- (b) We separate the quotient into two terms

$$\begin{aligned} \int \frac{3x + 4}{1 + x^2} dx &= \int \left(\frac{3x}{1 + x^2} + \frac{4}{1 + x^2} \right) dx = \frac{3}{2} \int \frac{2x}{1 + x^2} dx + 4 \int \frac{1}{1 + x^2} dx \\ &= \frac{3}{2} \log(1 + x^2) + 4 \arctan(x) + C. \end{aligned}$$

4. Calculate the following integrals.

- (a) $\int_2^3 \frac{x^2+1}{x^2-1} dx$
 (b) $\int_2^3 \frac{\sqrt{x+1}}{x} dx$

Solution:

(a) This is an integral of a rational function. We want

$$\frac{x^2+1}{x^2-1} = A + \frac{B}{x-1} + \frac{C}{x+1} = \frac{Ax^2 + (B+C)x + (-A+B-C)}{x^2-1}.$$

And it turns out that $A = 1$, $B = 1$ and $C = -1$. So the integral turns into

$$\begin{aligned} \int_2^3 \frac{x^2+1}{x^2-1} dx &= \int_2^3 dx + \int_2^3 \frac{1}{x-1} dx - \int_2^3 \frac{1}{x+1} dx \\ &= x \Big|_2^3 + \log(x-1) \Big|_2^3 - \log(x+1) \Big|_2^3 = 1 + \log\left(\frac{3}{2}\right). \end{aligned}$$

(b) Take $u = \sqrt{1+x}$ which means that $x = u^2 - 1$ and that $dx = 2u du$. Also note that $u(2) = \sqrt{3}$ and $u(3) = 2$. Now we have

$$\begin{aligned} \int_2^3 \frac{\sqrt{x+1}}{x} dx &= 2 \int_{\sqrt{3}}^2 \frac{u^2}{u^2-1} du = 2 \int_{\sqrt{3}}^2 \left(1 + \frac{1}{u^2-1}\right) du \\ &= 2 \int_{\sqrt{3}}^2 du + \int_{\sqrt{3}}^2 \frac{u+1-(u-1)}{(u+1)(u-1)} du \\ &= 2 \int_{\sqrt{3}}^2 du + \int_{\sqrt{3}}^2 \frac{1}{u-1} du - \int_{\sqrt{3}}^2 \frac{1}{u+1} du \\ &= \left[2u + \log\left(\left|\frac{u-1}{u+1}\right|\right)\right]_{\sqrt{3}}^2 = 4 - 2\sqrt{3} + \log\left(\frac{\sqrt{3}+1}{3(\sqrt{3}-1)}\right). \end{aligned}$$

5. Let $f: E \rightarrow \mathbb{R}$, $E =]a, b[$ an open interval. Assume that f is differentiable on E and that there exists a positive real number $C > 0$ such that $|f'(x)| \leq C$, $\forall x \in E$. Show that f is Lipschitz.

Solution: By the Mean Value Theorem, $\forall x, y \in E$, there exists $c \in]x, y[$ such that $f(x) - f(y) = f'(c)(x - y)$. Thus, as $|f'(c)| \leq C$, by assumption, then

$$|f(x) - f(y)| = |f'(c)| \cdot |x - y| \leq C|x - y|.$$

6. Compute the following improper integrals

- (a) $\int_1^{+\infty} \frac{x}{\sqrt{(x^2+5)^3}} dx$
 (b) $\int_0^{+\infty} \frac{\operatorname{arccot} x}{1+x^2} dx$
 (c) $\int_0^{+\infty} (x^3(8+x^4)^{-5/3} + 2xe^x) dx$

Solution:

(a) We have

$$\int \frac{x}{\sqrt{(x^2 + 5)^3}} dx = \frac{1}{2} \int 2x(x^2 + 5)^{-3/2} dx = -\frac{1}{\sqrt{x^2 + 5}} + c$$

And

$$\lim_{t \rightarrow +\infty} \left(-\frac{1}{\sqrt{x^2 + 5}} \Big|_1^t \right) = \lim_{t \rightarrow \infty} \left(-\frac{1}{\sqrt{t^2 + 5}} + \frac{1}{\sqrt{6}} \right) = \frac{1}{\sqrt{6}}$$

(b) We have

$$\int \frac{\operatorname{arccot}(x)}{1 + x^2} dx = - \int \operatorname{arccot}(x) \operatorname{arccot}'(x) dx = -\frac{1}{2} \operatorname{arccot}^2(x) + c$$

so the result is

$$\lim_{t \rightarrow +\infty} \left(-\frac{1}{2} \operatorname{arccot}^2(t) + \frac{1}{2} \operatorname{arccot}^2(0) \right) = \frac{1}{2} \operatorname{arccot}^2(0) = \frac{\pi^2}{8}.$$

(c) We can split by linearity this integral into the sum of two integral, the second can be done integrating by part, the result is

$$\int \left(x^3(8 + x^4)^{-5/3} + 2xe^x \right) dx = -\frac{3}{8} \frac{1}{\sqrt[3]{(8 + x^4)^2}} + 2xe^x - 2e^x + c$$

Now we compute the limit

$$\lim_{t \rightarrow +\infty} \left(-\frac{3}{8} \frac{1}{\sqrt[3]{(8 + t^4)^2}} + 2e^t(t - 1) + \frac{3}{32} + 2 \right) = +\infty$$

7. Find a recursive formula for $T_n = \int \cos^{2n}(x) dx \quad \forall n \in \mathbb{N}$.

Solution: Using integration by parts and the relation $\sin^2(x) = 1 - \cos^2(x)$ we get

$$\begin{aligned} T_n &= \int \cos^{2n}(x) dx \\ &= \int (1 - \sin^2(x)) \cos^{2n-2}(x) dx \\ &= \int \cos^{2n-2}(x) dx - \int \sin^2(x) \cos^{2n-2}(x) dx \\ &= T_{n-1} + \frac{1}{2n-1} \int \sin(x)(2n-1)(-\sin(x)) \cos^{2n-2}(x) dx \\ &= T_{n-1} + \frac{\sin(x) \cos^{2n-1}(x)}{2n-1} - \frac{1}{2n-1} \int \cos(x) \cos^{2n-1}(x) dx \\ &= T_{n-1} + \frac{\sin(x) \cos^{2n-1}(x)}{2n-1} - \frac{T_n}{2n-1}. \end{aligned}$$

Hence, rearranging the equation, we get

$$\frac{2n}{2n-1} T_n = T_{n-1} + \frac{\sin(x) \cos^{2n-1}(x)}{2n-1},$$

which can then be disguised as

$$T_n = \frac{\sin(x) \cos^{2n-1}(x)}{2n} + \frac{2n-1}{2n} T_{n-1}.$$

8. Find the radius and interval of convergence for each series.

(a) $\sum_{k=1}^{\infty} \frac{7k-22}{k^2(55k+94)} (x+2)^k$

(b) $\sum_{k=0}^{\infty} \frac{4}{(k+5)!} (x+6)^k$

(c) $\sum_{k=1}^{\infty} \frac{k \cdot k!}{k^4+3k^2} (x-1)^k$

Solution:

(a) Let $a_k = \frac{7k-22}{k^2(55k+94)}$, then

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{(7(k+1)-22) \cdot k^2(55k+94)}{(k+1)^2(55(k+1)+94)(7k-22)} = 1.$$

So the radius of convergence is $R = 1$. If $x = -1$ or $x = -3$ the series is absolutely convergent by comparison criterion because $0 \leq \frac{7k-22}{k^2(55k+94)} \leq \frac{7}{55k^2}$. So the interval of convergence is $[-3, -1]$.

(b) Let $a_k = \frac{4}{(k+5)!}$, then

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{1}{k+6} = 0.$$

So the radius of convergence is $R = +\infty$ and the interval of convergence is all of the real line $]-\infty, +\infty[$.

(c) Let $a_k = \frac{k \cdot k!}{k^4+3k^2}$, then

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{(k+1)^2(k^4+3k^2)}{k((k+1)^4+3(k+1)^2)} = +\infty.$$

So the radius of convergence is $R = 0$ and the interval of convergence is $\{1\} = [1, 1]$.

9. State if the following integrals converge or diverge.

(a) $\int_7^{+\infty} \frac{x}{(\sqrt{x^2-33})^2} dx$

(b) $\int_0^{+\infty} x e^{-x^2} dx$

Solution:

(a) Remark that when $x > 7$ we have $\frac{x}{(\sqrt{x^2-33})^2} > \frac{1}{x}$ so by comparison the integral is divergent.

(b) We compute the integral

$$\int_0^z x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_0^z = -\frac{1}{2} e^{-z^2} + \frac{1}{2}.$$

Since $\lim_{z \rightarrow +\infty} -\frac{1}{2} e^{-z^2} + \frac{1}{2} = \frac{1}{2}$, the integral converges.

10. For each of the following functions compute the Taylor series at 0, compute the radius of convergence and show that the function equals its Taylor series.

(a) e^x

(b) $\sin(x)$

Solution:

(a) We observe that $(e^x)' = e^x$, so $(e^x)^{(n)} = e^x$ for all $n > 0$ and has value 1 at $x = 0$. So the Taylor series of e^x is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. We compute the radius of convergence: $\lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$, so the Taylor series converges for all $x \in \mathbb{R}$. Let $]a, b[$ be an open interval containing 0. To prove that $f(x) = e^x$ equals its Taylor series for $x \in]a, b[$, we need to compute the limit

$$\lim_{n \rightarrow \infty} \sup_{y \in]a, b[} |f^{(n)}(y)| \frac{|x|^n}{n!} = \lim_{n \rightarrow \infty} e^b \frac{|x|^n}{n!} = e^b \lim_{n \rightarrow \infty} \frac{|x|^n}{n!},$$

which estimates the error of the $(n-1)$ -st order Taylor polynomial (remember that, for a function of class C^n , the error for the $(n-1)$ -st Taylor polynomial is given by $\frac{f^{(n)}(\xi)}{n!} (x - x_0)^n$, where x_0 is the center of the expansion, which is 0 in this case, and ξ is a suitable point between x and x_0).

But $\lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0$ for all $x \in \mathbb{R}$ because we just proved that the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all $x \geq 0$. Then e^x equals its Taylor series for $x \in]a, b[$ for all open intervals $]a, b[$, and hence for all $x \in \mathbb{R}$.

(b) We recall that

$$\sin^{(n)}(x) = \begin{cases} (-1)^{\frac{n-1}{2}} \cos(x) & n \text{ odd} \\ (-1)^{\frac{n}{2}} \sin(x) & n \text{ even} \end{cases}$$

So the Taylor series of $\sin(x)$ at 0 is $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$, it converges for all $x \in \mathbb{R}$ because the radius of convergence is $+\infty$, as

$$\lim_{k \rightarrow \infty} \frac{\left| \frac{(-1)^{k+1}}{(2(k+1)+1)!} \right|}{\left| \frac{(-1)^k}{(2k+1)!} \right|} = \lim_{k \rightarrow \infty} \frac{1}{(2k+3)(2k+2)} = 0.$$

To prove that $f(x) = \sin(x)$ equals its Taylor series for $x \in \mathbb{R}$, we use the fact that all derivatives of $\sin(x)$ are bounded above in absolute value by the constant function 1. So

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} |f^{(n)}(y)| \frac{|x|^n}{n!} = \lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0.$$

11. Compute the Taylor series of the following functions at 0 and determine the radius of convergence. (Hint: Recall the following Taylor series $\log(y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (y-1)^k$ and use the uniqueness of the Taylor series.)

- (a) $f(x) = \log(1+x)$
 (b) $f(x) = \frac{1}{1+x}$
 (c) $f(x) = \frac{1}{(1-x)^2}$

Solution:

- (a) Recall from the lecture that $\log(y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (y-1)^k$ around 1 with radius of convergence 1. So the equality holds for all $y \in]0, 2[$. Let $x = y - 1$, then for $x \in]-1, 1[$, we have $\log(1+x) = \log y = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (y-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$. The radius of convergence is the same, i.e. 1, because the coefficients of the power series are the same.

- (b) We observe that $(\log(1+x))' = \frac{1}{1+x}$. So we can use the derivative of the Taylor series of $(\log(1+x))'$ computed above to determine the Taylor series of $\frac{1}{1+x}$ around 0.

$$\frac{1}{1+x} = (\log(1+x))' = \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \right)' = \sum_{k=1}^{\infty} (-1)^{k+1} x^{k-1} = \sum_{k=0}^{\infty} (-1)^k x^k.$$

The radius of convergence remains equal to 1, because the function f is not defined at -1 .

- (c) We observe that $\frac{1}{(1-x)^2} = \left(\frac{1}{1-x} \right)'$. By uniqueness of the Taylor series, the Taylor series of $\frac{1}{1-x}$ can be computed by substituting $y = -x$ into the Taylor series of $\frac{1}{1+y}$. So $\frac{1}{1-x} = \sum_{k=0}^{\infty} (-1)^k (-x)^k = \sum_{k=0}^{\infty} x^k$ at 0 with radius of convergence 1, and

$$\frac{1}{(1-x)^2} = \left(\sum_{k=0}^{\infty} x^k \right)' = \sum_{k=1}^{\infty} k x^{k-1} = \sum_{k=0}^{\infty} (k+1) x^k.$$

with radius of convergence 1 because the function f is not defined at 1.

12. Compute the following integrals using Taylor series. Your answer should be in the form of an explicit numerical series, but you do not need to compute the sum of the series.

- (a) $\int_0^{\frac{1}{2}} \log(1+2x^2) dx$
 (b) $\int_0^1 \sin(x^3) dx$

Solution:

- (a) By uniqueness of the Taylor series, the Taylor series of $\log(1+2x^2)$ on the interval $] -1, 1[$ can be computed by substituting $y = 2x^2$ into the Taylor series of $\log(1+y)$ computed in a previous exercise. So

$$\log(1+2x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (2x^2)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^k}{k} x^{2k},$$

and we can compute the integral termwise because $[0, \frac{1}{2}] \subseteq]-1, 1[$:

$$\begin{aligned} \int_0^{\frac{1}{2}} \log(1+2x^2) dx &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^k}{k} \int_0^{\frac{1}{2}} x^{2k} dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^k}{k} \frac{x^{2k+1}}{2k+1} \Big|_0^{\frac{1}{2}} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(2k+1)} \frac{1}{2^{k+1}}. \end{aligned}$$

(b) By uniqueness of the Taylor series, the Taylor series of $\sin(x^3)$ can be computed by substituting $y = x^3$ into the Taylor series of $\sin(y)$ computed in a previous exercise.

So $\sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{3(2k+1)}$, and we can compute the integral termwise:

$$\int_0^1 \sin(x^3) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^1 x^{6k+3} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{6k+4}.$$

13. Compute the following improper integrals

- (a) $\int_{1/2}^{+\infty} \frac{1}{\sqrt{2x(2x+1)}} dx$
 (b) $\int_0^{+\infty} \frac{9x+8}{(x+2)(x^2+1)} dx$

Solution:

(a) Making the change of variables $u = \sqrt{2x}$ (hence $u^2 = 2x$, because $x > 0$, and $u du = dx$) we get

$$\int \frac{1}{u^2+1} du = \arctan(u) + c$$

hence we have to compute the following

$$\lim_{t \rightarrow \infty} \left(\arctan(\sqrt{2t}) - \arctan(1) \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

(b) Write

$$\frac{9x+8}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}.$$

We get $A = -2$, $B = 2$ and $C = 5$, so

$$\int \frac{9x+8}{(x+2)(x^2+1)} dx = \log \left(\frac{x^2+1}{(x+2)^2} \right) + 5 \arctan(x) + c$$

so the final result is

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\log \left(\frac{t^2+1}{(t+2)^2} \right) - \log \left(\frac{1}{4} \right) + 5 \arctan(t) - \arctan(0) \right) &= \log(1) + \log(4) + 5 \frac{\pi}{2} - 0 \\ &= \frac{5}{2} \pi + \log(4). \end{aligned}$$

14. Find the radius and interval of convergence for each series.

- (a) $\sum_{n=0}^{\infty} n(x-5)^n$

- (b) $\sum_{n=0}^{\infty} \left(1 + \frac{1}{n}\right)^n (x+3)^n$
 (c) $\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$ (without checking the boundary points)

Solution:

- (a) Define $a_n = n$. We know that if $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| = L$ then the radius of convergence is $R = 1/L$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

So the radius of convergence is $R = 1$ and the convergence interval is $T =]4, 6[$. Note that we removed the boundaries 4 and 6 since the power series will be divergent for $x = 4$ and $x = 6$, since the sequences n and $(-1)^n n$ do not converge to 0, as so their series cannot possibly converge.

- (b) Define $a_n = \left(1 + \frac{1}{n}\right)^n$. We know that if $\lim_{n \rightarrow \infty} |a_n|^{1/n} = L$ then the radius of convergence is $R = 1/L$. We have

$$\lim_{n \rightarrow \infty} \left| \left(\left(1 + \frac{1}{n}\right)^n \right)^{1/n} \right| = \lim_{n \rightarrow \infty} \left| 1 + \frac{1}{n} \right| = 1.$$

So the radius of convergence is $R = 1$ and the interval of convergence is $I =]-4, -2[$ and the boundaries of the interval are excluded because the series will be divergent for $x = -4$ and $x = -2$. Indeed, if $x = -2$, we consider $\sum_{n=0}^{\infty} \left(1 + \frac{1}{n}\right)^n$, and if $x = -4$, we consider $\sum_{n=0}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$, and in both cases the sequence we are taking the series of does not converge to 0.

- (c) Define $a_n = \frac{n!}{n^n}$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n \rightarrow \infty} (n+1) \cdot \frac{1}{n+1} \cdot \left(\frac{n}{n+1} \right)^n \\ &= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} = L, \end{aligned}$$

so the radius of convergence is $R = \frac{1}{L} = e$.

15. State if the following integrals converge or diverge. Motivate your answer.

- (a) $\int_1^{+\infty} \frac{\log x}{x^2} dx$
 (b) $\int_1^{+\infty} \frac{1}{\sqrt{1+x}} dx$.

Solution:

- (a) We compute the truncated integral by parts:

$$\int_1^z \frac{\log x}{x^2} dx = -\frac{\log x}{x} \Big|_1^z + \int_1^z \frac{1}{x^2} dx = -\frac{\log z}{z} - \frac{1}{z} + 1.$$

Since $\lim_{z \rightarrow +\infty} \frac{\log z}{z} = 0$ by L'Hôpital, we have

$$\int_1^{+\infty} \frac{\log x}{x^2} dx = \lim_{z \rightarrow +\infty} \frac{\log z}{z} - \frac{1}{z} + 1 = 1.$$

So the integral converges.

(b) We observe that a primitive of $(1+x)^{-1/2}$ is $2(1+x)^{1/2}$, so

$$\int_1^{+\infty} \frac{1}{\sqrt{1+x}} dx = \lim_{z \rightarrow +\infty} \int_1^z \frac{1}{\sqrt{1+x}} dx = \lim_{z \rightarrow +\infty} 2(1+z)^{1/2} - 2\sqrt{2} = +\infty$$

and the integral diverges.

16. For each of the following functions compute the Taylor series at 0, compute the radius of convergence and show that the function equals its Taylor series.

(a) $\cos(x)$

(b) $\sinh(x)$

Solution:

(a) For the function $\cos(x)$ it suffices to recall that it is the derivative of the function $\sin(x)$, that we can compute derivatives of converging power series termwise, so the uniqueness of the derivative gives

$$\cos(x) = (\sin(x))' = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right)' = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

for all $x \in \mathbb{R}$. By uniqueness of the Taylor series, this power series is the Taylor series of $\cos(x)$.

(b) We recall that

$$(\sinh(x))^{(n)} = \begin{cases} \cosh(x) & n \text{ odd} \\ \sinh(x) & n \text{ even.} \end{cases}$$

So the Taylor series of $\sinh(x)$ at 0 is $\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$ and converges for all $x \in \mathbb{R}$ because $\lim_{k \rightarrow \infty} \frac{\frac{1}{(2(k+1)+1)!}}{\frac{1}{(2k+1)!}} = 0$. Let $a > 0$. To prove that $f(x) = \sinh(x)$ equals its Taylor series for $x \in]-a, a[$, we compute the limit

$$\lim_{n \rightarrow \infty} \sup_{y \in]-a, a[} |f^{(n)}(y)| \frac{|x|^n}{n!} \leq \lim_{n \rightarrow \infty} \cosh(a) \frac{|x|^n}{n!} = \cosh(a) \lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0.$$

In the computation, we used that the n -th derivative is either $\cosh$ or $\sinh$, but $\cosh(t) > \sinh(t)$ for all t , so we can estimate any derivative (regardless of n even or odd) with $\cosh$. Since it holds for all intervals $] -a, a[$, then $\sinh(x)$ equals its Taylor series for all $x \in \mathbb{R}$.

17. Compute the Taylor series of the following functions at 0 and determine the radius of convergence.

- (a) $f(x) = \frac{15}{6-8x}$
 (b) $f(x) = 2xe^{4x^2}$
 (c) $f(x) = \operatorname{arccot} x$

Solution:

- (a) We observe that $\frac{15}{6-8x} = \frac{15}{6} \frac{1}{1-\frac{4}{3}x}$. By uniqueness of the Taylor series, we can compute the Taylor series by substituting $y = -\frac{4}{3}x$ in the Taylor series of $\frac{1}{1+y}$ at 0 computed above. So

$$\frac{15}{6-8x} = \frac{15}{6} \sum_{k=0}^{\infty} (-1)^k \left(-\frac{4}{3}x\right)^k = \sum_{k=0}^{\infty} \frac{15}{6} \left(\frac{4}{3}\right)^k x^k.$$

Since $\lim_{k \rightarrow +\infty} \sqrt[k]{\frac{15}{6} \left(\frac{4}{3}\right)^k} = \frac{4}{3}$, the radius of convergence is $\frac{3}{4}$.

- (b) We observe that $2xe^{4x^2} = \frac{1}{4}(e^{4x^2})'$. So by substituting $y = 4x^2$ in the Taylor series of e^y we get

$$2xe^{4x^2} = \frac{1}{4} \left(\sum_{k=0}^{\infty} \frac{(4x^2)^k}{k!} \right)' = \left(\sum_{k=0}^{\infty} \frac{4^{k-1}}{k!} x^{2k} \right)' = \sum_{k=1}^{\infty} \frac{2 \cdot 4^{k-1}}{(k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{2 \cdot 4^k}{k!} x^{2k+1}.$$

It is the Taylor series of $2xe^{4x^2}$ by uniqueness of the Taylor series. For the radius of convergence we need to compute the limit

$$\lim_{k \rightarrow \infty} \frac{2 \cdot 4^{k+1}}{(k+1)!} \left(\frac{2 \cdot 4^k}{k!} \right)^{-1} = \lim_{k \rightarrow \infty} \frac{4}{k+1} = 0.$$

So the radius of convergence is ∞ .

- (c) We observe that $\operatorname{arccot} x$ is a primitive of the function $-\frac{1}{1+x^2}$. By uniqueness of the Taylor series, we can compute the Taylor series of $\frac{1}{1+x^2}$ by substituting $y = x^2$ in the Taylor series of $\frac{1}{1+y}$ at 0. So

$$-\frac{1}{1+x^2} = -\sum_{k=0}^{\infty} (-1)^k x^{2k} = \sum_{k=0}^{\infty} (-1)^{k+1} x^{2k}$$

at 0 with radius of convergence $\sqrt{1} = 1$. We are ready to compute the Taylor series of $\operatorname{arccot} x$ as a formal integral

$$\operatorname{arccot} x = \int \sum_{k=0}^{\infty} (-1)^{k+1} x^{2k} dx + \operatorname{arccot}(0) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} x^{2k+1} + \operatorname{arccot} 0 = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} x^{2k+1}$$

with the same radius of convergence, i.e. $R = 1$.

18. Compute the following integrals using Taylor series. Your answer should be in the form of an explicit numerical series, but you do not need to compute the sum of the series.

- (a) $\int_0^{\frac{1}{3}} \frac{1}{1+x^3} dx$
 (b) $\int_0^1 \cosh(\sqrt{x}) dx$

(c) $\int_0^1 x^4 \sin(x^3) dx$

Solution:

- (a) By uniqueness of the Taylor series, the Taylor series of $\frac{1}{1+x^3}$ on the interval $] -1, 1[$ can be computed by substituting $y = x^3$ into the Taylor series of $\frac{1}{1+y}$ computed above. So $\frac{1}{1+x^3} = \sum_{k=0}^{\infty} (-1)^k x^{3k}$ and

$$\int_0^{\frac{1}{3}} \frac{1}{1+x^3} dx = \sum_{k=0}^{\infty} (-1)^k \int_0^{\frac{1}{3}} x^{3k} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{3k+1}(3k+1)}.$$

- (b) The Taylor series of $\cosh(y)$ can be computed as derivative of the Taylor series of $\sinh(y)$, which was computed on an exercise above. So $\cosh(y) = \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} y^{2k+1} \right)' = \sum_{k=0}^{\infty} \frac{1}{(2k)!} y^{2k}$. By uniqueness of the Taylor series, the Taylor series of $\cosh(\sqrt{x})$ on $[0, +\infty[$ can be computed by substituting $y = \sqrt{x}$ in the Taylor series of $\cosh(y)$. So $\cosh(\sqrt{x}) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^k$, and

$$\int_0^1 \cosh(\sqrt{x}) dx = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_0^1 x^k dx = \sum_{k=0}^{\infty} \frac{1}{(2k)!(k+1)}.$$

- (c) We recall that in a previous exercise we computed the Taylor series $\sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+3}$. Then $x^4 \sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+7}$ and

$$\int_0^1 x^4 \sin(x^3) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^1 x^{6k+7} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(6k+8)}.$$

Review Exercises

19. (Multiple Choice Question) The integral $\int_1^2 x^2 \log x \, dx$ is

- (a) $\frac{8}{3} \log(2) + \frac{7}{9}$
- (b) $\frac{8}{3} \log(2) - \frac{7}{9}$
- (c) $8 \log(2) - \frac{7}{9}$
- (d) $8 \log(2) + \frac{7}{9}$

Solution: (b) is correct. We use integration by parts to get

$$\int_1^2 x^2 \log x \, dx = \frac{x^3}{3} \log x \Big|_1^2 - \int_1^2 \frac{x^3}{3} \frac{1}{x} dx = \frac{x^3}{3} \log x \Big|_1^2 - \frac{x^3}{9} \Big|_1^2 = \frac{8}{3} \log(2) - \frac{7}{9}.$$

20. (Multiple Choice Question) The integral

$$\int_{2/\pi}^{3/\pi} \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx$$

is

- (a) $\frac{1}{2}$
- (b) $\frac{\sqrt{3}}{2}$
- (c) $-\frac{1}{2}$
- (d) $-\frac{\sqrt{3}}{2}$

Solution: (a) is correct. Take $u = \frac{1}{x}$ then $dx = -\frac{1}{u^2}du$ so

$$\int_{2/\pi}^{3/\pi} \frac{\sin(\frac{1}{x})}{x^2} dx = \int_{\pi/2}^{\pi/3} u^2 \sin(u) \left(-\frac{1}{u^2}\right) du = - \int_{\pi/2}^{\pi/3} \sin(u) du = \cos(u)|_{\pi/2}^{\pi/3} = \frac{1}{2}.$$

21. The integral

$$\int_0^1 x^2 e^x dx$$

is

- (a) e
- (b) $2 - e$
- (c) $e + 2$
- (d) $e - 2$

Solution: (d) is correct. We use integration by parts twice:

$$\begin{aligned} \int_0^1 x^2 e^x dx &= (x^2 e^x)|_0^1 - \int_0^1 2x e^x dx = (x^2 e^x)|_0^1 - 2 \left((x e^x)|_0^1 - \int_0^1 e^x dx \right) \\ &= (x^2 e^x)|_0^1 - 2 \left((x e^x)|_0^1 - e^x|_0^1 \right) = e - 2. \end{aligned}$$

22. For each of the following sequences defined by recursion, show the convergence and find the limit $\lim_{n \rightarrow \infty} a_n$.

- (a) $a_{n+1} = \frac{1+a_n}{2+a_n}, \quad a_0 = 1.$
- (b) $a_{n+1} = 1 + \frac{1}{2}a_n^2 - \frac{1}{2}a_n, \quad a_0 = \frac{3}{2}.$

Solution:

- (a) First we notice that if the sequence (a_n) is convergent then its limit $\lim_{n \rightarrow \infty} a_n = a$ must satisfy the relation:

$$a = \frac{1+a}{2+a}.$$

Assuming that $a \neq -2$, the relation above is equivalent to the relation

$$a^2 + a - 1 = 0$$

Which has the solutions $\frac{-1 \pm \sqrt{5}}{2}$. Now we show by induction that 0 is a lower bound for the sequence. We notice that $a_0 = 1 > 0$ so the base of induction hold. Now we assume that $a_n > 0$ then we have $a_{n+1} = \frac{1+a_n}{2+a_n} > 0$. So by induction all $a_n > 0$. So the only possible limit for the sequence is $\frac{-1+\sqrt{5}}{2}$. Now we have to check if the sequence is convergent or not. To show this, we use the fact that monotone and bounded sequences are convergent. We showed that the sequence is bounded below by zero, we just need to show that the sequence is monotone decreasing. Define the difference sequence:

$$d_n = a_n - a_{n-1}$$

we have to show that $d_n < 0$ for all n for monotonicity of (a_n) . We use induction. For the base of induction we have $d_1 = a_1 - a_0 = 2/3 - 1 = -1/3$. Now suppose that $d_n = a_n - a_{n-1} < 0$, then we have:

$$d_{n+1} = a_{n+1} - a_n = \frac{1+a_n}{2+a_n} - \frac{1+a_{n-1}}{2+a_{n-1}} = \frac{a_n - a_{n-1}}{(2+a_n)(2+a_{n-1})} = \frac{d_n}{(2+a_n)(2+a_{n-1})} < 0.$$

So by induction, all $d_n < 0$ meaning that (a_n) is a monotone decreasing sequence. By putting everything together, (a_n) is a convergent sequence and it converges to $\frac{-1+\sqrt{5}}{2}$.

- (b) If the limit $a = \lim_{n \rightarrow \infty} a_n$ exists, it satisfies the equation

$$a = 1 + \frac{1}{2}a^2 - \frac{1}{2}a, \quad (1)$$

This equation is equivalent to

$$a^2 - 3a + 2 = (a-1)(a-2) = 0,$$

So $a = 1$ or $a = 2$.

We have

$$a_2 = 1 + \frac{1}{2} \left(\frac{3}{2} \right)^2 - \frac{1}{2} \cdot \frac{3}{2} = 1 + \frac{9}{8} - \frac{3}{4} = \frac{11}{8} < \frac{12}{8} = \frac{3}{2} = a_1.$$

We show by induction that 1 is a lower bound for the sequence (a_n) . We have

$$a_1 = \frac{3}{2} \geq 1,$$

and if $a_{n-1} \geq 1$, it follows that

$$a_n = 1 + \frac{1}{2}a_{n-1}^2 - \frac{1}{2}a_{n-1} = 1 + \frac{1}{2}a_{n-1}(a_{n-1} - 1) \geq 1.$$

We show by induction that the sequence is monotone decreasing. We already showed that $a_2 \leq a_1$.

Suppose that $a_n \leq a_{n-1}$. Since 1 is a lower bound for the sequence, we obtain

$$0 \leq a_n - 1 \leq a_{n-1} - 1,$$

and so

$$a_n(a_n - 1) \leq a_{n-1}(a_{n-1} - 1),$$

since $0 < a_n \leq a_{n-1}$ (here it is important all quantities are positive to preserve inequalities). Finally, we have

$$a_{n+1} = 1 + \frac{1}{2}a_n^2 - \frac{1}{2}a_n = 1 + \frac{1}{2}a_n(a_n - 1) \leq 1 + \frac{1}{2}a_{n-1}(a_{n-1} - 1) = a_n.$$

The sequence $(a_n)_{n \geq 1}$ is bounded and monotone decreasing. So it is convergent and the limit is $a = \lim_{n \rightarrow \infty} a_n = 1$.

23. **True/False** If the statement is true, you should prove it. If the statement is false, you should provide a counterexample.

- (a) If $(a_n) \subset \mathbb{R}$ is a Cauchy sequence, then (a_n^2) is also a Cauchy sequence.
- (b) If (a_n^2) is a Cauchy sequence then (a_n) is also a Cauchy sequence.

Solution:

- (a) True.

Solution 1: Since (a_n) is Cauchy, it is convergent. Since the product of two convergent sequences is convergent the sequence (a_n^2) is convergent and therefore is Cauchy.

Solution 2: Using the definition of a Cauchy sequence for (a_n^2) , we must show that for a given $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $m, n > N$ we have $|a_m^2 - a_n^2| \leq \epsilon$.

To show this we note that (a_n) is Cauchy and so is convergent and bounded. Now let $\epsilon > 0$ be given and define $M = \max\{a_n\}_{n=1}^{\infty}$. Since (a_n) is Cauchy, we can find N such that $|a_m - a_n| \leq \epsilon/(2M)$ for all $m, n > N$. We now have

$$|a_m^2 - a_n^2| = |a_m + a_n||a_m - a_n| \leq 2M|a_m - a_n| \leq 2M \cdot \frac{\epsilon}{2m} = \epsilon$$

for all $m, n > N$. This shows that (a_n^2) is a Cauchy sequence.

- (b) False. The inverse statement is not necessarily true! For example let $a_n = (-1)^n$ and so $a_n^2 = 1$. (a_n^2) is a constant sequence, therefore it is convergent and so a Cauchy sequence, on the other hand (a_n) does not converge, and so it cannot possibly be a Cauchy sequence.

24. Check the convergence of the following series.

- (a) $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$.
- (b) $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3 + 1}$
- (c) $\sum_{n=1}^{\infty} \frac{n+1}{n^2 + 2n + 3}$
- (d) $\sum_{n=1}^{\infty} e^{-n}$
- (e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$
- (f) $\sum_{n=1}^{\infty} \frac{n(\sqrt{n}+1)}{\sqrt{n^3+2n^2-1}}$
- (g) $\sum_{n=1}^{\infty} \frac{n^3}{(\log 3)^n}$
- (h) $\sum_{n=1}^{\infty} \frac{(-5)^n}{4^{2n+1}(n+1)}$
- (i) $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(n+2)!}$

Solution:

- (a) We use the comparison test. Notice that $0 \leq \frac{e^{-n}}{n^2} < \frac{1}{n^2}$. And since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, then $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ is also convergent.

(b) We use the comparison test. Notice that $0 \leq \frac{n \sin^2 n}{n^3+1} < \frac{n}{n^3+1} < \frac{1}{n^2}$. And since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, then $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3+1}$ is also convergent.

(c) We have

$$\frac{n+1}{n^2+2n+3} > \frac{n+1}{n^2+4n+3} = \frac{n+1}{(n+1)(n+3)} = \frac{1}{n+3}$$

and $\sum_{n=1}^{\infty} \frac{1}{n+3}$ is the harmonic series (just shifted in the indexes) and is divergent.

(d) This is a geometric series and is convergent since $0 < e^{-1} < 1$.

(e) We use the Leibniz criterion. Define $a_n = \frac{1}{\sqrt{3n-1}}$. Then (a_n) is monotone decreasing and converges to 0, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$ is convergent.

(f) Note that

$$\lim_{n \rightarrow \infty} \frac{n(\sqrt{n}+1)}{\sqrt{n^3+2n^2-1}} = 1.$$

So this series is not convergent.

(g) We use the ratio test. Define the sequence $a_n = \frac{n^3}{(\log 3)^n}$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^3}{(\log 3)^{n+1}}}{\frac{n^3}{(\log 3)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\log 3} \cdot \frac{(n+1)^3}{n^3} = \frac{1}{\log 3} < 1,$$

as $3 > e$. So the series is convergent.

(h) We use the ratio test. Define the sequence $a_n = \frac{(-5)^n}{4^{2n+1}(n+1)}$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-5)^{n+1}}{4^{2n+3}(n+2)}}{\frac{(-5)^n}{4^{2n+1}(n+1)}} \right| = \lim_{n \rightarrow \infty} \left| (-5) \cdot \frac{1}{4^2} \cdot \frac{n+1}{n+2} \right| = \frac{5}{16} < 1.$$

So the series is convergent.

(i) We use the ration test. Define the sequence $a_n = \frac{4^n (n!)^2}{(n+2)!}$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{4^{n+1}((n+1)!)^2}{(n+3)!}}{\frac{4^n (n!)^2}{(n+2)!}} = \lim_{n \rightarrow \infty} 4 \cdot (n+1)^2 \frac{1}{n+2} = \infty.$$

The series is divergent.

25. **True/False.** If the statement is true, you should prove it. If the statement is false, you should provide a counterexample. Let (a_n) and (b_n) be numerical sequences.

(a) If $\lim_{n \rightarrow \infty} |a_n| = a$, then $\limsup_{n \rightarrow \infty} a_n = a$ and $\liminf_{n \rightarrow \infty} a_n = -a$.

(b) If $\limsup_{n \rightarrow \infty} |a_n| = 0$, then (a_n) converges to zero.

(c) If $\limsup_{n \rightarrow \infty} a_n = 0$, then $a_n \leq 0$ for all $n \in \mathbb{N}$.

(d) If $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} b_n = 0$, then $\limsup_{n \rightarrow \infty} (a_n - b_n) = 0$.

Solution:

- (a) False. Take for example the constant sequence $a_n = 1$ for all $n \in \mathbb{N}$. Then $1 = \lim_{n \rightarrow \infty} |a_n| = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n$.
- (b) True. Since $0 \leq \liminf_{n \rightarrow \infty} |a_n| \leq \limsup_{n \rightarrow \infty} |a_n|$, we have $\liminf_{n \rightarrow \infty} |a_n| = \limsup_{n \rightarrow \infty} |a_n| = 0$. So $\lim_{n \rightarrow \infty} |a_n| = 0$ and (a_n) converges to zero too.
- (c) False. Take for example $a_n = \frac{1}{n} \geq 0$ for all $n \in \mathbb{N}^*$. Then $\sup \left\{ \frac{1}{n}, \frac{1}{n+1}, \dots \right\} = \frac{1}{n}$, and so $\limsup_{n \rightarrow \infty} a_n = 0$.
- (d) False. Take for example $a_n = (-1)^n - 1$ and $b_n = (-1)^n + 1$. Then $\sup \{a_n, a_{n+1}, \dots\} = \sup \{0, -2\} = 0$ and $\inf \{b_n, b_{n+1}, \dots\} = \inf \{2, 0\} = 0$ for all $n \in \mathbb{N}$, but $a_n - b_n = -2$ for all $n \in \mathbb{N}$.

26. Check if the limit of the following sequences exist.

- (a) $a_n = \frac{(-1)^n}{n+1}$
- (b) $a_n = (-1)^n + (-1)^{n+2}$
- (c) $a_n = \sin n\pi + \cos n\pi$
- (d) $a_n = 2(-1)^n + \frac{n}{n+1}$

Solution:

- (a) Note that this sequence converges to 0.
- (b) We see that $a_n = -2$ if n is odd and $a_n = 2$ if n is even. So $\lim_{n \rightarrow \infty} a_n$ does not exist.
- (c) We have
$$a_n = \begin{cases} 1 & \text{if } n = 4k \text{ or } n = 4k + 1 \\ -1 & \text{if } n = 4k + 2 \text{ or } n = 4k + 3 \end{cases},$$
so $\lim_{n \rightarrow \infty} a_n$ does not exist.
- (d) The subsequence a_{2n} is equal to $2 + \frac{n}{n+1}$, so it converges to 3; the subsequence a_{2n+1} is equal to $-2 + \frac{1}{n+1}$, hence it converges to -1 . We conclude that $\lim_{n \rightarrow \infty} a_n$ does not exist.