*Analysis I* (English)
Roberto Svaldi and Stefano Filipazzi
Fall Semester 2021–2022

# Analysis 1 - Exercise Set 13

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State on which closed intervals the following functions are integrable and compute the antiderivatives.
   (a) $f(x) = e^x$;
   (b) $f(x) = \sinh(x)$;
   (c) $f(x) = (ax+b)^s$ with $s \in \mathbb{Z}$ and $a, b \in \mathbb{R} \setminus \{0\}$;
   (d) $f(x) = \cos(x)^3$;
   (e) $f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0; \end{cases}$
   (f) $f(x) = \cot(x)$, $\cot(x) := \frac{\cos(x)}{\sin(x)}$.
   (g) $f(x) = |x|^s$, $s > 0$.

**Solution:**

(a) The function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. The antiderivative is $F(x) = e^x + c$.

(b) The function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. We compute the antiderivative formally as follows:

$$F(x) = \int \frac{e^x - e^{-x}}{2} dx = \frac{e^x + e^{-x}}{2} + c = \cosh(x) + c$$

where $c$ is the constant of integration.

(c) If $s \geq 0$ the function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. If $s \leq -1$ the function is continuous and hence integrable on every closed interval contained in $\mathbb{R} \setminus \{-\frac{b}{a}\}$. We compute the antiderivative formally as follows: if $s \neq -1$ then

$$F(x) = \int (ax+b)^s dx = \frac{1}{a} \int a(ax+b)^s dx = \frac{1}{a(s+1)} (ax+b)^{s+1} + c.$$

If $s = -1$, we make the substitution $u = ax + b$ so $x = \frac{1}{a}(u - b)$ with derivative $\frac{1}{a}$

$$\int (ax+b)^{-1} dx = \int \frac{u^{-1}}{a} du = \frac{1}{a} \log|u| + c = \frac{1}{a} \log|ax+b| + c.$$

(d) The function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval.

We have two possible strategies. The first one consists in writing $\cos^3(x) = \cos(x)(1-\sin^2(x))$, so that the integral becomes

$$\int cos(x)\, dx - \int \sin^2(x)\cos(x)\, dx,$$

where the first integral can be computed directly, while for the second one we set $u(x) = \sin(x)$ and $u'(x) = \cos(x)$. Then, we get

$$\begin{aligned}\int cos(x)\, dx - \int \sin^2(x)\cos(x)\, dx &= \sin(x) - \int u^2(x)u'(x)\, dx \\ &= \sin(x) - \frac{1}{3}u^3(x) + c \\ &= \sin(x) - \frac{\sin^3(x)}{3} + c.\end{aligned}$$

Alternatively, we use integration by parts with $g(x) = \cos(x)^2$ and $h(x) = \sin(x)$, so that $g'(x) = -2\cos(x)\sin(x)$ and $h'(x) = \cos(x)$

$$\int \cos(x)^3 dx = \int gh'dx = gh - \int g'hdx = \cos(x)^2\sin(x) + 2\int \cos(x)\sin(x)^2 dx$$
$$= \cos(x)^2\sin(x) + 2\int \cos(x)(1-\cos(x)^2)dx = \cos(x)^2\sin(x) + 2\sin(x) - 2\int \cos(x)^3 dx$$

We use the equation

$$\int \cos(x)^3 dx = \cos(x)^2\sin(x) + 2\sin(x) - 2\int \cos(x)^3 dx$$

to solve for $\int \cos(x)^3 dx$, and we get

$$F(x) = \frac{1}{3}(\sin(x)(2+\cos^2(x))) + c.$$

Notice that the two answers are the same, we just need to rewrite $\cos^2(x) = 1 - \sin^2(x)$ to go from the second answer to the first answer.

(e) Since $f$ is continuous everywhere except at $x = 0$, $f$ is clearly integrable on any closed interval not containing 0. We claim that $f$ is also integrable on intervals containing 0. Let $I = [a, b]$ be such interval, and let $\sigma_n$ be the even partition of $I$ into $n$ subintervals. Notice that 0 belongs to at least one of these sub-intervals, and to at most two of these sub-intervals (this only happens if 0 is the right-end point of a sub-interval that is not the last sub-interval, so that it is also the left-end point of the subsequent one). Notice that, on every sub-interval, the infimum of $f$ is 0, as the function is identically 0 except at 0, where it is strictly positive. To conclude, we need to show that the upper sums converge to 0. If $I_i$ is a sub-interval and $0 \notin I_i$, then the supremum of $f$ on $I_i$ is 0. So, we only need to consider the sub-intervals containing 0. There, the supremum is 1, as $f(0) = 1$. Then, each such sub-interval would contribute with $1 \cdot \frac{b-a}{n}$. Since there are at most 2 such sub-intervals, the upper sum is bounded above by $\frac{2(b-a)}{n}$. As $b$ and $a$ are fixed, this quantity converges to 0 as $n \to +\infty$. So, $f$ is integrable on every closed interval.

There is no antiderivative because if there was it would be $g(x) = \int_0^x f(t)dt + C = C$ and $g'(x) = 0 \quad \forall x \in \mathbb{R}$. Hence, $g'(0) \neq f(0)$.

(f) One can quickly verify that $g(x) = \ln|\sin(x)| + C$ is an antiderivative. This can be seen from $\cot(x) = \frac{(\sin(x))'}{\sin(x)}$. $\cot(x)$ is also continuous on the intervals $S_i = (k\pi, (k+1)\pi)$ and therefore integrable in any interval contained in any of the sets $S_i$. We will show that it is not integrable at any interval containing the endpoints of $S_i$. By symmetry it suffices to consider $[-\frac{\pi}{4}, \frac{\pi}{4}]$. On $[-\frac{\pi}{4}, \frac{\pi}{4}]$ we have $\cos(x) > \frac{1}{2}$. We also know $|\sin(x)| \leq x$.

Let $\Pi = \{\pi_1, \cdots, \pi_n\}$ be a partition of $[-\frac{\pi}{4}, \frac{\pi}{4}]$. There will be at least one of the intervals in $\Pi$ containing 0. Call it $\pi_i$. Then $\sup\limits_{x\in\pi_i} \cot(x) = +\infty$ and $\inf\limits_{x\in\pi_i} = -\infty$ which shows that the Darboux sums do not converge. Hence, $\cot(x)$ is not Riemann integrable in an interval containing a discontinuity of $\cot(x)$.

(g) Since $s > 0$, the function is continuous on $\mathbb{R}$ hence it is integrable on every closed interval. Moreover, we know that the derivative of the function $g\colon \mathbb{R}_+^* \to \mathbb{R}$, $g(x) := x^t$, $t \in \mathbb{R}$ is $g'(t) = tx^{t-1}$. Hence, an antiderivative of $f$ is given by $G(x) = \frac{\text{sgn(x)}}{s+1}|x|^{s+1}$.

2. Determine the number $c$ that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

**Solution:** First let's notice that the function is a polynomial and so is continuous on the given interval. This means that we can use the Mean Value Theorem. We have

$$\begin{aligned}
&\int_1^4 x^2 + 3x + 2 \; dx = (c^2 + 3c + 2)(4 - 1) \\
&\Rightarrow \left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\Big|_1^4 = 3(c^2 + 3c + 2) \\
&\Rightarrow \frac{99}{2} = 3c^2 + 9c + 6 \\
&\Rightarrow 0 = 3c^2 + 9c - \frac{87}{2}
\end{aligned}$$

This equation has the two solutions $c_1 = (-3 + \sqrt{67})/2$ and $c_2 = (-3 - \sqrt{67})/2$. Clearly the second number is not in the interval $[1, 4]$ so $c_1$ is the acceptable value. Note that it is possible for both numbers to be in the interval so don't expect only one to be in the interval.

3. Let

$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq x \leq 3 \end{cases}$$

Compute $\int_0^3 f(x)dx$.

**Solution:**

$$\int_0^3 f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^3 f(x)dx = \int_0^{\frac{\pi}{2}} \sin(x)dx + \int_{\frac{\pi}{2}}^3 1dx = 1 + \left(3 - \frac{\pi}{2}\right).$$

4. **True/False:** If the statement is true you should prove it. If it is false you should give a counterexample. Let $F$ be an anti-derivative of $f$ on $[a, b]$.

   (a) If $f(x) \leq 0$ for all $x \in [a, b]$, then $F(x) \leq 0$ for all $x \in [a, b]$.

   (b) For all $x \in [a, b]$, we have $F(x) = \displaystyle\int_a^x f(t)\, dt$.

> **Solution:**
>
> (a) False. Take for example $f(x) = x$ on the interval $[-2, -1]$. Then $f(x) \leq 0$ on $[-2, -1]$ but $F(x) = \frac{1}{2}x^2 > 0$ For all $x \in [-2, -1]$.
>
> (b) False. Consider for example the constant function $f(x) = 1$ on the interval $[0, 1]$. Then $F(x) = x + 1$ is an anti-derivative of $f$ but
>
> $$\int_0^x f(t)\, dt = \int_0^x dt = x - 0 = x \neq x + 1 = F(x).$$
>
> Indeed, remember that, once we know there is one anti-derivative, then there are infinitely many (we may add any constant!).

5. Show that:

   (a) if $f : [-a, a] \to \mathbb{R}$ is an integrable odd function then $\int_{-a}^{a} f(x)dx = 0$;

   (b) if $f : [-a, a] \to \mathbb{R}$ is an integrable even function then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

> **Solution:**
>
> (a) Using Algebra of Integrals we see
>
> $$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_0^a f(x)dx + \int_{-a}^{0} f(x)dx \\ &= \int_0^a f(x)dx - \int_a^0 f(-t)dt \\ &= \int_0^a f(x)dx + \int_a^0 f(t)dt \\ &= \int_0^a f(x)dx - \int_0^a f(t)dt = 0,\end{aligned}$$
>
> where in the second line we make the change of variable $t = -x$ and $dt = -dx$, in the third line we use that $f$ is odd to get $f(-t) = -f(t)$, and in the last step we flip sign by swapping the extrema of integration.
>
> (b) Using Algebra of Integrals we see
>
> $$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_0^a f(x)dx + \int_{-a}^{0} f(x)dx \\ &= \int_0^a f(x)dx - \int_a^0 f(-t)dt \\ &= \int_0^a f(x)dx - \int_a^0 f(t)dt \\ &= \int_0^a f(x)dx + \int_0^a f(t)dt = 2\int_0^a f(x)dx,\end{aligned}$$

where in the second line we make the change of variable $t = -x$ and $dt = -dx$, in the third line we use that $f$ is even to get $f(-t) = f(t)$, and in the last step we flip sign by swapping the extrema of integration.

6. Calculate the following formal integrals.
   (a) $\int \sin(x)^2 dx$
   (b) $\int \arcsin(x)\ dx$
   (c) $\int \frac{\sinh(x)}{e^x+1}\ dx$
   (d) $\int e^{ax}\cos(bx)\, dx$ $\quad (a \neq 0)$, (*Hint: apply integration by parts multiple times until you see a pattern.*)

**Solution:**

(a) We apply integration by parts with $f = \sin(x)$, $g = -\cos(x)$ and $f' = \cos(x)$, $g' = \sin(x)$.

$$\int \sin(x)^2 dx = -\sin(x)\cos(x) + \int \cos(x)^2 dx$$
$$= -\sin(x)\cos(x) + \int 1 - \sin(x)^2 dx = -\sin(x)\cos(x) + x - \int \sin(x)^2 dx.$$

From the equation we just obtained we get $\int \sin(x)^2 dx = \frac{1}{2}(x - \sin(x)\cos(x)) + C$.

(b) We apply integration by parts. Take $f'(x) = 1$ so $f(x) = x$ and $g(x) = \arcsin x$ so $g'(x) = \frac{1}{\sqrt{1-x^2}}$. We get

$$\int \arcsin x\ dx = \int 1 \cdot \arcsin x\ dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}\ dx}.$$

To figure out the last integral we notice that if we define $u(x) = (1 - x^2)$ then $u'(x) = -2x$. So we can write

$$\int x \cdot \frac{1}{\sqrt{1-x^2}}\ dx = -\frac{1}{2}\int (-2x)(1-x^2)^{-1/2}\ dx = -\frac{1}{2}\int u'(x)u(x)^{-1/2} dx$$
$$= -u(x)^{1/2} + C = -(1-x^2)^{1/2} + C = -\sqrt{1-x^2} + C.$$

Now combine everything together and write:

$$\int \arcsin x\ dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}\ dx} = x \cdot \arcsin x + \sqrt{1-x^2} + C.$$

(c) We use the definition of sinh:

$$\int \frac{\sinh(x)}{e^x+1}\, dx = \frac{1}{2}\int \frac{e^x - e^{-x}}{e^x+1}\, dx = \frac{1}{2}\int \frac{1-(e^{-x})^2}{1+e^{-x}}\, dx$$
$$= \frac{1}{2}\int (1 - e^{-x})\, dx = \frac{1}{2}\left(x + e^{-x}\right) + C\,.$$

(d) We apply integration by parts twice. Let $I_{a,b} = \int e^{ax} \cos(bx)\, dx$. We take $f'(x) = e^{ax}$ [$\Rightarrow f(x) = \frac{1}{a} e^{ax}$] and also $g(x) = \cos(bx)$ [$\Rightarrow g'(x) = -b\sin(bx)$] we get

$$I_{a,b} = \frac{1}{a} e^{ax} \cos(bx) + \frac{b}{a} \int e^{ax} \sin(bx)\, dx.$$

We apply integration by parts one more time on the last integral. Take $f'(x) = e^{ax}$ and $g(x) = \sin(bx)$ [$\Rightarrow g'(x) = b\cos(bx)$]

$$\int e^{ax} \sin(bx)\, dx = \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a} \int e^{ax} \cos(bx)\, dx.$$

Note that we recovered $I_{a,b}$ again in last integral. So we can combine the two equations and compute $I_{a,b}$ as follows:

$$\begin{aligned} I_{a,b} &= \frac{1}{a} e^{ax} \cos(bx) + \frac{b}{a} \left( \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a} I_{a,b} \right) \\ \Leftrightarrow \quad \left(1 + \frac{b^2}{a^2}\right) I_{a,b} &= \frac{e^{ax}}{a} \left( \cos(bx) + \frac{b}{a} \sin(bx) \right) \\ \Leftrightarrow I_{a,b} &= \frac{e^{ax}}{a^2 + b^2} \Big( a\cos(bx) + b\sin(bx) \Big) + C. \end{aligned}$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with a period $T > 0$. Let $F$ be defined by

$$F(x) = \int_0^x f(t)\, dt.$$

Show that $F$ is periodic with period $T$ if and only if

$$\int_0^T f(t)\, dt = 0.$$

**Solution:** We first assume $F$ is periodic with period $T$, and then show this implies $\int_0^T f(t)\, dt = 0$. This is easy, we just insert the definition of $F$

$$0 = F(T) - F(0) = \int_0^T f(t)\, dt - \int_0^0 f(t)\, dt = \int_0^T f(t)\, dt.$$

Now we must show that $\int_0^T f(t)\, dt = 0$ implies that $F$ is periodic i.e. $F(x+T) - F(x) = 0$ for all $x \in \mathbb{R}$. Fix one $x \in \mathbb{R}$. We can use the definition of $F$ to write

$$F(x+T) - F(x) = \int_0^{x+T} f(t)\ dt - \int_0^x f(t)\ dt = \int_x^{x+T} f(t)\ dt = \int_x^{x+T} f(t)\ dt.$$

In the last equality, we used the fact that the integral of a periodic function with period $T$ is always the same on any interval of length $T$; we will prove this claim below. But by the assumption $\int_0^T f(t)\, dt = 0$. So $F(x+T) - F(x) = 0$.

So, we are left to show that the integral of a periodic function with period $T$ is always the same on any interval of length $T$. Said otherwise, we need to show that

$$\int_0^T f(s)ds = \int_x^{x+T} f(t)dt$$

for every $x \in \mathbb{R}$. First, we consider

$$k = \sup\{m \in \mathbb{Z} | mT < x + T\},$$

which exists, as the set is non-empty (e.g., take $m$ very negative so that $mT < x$) and bounded above (e.g., bounded above by $|x|/T + 1$). Then, we use the change of variable $s = t - kT$, and we get

$$\int_x^{x+T} f(t)dt = \int_{x-kT}^{x+T-kT} f(s + kT)ds = \int_{x-kT}^{x+T-kT} f(s)ds,$$

where in the last equality we used that $f(s) = f(s + kT)$, by the $T$-periodicity. Now, by the definition of $k$, we have $x + T - kT > 0$ and $x - kT = x + T - (k + 1)T \leq 0$. So, if we define $y = x - kT$, we have $y \leq 0$ and $0 < y + T \leq T$. So, we have

$$\int_{x-kT}^{x+T-kT} f(s)ds = \int_y^{y+T} f(s)ds = \int_y^0 f(s)ds + \int_0^{y+T} f(s)ds. \tag{1}$$

Now, on the first summand, we make the change of variable $u = s + T$. So, we have

$$\int_y^0 f(s)ds = \int_{y+T}^T f(u - T)du = \int_{y+T}^T f(u)du,$$

where in the second equality we used that $f(u) = f(u - T)$. If we substitute it back in (1), we get

$$\int_{x-kT}^{x+T-kT} f(s)ds = \int_y^0 f(s)ds + \int_0^{y+T} f(s)ds = \int_{y+T}^T f(s)ds + \int_0^{y+T} f(s)ds = \int_0^T f(s)ds,$$

where we used that the variable of integration is just a dummy variable, so we can change the letter we used.

8. Calculate the following integrals.

   (a) $\int_{\pi^2/16}^{\pi^2/9} \cos(\sqrt{x})\, dx$

   (b) $\int_0^{\pi^{1/2017}} \sin(\sin(x^{2017}))\, \cos(x^{2017})\, x^{2016}\, dx$

**Solution:**

(a) We change the variable using $u = \sqrt{x}$ which gives $x = u^2$ and $dx = 2u\ du$. Note that $u(\pi^2/16) = \pi/4$ and $u(\pi^2/9) = \pi/3$. We have

$$\begin{aligned}\int_{\pi^2/16}^{\pi^2/9} \cos(\sqrt{x})\, dx &= 2\int_{\pi/4}^{\pi/3} u\cos(u)\, du \overset{(*)}{=} 2\Big[u\sin(u)\Big]_{\pi/4}^{\pi/3} - 2\int_{\pi/4}^{\pi/3} \sin(u)\, du \\ &= 2\Big[u\sin(u) + \cos(u)\Big]_{\pi/4}^{\pi/3} = 1 - \sqrt{2} - \frac{\pi\sqrt{2}}{4} + \frac{\pi\sqrt{3}}{3}\ .\end{aligned}$$

For $(*)$ we used integration by parts by taking $f'(u) = \cos(u)$, $g(u) = u$.

(b) We change the variable with $u = x^{2017}$ which gives $du = 2017x^{2016}\, dx$. Note that $u(0) = 0$ and $u(\pi^{1/2017}) = \pi$. We get

$$\begin{aligned}
\int_0^{\pi^{1/2017}} \sin(\sin(x^{2017}))\,\cos(x^{2017})\,x^{2016}\,dx &= \frac{1}{2017}\int_0^{\pi} \sin(\sin(u))\cos(u)\,du \\
&= \frac{1}{2017}\Big[-\cos(\sin(u))\Big]_0^{\pi} \qquad \text{since } \big(\sin(u)\big)' = \cos(u) \\
&= \frac{1}{2017}\Big(-\cos(\sin(\pi)) + \cos(\sin(0))\Big) \\
&= \frac{1}{2017}(-\cos(0) + \cos(0)) = 0\ .
\end{aligned}$$

9. **True/False:** Let $I \subset \mathbb{R}$ be an open non-empty and bounded interval and let $f\colon I \to \mathbb{R}$ be a continuous function. Let $[a,b] \subseteq I$. If the statement is true you should prove it. If the statement is false you should give a counter example.

   (a) If $\displaystyle\int_a^b f(x)\,dx = 0$, then $f$ has a zero $[a,b]$.

   (b) If $\displaystyle\int_a^b f(x)\,dx \geq 0$, then $f(x) \geq 0$ for all $x \in [a,b]$.

   (c) If $f(x) < 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx < 0$.

**Solution:**

(a) True. By the mean value theorem for integrals, there exists $u \in\, ]a,b[$ such that $0 = \int_a^b f(x)\,dx = f(u)(b-a)$. Since $b > a$, we must have $f(u) = 0$.

(b) False. Take for example $f(x) = x$ on the interval $[-1,2]$. Then $\int_{-1}^2 f(x)\,dx = \left(\frac{x^2}{2}\right)\Big|_{-1}^{2} = \frac{3}{2} \geq 0$ but $f(-1) = -1 < 0$.

(c) True. By the mean value theorem for integrals, there exists $u \in\, ]a,b[$ such that $\int_a^b f(x)\,dx = f(u)(b-a)$. Since we have $f(u) < 0$ and $b > a$, the result holds.

10. Calculate the following formal integrals.

    (a) $\int \frac{\sin(x)}{\cos(x)^3}\,dx$

    (b) $\int x^2\cos(x)\,dx$

    (c) $\int x\log x\ dx$

    (d) $\int \frac{1}{\sqrt{4-3x^2}}\,dx$ (*Hint: recall* $(\arcsin x)'$)

    (e) $\int (2x+2)e^{x^2+2x+3}\,dx$

    (f) $\int \frac{x^2+1}{x^3+3x}\,dx$

    (g) $\int \frac{x^3}{(1+x^4)^{\frac{1}{3}}}\,dx$

(h) $\int \frac{\sin(\log(x))}{x}\,dx$

(i) $\int (2x+5)(x^2+5x)^7\,dx$

**Solution:**

(a) We take $u(x) = \cos(x)$, so $u'(x) = -\sin(x)$. We get

$$\int \frac{\sin(x)}{\cos(x)^3}\,dx = \int \frac{-u'(x)}{u(x)^3}\,dx = -\int u'(x)u^3(x)\,dx = -\int \left(-\frac{u^{-2}(x)}{2}\right)'\,dx$$
$$= \frac{u^{-2}}{2}(x) + C = \frac{1}{2\cos^2(x)} + C.$$

(b) We use integration by parts twice: first with $f'(x) = \cos(x)$ [$\Rightarrow f(x) = \sin(x)$], $g(x) = x^2$ [$\Rightarrow g'(x) = 2x$] and then with $f'(x) = \sin(x)$ [$\Rightarrow f(x) = -\cos(x)$], $g(x) = x$ [$\Rightarrow g'(x) = 1$], we get

$$\int x^2\cos(x)\,dx = \sin(x)\,x^2 - 2\int \sin(x)\,x\,dx = \sin(x)\,x^2 - 2\left(-\cos(x)\,x + \int \cos(x)\,dx\right)$$
$$= \left(x^2 - 2\right)\sin(x) + 2x\cos(x) + C.$$

(c) Take $f'(x) = x$ and so $f(x) = x^2/2$ and take $g(x) = \log x$ so $g'(x) = \frac{1}{x}$ We have:

$$\int x\log x\;dx = \frac{x^2}{2}\log x - \int \frac{x^2}{2}\frac{1}{x}\;dx = \frac{x^2}{2}\log x - \frac{x^2}{4} + C.$$

(d) Using the change of variable $y = \sqrt{3}x/2$ we obtain

$$\frac{1}{\sqrt{4-3x^2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{1-y^2}}$$

and we also have that $dx = \frac{2}{\sqrt{3}}dy$. So

$$\int \frac{1}{\sqrt{4-3x^2}}\,dx = \int \frac{1}{2}\cdot\frac{1}{\sqrt{1-y^2}}\left(\frac{2}{\sqrt{3}}dy\right) = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-y^2}}\;dy = \frac{1}{\sqrt{3}}\arcsin y + C$$
$$= \frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}x}{2}\right) + C.$$

(e) We set $u(x) = x^2 + 2x + 3$, so that $u'(x) = 2x + 2$. Thus, our integral becomes

$$\int (2x+2)e^{x^2+2x+2}\,dx = \int e^{u(x)}u'(x)\,dx = e^{u(x)} + C = e^{x^2+2x+3} + C.$$

(f) We set $u(x) = x^3 + 3x$, then $u'(x) = 3x + 3$. Then, our integral becomes

$$\int \frac{x^2+1}{x^3+3x}\,dx = \int \frac{u'(x)}{3u(x)}\,dx = \frac{1}{3}\int \frac{u'(x)}{u(x)}\,dx = \frac{1}{3}\log|u(x)| + c = \frac{1}{3}\log|x^3+3x| + C.$$

(g) We set $u(x) = 1 + x^4$, so that $u'(x) = 4x^3$. Then, we have

$$\int \frac{x^3}{(1+x^4)^{\frac{1}{3}}}\,dx = \frac{1}{4}\int \frac{u'(x)}{u^{\frac{1}{3}}(x)}\,dx = \frac{1}{4}\frac{3}{2}u^{\frac{2}{3}}(x) + C = \frac{3}{8}(1+x^4)^{\frac{2}{3}} + C.$$

(h) We set $u(x) = \log(x)$, so that $u'(x) = \frac{1}{x}$. Then, we have

$$\int \frac{\sin(\log(x))}{x}\, dx = \int \sin(u(x))u'(x)\, dx = -\cos(u(x)) + C = -\cos(\log(x)) + C.$$

(i) This is the integral of a polynomial, so we could expand the product and use the power rule. Yet, in this case $u$-substitution makes it much faster. We set $u(x) = x^2 + 5x$, so that $u'(x) = 2x + 5$. Then, we have

$$\int (2x+5)(x^2+5x)^7\, dx = \int u^7(x)u'(x)\, dx = \frac{1}{8}u^8(x) + C = \frac{1}{8}(x^2+5x)^8 + C.$$

11. Let $f$ be a continuous function on a closed interval $[a, b]$. Show that $|f|$ is integrable and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

**Solution:** Since $f$ is continuous also $|f|$ is continuous on $[a, b]$, hence it is integrable. We distinguish two cases.

If $\int_a^b f(x)dx \geq 0$, then $|\int_a^b f(x)dx| = \int_a^b f(x)dx = \overline{S}(f) = \inf_\sigma \overline{S}_\sigma(f)$, and for every partition $\sigma$ of $[a, b]$ given by $a = x_0 < \cdots < x_n = b$ we have

$$\overline{S}_\sigma(f) = \sum_{i=1}^{n} (\sup_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} |f(x)|(x_i - x_{i-1}) = \overline{S}_\sigma(|f|)$$

So $\inf_\sigma \overline{S}_\sigma(f) \leq \inf_\sigma \overline{S}_\sigma(|f|) = \int_a^b |f(x)|dx$.

If $\int_a^b f(x)dx < 0$, then $|\int_a^b f(x)dx| = -\int_a^b f(x)dx = \int_a^b -f(x)dx$. Since $|-f| = |f|$ and $\int_a^b -f(x)dx \geq 0$, by applying the previous case, we obtain $\int_a^b -f(x)dx \leq \int_a^b |f(x)|dx$.

12. Calculate the following integral:

$$\int_0^{\pi/2} \sin(x)^5\, dx$$

(*Hint: remember* $\cos^2(x) + \sin^2(x) = 1$ *and try to make a substitution of the form* $u = \cos x$).

**Solution:**

We want to use the fact that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. For this we use the formula $\sin^2 x = 1 - \cos^2 x$ to write

$$\int_0^{\pi/2} \sin(x)^5\ dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\ dx.$$

Take $u(x) = \cos(x)$ and we see that $u'(x) = -\sin x$ which means $du = -\sin x dx$. Note that $\cos(0) = 1$ and $\cos(\pi/2) = 0$, so

$$\int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\ dx = -\int_1^0 (1-u^2)^2 du = \int_0^1 (1-u^2)^2\ du$$

$$= \int_0^1 1 - 2u^2 + u^4\ du = (t - \frac{2}{3}t^3 + \frac{1}{5}t^5)\Big|_0^1 = \frac{8}{15}$$

13. Prove that if $f, g : I \to \mathbb{R}$ are square-integrable continuous functions over $I$[1], then

$$\left|\int_I f(x)g(x)dx\right| \le \left(\int_I f(x)^2 dx\right)^{\frac{1}{2}} \left(\int_I g(x)^2 dx\right)^{\frac{1}{2}}$$

This is known as the Cauchy–Schwarz inequality. (*Hint: If at least one of the functions is zero, then there is nothing to prove. Suppose both are non-zero. Evaluate $\int_I (f(x) - \lambda g(x))^2 dx$ and choose $\lambda \in \mathbb{R}$ carefully.*)

**Solution:** As mentioned above, we assume that both functions are non-zero. Then none of the integrals on the right-hand side of the inequality are 0 because each integral is the integral of a non-negative continuous function that is not identically 0: a theorem guarantees that such an integral is strictly positive. Using the hint:

$$0 \le \int_I (f(x) - \lambda g(x))^2 dx = \int_I (f(x)^2 - 2\lambda f(x)g(x) + \lambda^2 g(x)^2) dx \tag{2}$$

$$= \int_I f(x)^2 dx - 2\lambda \int_I f(x)g(x)dx + \lambda^2 \int_I g(x)^2 dx. \tag{3}$$

Let

$$\lambda = \frac{\int_I f(x)g(x)dx}{\int_I g(x)^2 dx}, \tag{4}$$

which is well defined as the denominator is not 0, and insert this into (2), which gives

$$0 \le \int_I f(x)^2 dx - \frac{\left(\int_I f(x)g(x)dx\right)^2}{\int_I g(x)^2 dx}. \tag{5}$$

Rearranging and taking the square root gives

$$\left|\int_I f(x)g(x)dx\right| \le \left(\int_I f(x)^2 dx\right)^{\frac{1}{2}} \left(\int_I g(x)^2 dx\right)^{\frac{1}{2}} \tag{6}$$

as required.

**Revision Exercises**

Questions 14-17 are multiple choice questions. In each of the questions you should explain why your choice is correct.

14. The equation $x(e^x - e^{-x}) - e^x = 0$

(a) has no solution belonging to the interval $[0, +\infty[$.
(b) has exactly one real solution.
(c) has no solution belonging to the interval $]-\infty, 0[$.
(d) has at least two real solutions.

**Solution:** (d) is correct. Let $f(x) = x(e^x - e^{-x}) - e^x$. Then $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = +\infty$ and $f(0) = -1 < 0$. By the intermediate value theorem, there must be at least two solutions to the equation.

[1] Meaning that $f^2$ and $g^2$ are integrable.

15. Let the one-to-one function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sinh(\sinh(x))$ and let $a = f(1)$. Then the derivative of the inverse function $g = f^{-1}$ at $a$ is

(a) $g'(a) = \frac{1}{\cosh(\sinh(1))}$.

(b) $g'(a) = \frac{1}{\cosh(\sinh(a))\cosh(a)}$.

(c) $g'(a) = \frac{1}{\cosh(\sinh(1))\cosh(1)}$.

(d) $g'(a) = \frac{1}{\cosh(\sinh(a))\cosh(1)}$.

**Solution:** Using the inverse function theorem we have

$$g'(x) = \frac{1}{f'(g(x))} \tag{7}$$

and using the chain rule we have

$$f'(x) = \cosh(x)\cosh(\sinh(x)) \tag{8}$$

and $g(a) = 1$ which gives

$$g'(x) = \frac{1}{\cosh(1)\cosh(\sinh(1))} \tag{9}$$

and thus (c) is correct.

16. The limit $\lim\limits_{x\to 0} \frac{e^{|x|}-1-|x|}{x^2}$ is

(a) 0.

(b) 1.

(c) $\frac{1}{2}$.

(d) Does not exist.

**Solution:** Consider the function $f(x) = \frac{e^x-1-x}{x^2}$. Applying L'Hôpital's rule twice gives

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2} \tag{10}$$

Applying a similar argument on $g(x) = f(-x)$ shows that $\lim\limits_{x\to 0} g(x) = \frac{1}{2}$. In particular, this implies that $\lim\limits_{x\to 0} \frac{e^{|x|}-1-|x|}{x^2} = \frac{1}{2}$ and thus (c) is correct.

17. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{e^x-1}$. The truncated expansion of order 2 of $f$ around $x = 0$ is

(a) $f(x) = 1 + x + x^2 + x^2\epsilon(x)$

(b) $f(x) = 2x + x^2 + x^2\epsilon(x)$

(c) $f(x) = 1 + x + \frac{1}{2}x^2 + x^2\epsilon(x)$

(d) $f(x) = 1 + x + 2x^2 + x^2\epsilon(x)$

where $\lim_{x \to 0} \epsilon(x) = 0$.

**Solution:** (a) is correct. Note that $f(0) = 1$ and

$$f'(x) = e^{e^x + x - 1} \Rightarrow f'(0) = 1$$
$$f''(x) = (e^x + 1)e^{e^x + x - 1} \Rightarrow f''(0) = 2$$

Hence,

$$f(x) = 1 + x + x^2 + x^2\epsilon(x). \tag{11}$$

Questions 18-22 are true or false questions. If the statement is true, you should prove it. If it is false, you should give a counter example.

18. For $a < b$ in $\mathbb{R}$, let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on $]a, b[$. If $f(a) = f(b) = 0$, then there exists $c \in ]a, b[$ such that $f''(c) = 0$.

**Solution:** False. Take $f : [-1, 1] \to \mathbb{R}, f(x) = 1 - x^2$.

19. Define $f : \mathbb{R} \to \mathbb{R}, f(x) = \int_0^x |t|dt$. Then $f'(x) = x \quad \forall x \in \mathbb{R}$.

**Solution:** False. By evaluating the integral we see that

$$f(x) = \frac{1}{2}\text{sign}(x)x^2 \tag{12}$$

and thus, $f'(x) = |x|$, since $f'(x) = x$ for $x \geq 0$ and $f(x) = -x$ for $x < 0$.

20. Let $f : I \to \mathbb{R}$ be differentiable over an open interval $I \subset \mathbb{R}$. Then the derivative of $f$ at the point $y \in I$ satisfies

$$f'(y) = \lim_{x \to 0} \frac{f(y + x) - f(y)}{x}.$$

**Solution:** True. This is the definition of the derivative of $f$ at $y$.

21. Let $f : ]0, 1[ \to \mathbb{R}$ be a differentiable function on $]0, 1[$. Then the function $f' : ]0, 1[ \mapsto \mathbb{R}$ is differentiable on $]0, 1[$.

**Solution:** False. Take the function defined in Question 19. $f'(x) = |x|$ is not differentiable at 0.

22. A function $f : \mathbb{R} \to \mathbb{R}$ s.t. $\forall \varepsilon > 0$ and $\forall x, y \in \mathbb{R}$ with the following property

$$|x - y| \leq 2\varepsilon \Rightarrow |f(x) - f(y)| \leq \varepsilon$$

is continuous on $\mathbb{R}$.

**Solution:** True. Let $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon$. Then, $|x - x_0| < \delta = \varepsilon \Rightarrow |f(x) - f(x_0)| \leq \frac{\varepsilon}{2} < \varepsilon$. Hence, $f$ is continuous.