Analysis I (English)
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Analysis 1 - Exercise Set 12

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Find the Taylor expansion of order n at $x = 0$ of the following functions.

   (a) $f(x) = e^{\sin(x)}, \quad n = 4$

   (b) $f(x) = \sqrt{1+\sin(x)}, \quad n = 3$

Solution:

(a) We use the 3rd order Taylor expansion around the point $x = 0$ of the exponential function which is valid for all $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + x^4\varepsilon(x) \qquad \text{and} \qquad \sin(x) = x - \frac{x^3}{3!} + x^4\varepsilon(x).$$

So

$$\begin{aligned}
e^{\sin(x)} &= 1 + \left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right)^3 \\
&\quad + \frac{1}{24}\left(x - \frac{x^3}{6} + x^4\varepsilon(x)\right)^4 + x^4\varepsilon(x) \\
&= 1 + \left(x - \frac{x^3}{6}\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{6}x^3 + \frac{1}{24}x^4 + x^4\varepsilon(x) \\
&= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + x^4\varepsilon(x)\,.
\end{aligned}$$

(b) The 3rd order Taylor expansion around the point $x = 0$ of $\sin(x)$ and $(1+y)^{1/2}$ are

$$\sin(x) = x - \frac{x^3}{6} + x^3\varepsilon(x) \qquad \text{and} \qquad \sqrt{1+y} = 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} + y^3\varepsilon(y)\,.$$

where by substituting $y = \sin(x)$, we get

$$\begin{aligned}
\sqrt{1+\sin(x)} &= 1 + \frac{1}{2}\left(x - \frac{x^3}{6} + x^3\varepsilon(x)\right) - \frac{1}{8}\left(x - \frac{x^3}{6} + x^3\varepsilon(x)\right)^2 \\
&\quad + \frac{1}{16}\left(x - \frac{x^3}{6} + x^3\varepsilon(x)\right)^3 + x^3\varepsilon(x) \\
&= 1 + \frac{1}{2}\left(x - \frac{x^3}{6}\right) - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x) \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + x^3\varepsilon(x)\,.
\end{aligned}$$

2. For each one of the following functions, determine whether the funciton is differentiable at $x = 0$. If yes, also compute the derivative at $x = 0$:

   (a) $f(x) = \begin{cases} x+1, & x \geq 0 \\ x, & x < 0 \end{cases}$;

   (b) $f(x) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$;

   (c) $f(x) = \begin{cases} \frac{\sin(x)-x}{x}, & x > 0 \\ 0, & x = 0 \\ \frac{\cos(x)-\frac{x^2}{2}}{x^4}, & x < 0 \end{cases}$.

Solution:

(a) The function is not continuous at 0, as the limit from the right is 1 and the limit from the left is 0. As a differentiable function is continuous, then f is not differentiable at 0.

(b) We claim that the function is differentiable at 0 with derivative $f'(0) = 0$. Indeed, we compute

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0.$$

Similarly, we have

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^3-0}{h} = \lim_{h\to 0^-} h^2 = 0.$$

Since the left-hand limit and the right-hand limit exist and agree, we conclude that

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0.$$

Thus, f is differentiable at 0 with $f'(0) = 0$.

(c) We claim that the function is not continuous at 0. Then, if that is the case, we can conclude as in case (a). One can see that $\lim_{x\to 0^+} f(x) = 0 = f(x)$. Yet, it suffices to show that $\lim_{x\to 0^-} f(x) \neq 0 = f(0)$. To this end, we compute the limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\cos(x)-\frac{x^2}{2}}{x^4} = +\infty$$

as the limit of the numerator is 1, while the denominator goes to 0 from above (hence the sign + for ∞).

3. Find the vertical and horizontal asymptotes of the function $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

Solution: A vertical asymptotes cannot exist in a point where the function is defined, so here potentially $x = 0$ is a candidate for a vertical asymptote. In fact, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty \qquad \text{and} \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

So f has a vertical asymptote at $x = 0$.

A horizontal asymptote (if it exists) is characterized by the limit of the function f at infinity ($\pm\infty$). Here we have

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x} = 0 \qquad \text{et} \qquad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0$$

So indeed f has an horizontal asymptote at $y = 0$.

4. State if the following statements are true or false. Let $f, g : I \to \mathbb{R}$ be two convex functions, where $I \subset \mathbb{R}$ is some interval. If it is true, prove it. If not, give a counter example.
   (a) The function $f + g$ is convex.
   (b) The function $h = f \cdot g$ is convex.
   (c) If g is increasing then the function $h = g \circ f$ is convex.

Solution:

(a) True. Using the definition we have

$$\begin{aligned}(f+g)(\lambda a + (1-\lambda)b) &= f(\lambda a + (1-\lambda)b) + g(\lambda a + (1-\lambda)b) \\ &\le \lambda f(a) + (1-\lambda)f(b) + \lambda g(a) + (1-\lambda)g(b) \\ &= \lambda(f+g)(a) + (1-\lambda)(f+g)(b)\end{aligned}$$

So $f + g$ is convex.

(b) False. Take for example $f(x) = -\sqrt{x}$ and $g(x) = 1/x$ on $I =]0, +\infty[$. Then we have $h = f \cdot g = -\frac{1}{\sqrt{x}}$ which is a concave function. So in general if f and g are convex functions, we cannot say much about $f \cdot g$.

(c) True. Note that if f is convex then $f(\lambda a + (1 - \lambda b)) \le \lambda f(a) + (1-\lambda)f(b)$ now take $x_1 = f(\lambda a + (1 - \lambda b))$ and $x_2 = \lambda f(a) + (1-\lambda)f(b)$. Since g is increasing and $x_1 \le x_2$ then $g(x_1) \le g(x_2)$. Finally, using convexity of g we can write:

$$g(x_1) \le g(x_2) \Rightarrow g(f(\lambda a + (1-\lambda b))) \le g(\lambda f(a) + (1-\lambda)f(b)) \le \lambda g(f(a)) + (1-\lambda)g(f(b))$$

Or we can write:

$$h(\lambda a + (1-\lambda)b) \le \lambda h(a) + (1-\lambda)h(b)$$

so h is convex.

5. Consider $f\colon]a, b[\mapsto \mathbb{R}$. Let $g\colon]c, d[\mapsto \mathbb{R}$ be the restriction to f to the interval $]c, d[\subset]a, b[$, i.e., $f(x) = g(x) \quad \forall x \in]c, d[$. Show that
   (a) If $f \in C^n(]a, b[, \mathbb{R})$ then $g \in C^n(]c, d[, \mathbb{R})$.
   (b) If f is Lipschitz continuous, then g is Lipschitz continuous.

Solution:

(a) Note that $\forall x \in]c,d[$ we have $g^{(k)}(x) = f^{(k)}(x) \quad \forall k = 1, \cdots, n$ which shows $g \in \mathbb{C}^n(c,d)$.

(b) Since f is Lipschitz continuous with constant L we have $\forall x, y \in]c,d[, \quad |g(x) - g(y)| = |f(x) - f(y)| \leq L|x-y|$. Hence, g is Lipschitz continuous.

6. Find the local extrema and the absolute maximum and minimum of $f(x) = x^2 - \left|x + \frac{1}{4}\right| + 1$ in $[-1,1]$.

Solution: Before calculating the derivatives, we rewrite f by distinguishing two cases. we have

$$f(x) = \begin{cases} x^2 + x + \frac{5}{4}, & -1 \leq x \leq -\frac{1}{4} \\ x^2 - x + \frac{3}{4}, & -\frac{1}{4} < x \leq 1 \end{cases}, \qquad f'(x) = \begin{cases} 2x+1, & -1 < x < -\frac{1}{4} \\ 2x-1, & -\frac{1}{4} < x < 1 \end{cases}$$

For $x_0 = -\frac{1}{4}$ we have

$$f_r'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to -\frac{1}{4}^+} \frac{x^2 - x - \frac{5}{16}}{x + \frac{1}{4}} = \lim_{x \to -\frac{1}{4}^+} \frac{\left(x - \frac{5}{4}\right)\left(x + \frac{1}{4}\right)}{x + \frac{1}{4}} = -\frac{3}{2}$$

$$f_l'(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to -\frac{1}{4}^-} \frac{x^2 + x + \frac{3}{16}}{x + \frac{1}{4}} = \lim_{x \to -\frac{1}{4}^-} \frac{\left(x + \frac{3}{4}\right)\left(x + \frac{1}{4}\right)}{x + \frac{1}{4}} = \frac{1}{2}$$

and so f is not differentiable in this point. Also $f''(x) = 2$ For all $x \in \left]-1, -\frac{1}{4}\right[\cup \left]-\frac{1}{4}, 1\right[$. So local and absolute extrema are listed bellow:

(a) Stationary points: $f'(x) = 0 \Rightarrow x_1 = -\frac{1}{2}$ or $x_2 = \frac{1}{2}$. Since $f''(x_1) = f''(x_2) > 0$, x_1 and x_2 are local minimums. We have $f(x_1) = 1$ and $f(x_2) = \frac{1}{2}$.

(b) Points where f' does not exist: The only point is $x_0 = -\frac{1}{4}$ since we have $f_r'(x_0) = -\frac{3}{2}$ and $f_l'(x_0) = \frac{1}{2}$. Looking at the sign of the derivative at x_0 we deduce that this point is a local maximum. We have $f(x_0) = \frac{17}{16}$.

(c) Boundaries of the domain: Since f is continuous on $[-1,1]$, we look at the sign of f' in the boundaries of $[-1,1]$ and notice that f has local maximums at $a = -1$ and $b = 1$. We have $f(a) = \frac{5}{4}$ and $f(b) = \frac{3}{4}$.

$$(a), (b), (c) \quad \Rightarrow \quad \begin{cases} \text{global maximum at} \quad x = -1, \quad f(-1) = \frac{5}{4} \\ \text{global minimum at} \quad x = \frac{1}{2}, \quad f\left(\frac{1}{2}\right) = \frac{1}{2} \end{cases}$$

7. Let $a, b \in \overline{\mathbb{R}}$, $a < b$. Let $f :]a,b[\to \mathbb{R}$ be a differentiable function. State if the following statements are true or false. If it is true, prove it. If not, give a counter example.

(a) If f' is bounded, then f is Lipschitz continuous with Lipschitz contstant $k = \sup_{x \in]a,b[} |f'(x)|$.

(b) If f is Lipschitz continuous, then it is uniformly continuous.

(c) If f' is bounded then f is uniformly continuous.

Solution:

(a) True. We use the Mean Value Theorem as follows. For every $x < y$ in $]a, b[$ there exists $c \in [x, y]$ such that $f(y) - f(x) = f'(c)(y - x)$. Hence

$$|f(y) - f(x)| = |f'(c)||y - x| \leq k|y - x|.$$

(b) True. Let $k \geq 0$ such that f is Lipschitz continuous with constant k. Let $\varepsilon > 0$ and $x, y \in]a, b[$. We compute that

$$|f(x) - f(y)| \leq k|x - y| \leq \varepsilon$$

holds if $|x - y| \leq \frac{\varepsilon}{k}$. So f is uniformly continuous with $\delta := \frac{\varepsilon}{k}$.

(c) True. Combine the previous statements.

8. Study the function $f(x) = \dfrac{x}{x^2 - 1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

Solution:

(a) $D(f) = \mathbb{R} \setminus \{-1, 1\}, \quad \mathrm{Im}(f) = \mathbb{R}$

(b) Odd, non-periodic

(c) $f(x) = 0 \Leftrightarrow x = 0$

(d) f is continuous since it is composition of continuous functions on $D(f)$.

(e) f is differentiable on $D(f)$

$$f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2}, \quad D(f') = D(f) \qquad \text{and} \qquad f''(x) = \frac{2x\left(x^2 + 3\right)}{(x^2 - 1)^3}, \quad D(f'') = D(f)$$

(f)
- $f'(x) < 0$ for all $x \in D(f')$, so there are no stationary points.
- $f''(x) = 0 \Leftrightarrow x = 0$. We also calculate f''':

$$f'''(x) = -\frac{6(x^4 + 6x^2 + 1)}{(x^2 - 1)^4} < 0 \quad \text{for all } x \in D(f).$$

Since $f'''(0) = -6 \neq 0$, f has an inflection point in $x = 0$.

(g)
- Monotonicity:

x	$]-\infty, -1[$	$]-1, 1[$	$]1, \infty[$
f'	< 0	< 0	< 0
f	decreasing	decreasing	decreasing

Note that f is strictly decreasing in the intervals in the table but not on the entire domain $D(f)$.

- Convexity and concavity:

x	$]-\infty, -1[$	$]-1, 0[$	$]0, 1[$	$]1, \infty[$
f''	< 0	> 0	< 0	> 0
f	concave	convex	concave	convex

(h)
- Vertical asymptotes: f is not defined in $x=\pm 1$ and we have $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to 1^+} f(x) = \infty$ also $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to 1^-} f(x) = -\infty$, so there of vertical asymptotes at $x=\pm 1$.
- horizontal asymptotes: $\lim\limits_{x\to\pm\infty} f(x) = 0$, so one horizontal asymptote at $y=0$.

9. Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=e^x$. Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

Solution: The regular partition σ_n of the interval $[0,1]$ is $0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}=1$. Since the function e^x is increasing, we have

$$\underline{S}_{\sigma_n} = \sum_{i=1}^{n}\left(\inf_{x\in[\frac{i-1}{n},\frac{i}{n}]} f(x)\right)\left(\frac{i}{n}-\frac{i-1}{n}\right) = \sum_{i=1}^{n} e^{\frac{i-1}{n}}\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}(e^{\frac{1}{n}})^{i-1} = \frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1},$$

$$\overline{S}_{\sigma_n} = \sum_{i=1}^{n}\left(\sup_{x\in[\frac{i-1}{n},\frac{i}{n}]} f(x)\right)\left(\frac{i}{n}-\frac{i-1}{n}\right) = \sum_{i=1}^{n} e^{\frac{i}{n}}\frac{1}{n} = \frac{e^{\frac{1}{n}}}{n}\sum_{i=1}^{n}(e^{\frac{1}{n}})^{i-1} = \frac{e^{\frac{1}{n}}}{n}\frac{e-1}{e^{\frac{1}{n}}-1}.$$

We want to compute the limits $\lim_{n\to+\infty}\underline{S}_{\sigma_n}$ and $\lim_{n\to+\infty}\overline{S}_{\sigma_n}$, but we compute first $\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1)$. We observe that since the function f is differentiable, we have

$$\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1) = \lim_{n\to+\infty}\frac{f(\frac{1}{n})-f(0)}{\frac{1}{n}-0} = f'(0) = 1$$

Then

$$\lim_{n\to+\infty}\underline{S}_{\sigma_n} = \lim_{n\to+\infty}\frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1} = (e-1)(\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1))^{-1} = e-1$$

$$\lim_{n\to+\infty}\overline{S}_{\sigma_n} = \lim_{n\to+\infty}\frac{e^{\frac{1}{n}}}{n}\frac{e-1}{e^{\frac{1}{n}}-1} = (e-1)(\lim_{n\to+\infty} e^{\frac{1}{n}})(\lim_{n\to+\infty} n(e^{\frac{1}{n}}-1))^{-1} = e-1$$

Since

$$\lim_{n\to+\infty}\underline{S}_{\sigma_n} \le \underline{S} \le \overline{S} \le \lim_{n\to+\infty}\overline{S}_{\sigma_n}$$

we obtain that $\underline{S}=\overline{S}$ and the function f is integrable.

10. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function on $[a,b]\subset D(f)$, $a<b$, and differentiable on $]a,b[$.

   (a) If $f'(x)\ge 0$ for all $x\in\,]a,b[$, then f is increasing on $[a,b]$.
   (b) If f is increasing on $[a,b]$, then $f'(x)\ge 0$ for all $x\in\,]a,b[$.
   (c) If f is strictly increasing on $[a,b]$, then $f'(x)>0$ for all $x\in\,]a,b[$.
   (d) If $f'(x)>0$ for all $x\in\,]a,b[$, then f is strictly increasing on $[a,b]$.
   (e) If $\lim\limits_{x\to a^+} f'(x)=\ell$ exists, then f is differentiable from right at a and the right derivative is $f'_d(a)=\ell$.

Solution:

(a) True. Direct consequence of $f'(x) \geq 0$ and MVT.

(b) True. For all $x \in \,]a, b[$, the derivative of f is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\,.$$

Since f is increasing on $[a, b]$, $f(x+h) - f(x)$ has the same sign as h. So the quotient inside the limit is always positive so $f'(x) \geq 0\,.$

(c) False. Take for example $f\colon [-1, 1] \to \mathbb{R}$ defined by $f(x) = x^3$. This function is strictly increasing on $[-1, 1]$ but $f'(0) = 0$.

(d) True. Direct consequence of $f'(x) > 0$ and MVT.

(e) True. We want to compute $f'_d(a)$, which by definition is

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

By the mean value theorem, for every $x \in \,]a, b[$ there exists $c = c(x) \in \,]a, x[$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c(x))$$

(Let us notice that $c(x)$ is a function of x, a fancy function but still a function; and let us stress again that $a \leq c(x) \leq x$).

To conclude the exercise we just have to show that

$$\lim_{x \to a^+} f'(x) = \lim_{x \to a^+} f'(c(x))\,,$$

This follows from the definition of limit and the fact that $a \leq c(x) \leq x$. Indeed, we have to show that for every ϵ ther exists a δ such that if $|a - x| < \delta$ then $|\ell - f(c(x))| \leq \epsilon$. We know that there exists δ such that if $|a - x| < \delta$ then $|\ell - f(x)| \leq \epsilon$, but for this very same δ we also have that $|a - c(x)| \leq \delta$, so $|\ell - f(c(x))| \leq \epsilon$, and this prove the claim.

11. Using the definition of convex functions, show that the function $f(x) = x^2$ is convex.

Solution: We must show that for any $\lambda \in [0, 1]$ and $a, b \in D_f = \mathbb{R}$ we have

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$$

for convex functions and

$$f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b)$$

for concave functions.

We have

$$\begin{aligned}
&f(\lambda a + (1 - \lambda)b) - (\lambda f(a) + (1 - \lambda)f(b)) \\
&= \lambda^2 a^2 + (1 - \lambda)^2 b^2 + 2\lambda(1 - \lambda)ab - \lambda a^2 - (1 - \lambda)b^2 \\
&= -\lambda(1 - \lambda)a^2 - \lambda(1 - \lambda)b^2 + 2\lambda(1 - \lambda)ab \\
&= -\lambda(1 - \lambda)(a - b)^2 \leq 0
\end{aligned}$$

The last equality is true for all $\lambda \in [0,1]$. So the function is convex.

12. Find the local extrema and the absolute maximum and minimum of $f(x) = (x-1)^2 - 2\,|2-x|$ in $]2,3[$

Solution: Since $2 - x < 0$ for all $x \in\,]2,3[\,=: I$, there is no need for distinguishing two cases for f. We have

$$f(x) = (x-1)^2 + 2(2-x) = x^2 - 4x + 5 \quad \text{and} \quad f'(x) = 2(x-2) \quad \text{for all } x \in I$$

The local and global extrema are listed below:

(a) Stationary points: $f'(x) \neq 0$ for all $x \in I$, so no stationary points.

(b) Points where f' does not exist: f' exists on I.

(c) Boundaries of the domain: The domain I is an open interval and so f does not take any extrema.

So the function f does not have any local or global extrema on I

13. Show that the following functions are Lipschitz continuous on the given domain.
    (a) $f(x) = |x|$, $f : \mathbb{R} \to \mathbb{R}$.
    (b) $f(x) = \sqrt{x}$, $f : [a, \infty[\to \mathbb{R}$, $a > 0$.
    (c) $f(x) = x^n$, $f : [a,b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

Solution:

(a) We need to find $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all x, y in $\mathbb{R}$. By applying the reversed triangle inequality:

$$\begin{aligned} |f(x) - f(y)| &= ||x| - |y|| \\ &\leq |x - y| \end{aligned}$$

We conclude that $f(x) = |x|$ is Lipschitz continuous with $L = 1$.

(b) For all $x, y \in [a, \infty[$ we can write

$$\begin{aligned} |f(x) - f(y)| &= \left|\sqrt{x} - \sqrt{y}\right| \\ &= \left|\left(\sqrt{x} - \sqrt{y}\right) \frac{\sqrt{y} + \sqrt{x}}{\sqrt{y} + \sqrt{x}}\right| \\ &= \left|\frac{x - y}{\sqrt{y} + \sqrt{x}}\right| \\ &= \frac{|x - y|}{\sqrt{y} + \sqrt{x}} \\ &\leq \frac{|x - y|}{2\sqrt{a}} \end{aligned}$$

So with $L = \frac{1}{2\sqrt{a}}$ we conclude that $f(x) = \sqrt{x}$ is Lipshitz continuous.

(c) We first prove it for $[a,b]=[-1,1]$. For all $x,y\in[-1,1]$ we have

$$\begin{aligned}|f(x)-f(y)| &= |x^n-y^n|\\ &= |x-y|\cdot\left|\sum_{k=0}^{n-1}x^{n-k-1}y^k\right|\\ &\le |x-y|\cdot\sum_{k=0}^{n-1}\left|x^{n-k-1}\right|\left|y^k\right|\\ &\le |x-y|\cdot\sum_{k=0}^{n-1}1\\ &= n\cdot|x-y|\end{aligned}$$

Now for the general interval $I=[-R,R]$ we can write:

$$\begin{aligned}|f(x)-f(y)| &= |x^n-y^n|\\ &= |x-y|\cdot\left|\sum_{k=0}^{n-1}x^{n-k-1}y^k\right|\\ &\le |x-y|\cdot\sum_{k=0}^{n-1}\left|x^{n-k-1}\right|\left|y^k\right|\\ &= |x-y|\cdot\sum_{k=0}^{n-1}|x|^{n-k-1}|y|^k\\ &\le |x-y|\cdot\sum_{k=0}^{n-1}|R|^{n-k-1}|R|^k\\ &\le |x-y|\cdot\sum_{k=0}^{n-1}|R|^{n-1}\\ &= nR^{n-1}|x-y|\end{aligned}$$

Thus for any interval $I=[a,b]$, there exists an $R\in\mathbb{R}$ such that $[a,b]\subset[-R,R]$, so $f(x)=x^n$ is Lipschitz continuous.

Alternatively, we can consider $f'(x)=nx^{n-1}$. Since f' is continuous everywhere, it achieves maximum and minimum on the closed and bounded interval $[a,b]$. In particular, f' is bounded on $[a,b]$. Then, we can conclude by Exercise 7.

14. Study the function $f(x)=\dfrac{3x^2-x}{2x-1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

Solution:

(a) $D(f)=\mathbb{R}\setminus\frac{1}{2}$, $\quad\mathrm{Im}(f)=\mathbb{R}\setminus\left]1-\frac{\sqrt{3}}{2},1+\frac{\sqrt{3}}{2}\right[$

(b) not even, not odd and not periodic.

(c) $f(x)=0 \quad\Leftrightarrow\quad 3x^2-x=0 \quad\Leftrightarrow\quad x=0$ or $x=\frac{1}{3}$.

(d) Continuous on $D(f)$ (composition of continuous functions)

(e) f is differentiable on $D(f)$

$$f'(x) = \frac{(6x-1)(2x-1) - 2(3x^2-x)}{(2x-1)^2} = \frac{6x^2-6x+1}{(2x-1)^2}\ , \qquad D(f') = D(f)$$

$$f''(x) = \frac{(12x-6)(2x-1)^2 - 4(6x^2-6x+1)(2x-1)}{(2x-1)^4}$$

$$= \frac{(12x-6)(2x-1) - 4(6x^2-6x+1)}{(2x-1)^3} = \frac{2}{(2x-1)^3}\ , \qquad D(f'') = D(f)$$

(f)
- $f'(x) = 0 \quad \Leftrightarrow \quad 6x^2-6x+1 = 0 \quad \Leftrightarrow \quad x = \frac{6\pm\sqrt{36-24}}{12} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$
  Sp f has a stationary point at $x_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ and $x_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Since
  $$f''(x_{1,2}) = \frac{2}{\left(2\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right) - 1\right)^3} = \frac{2}{\left(\pm\frac{\sqrt{3}}{3}\right)^3} = \pm\frac{2}{3^{-3/2}} = \pm 6\sqrt{3}\ ,$$
  It follows that x_1 is a local minimum (since $f''(x_1) > 0$) and x_2 a local maximum (since $f''(x_2) < 0$) of f.
- Since $f''(x) \neq 0$ for all $x \in D(f)$, f does not have an inflection point.

(g)
- Monotonicity:

x	$\left]-\infty, \frac{1}{2} - \frac{\sqrt{3}}{6}\right[$	$\left]\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right[$	$\left]\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{3}}{6}\right[$	$\left]\frac{1}{2} + \frac{\sqrt{3}}{6}, \infty\right[$
f'	> 0	< 0	< 0	> 0
f	increasing	decreasing	decreasing	increasing

- Convexity and concavity:

x	$\left]-\infty, \frac{1}{2}\right[$	$\left]\frac{1}{2}, \infty\right[$
f''	< 0	> 0
f	concave	convex

(h)
- Vertical asymptotes: f is not defined in $x = \frac{1}{2}$ and
  $$\lim_{x\to\frac{1}{2}^\pm} f(x) = \lim_{x\to\frac{1}{2}^\pm} \frac{x(3x-1)}{2x-1} = \pm\infty$$
  Since $x(3x-1) > 0$ for x close to $\frac{1}{2}$. So f has a vertical asymptote at $x = \frac{1}{2}$.
- Horizontal asymptotes:
  $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x(3x-1)}{2x-1} = \lim_{x\to\pm\infty} \frac{3x-1}{2-\frac{1}{x}} = \pm\infty,$$
  So f does not have any horizontal asymptotes.
- obliqued asymptotes:
  $$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{3x-1}{2x-1} = \frac{3}{2} \qquad \text{et}$$
  $$b = \lim_{x\to\pm\infty} \big(f(x) - ax\big) = \lim_{x\to\pm\infty} \left(\frac{x(2x-1+x)}{2x-1} - \frac{3}{2}x\right) = \lim_{x\to\pm\infty} \left(-\frac{x}{2} + \frac{x^2}{2x-1}\right)$$
  $$= \lim_{x\to\pm\infty} \frac{x}{2(2x-1)} = \frac{1}{4}$$

So f has an obliqued asymptotes with equation $y = ax + b = \frac{3}{2}x + \frac{1}{4}$.

(i) We find the value of f at $x_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$:

$$f(x_{1,2}) = \frac{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)\left(3\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right) - 1\right)}{\pm 3^{-1/2}} = \pm\sqrt{3}\left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$$
$$= \left(\pm\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)^2 = 1 \pm \frac{\sqrt{3}}{2},$$

Since $f(x_1) > f(x_2)$ and knowing the nature of the local extrema of f in x_1 and x_2 that $\mathrm{Im}(f) = \mathbb{R} \setminus \left]1 - \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}\right[$.

15. (a) Show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ \frac{1}{2}, & x \notin \mathbb{Q}. \end{cases}$$

Compute the upper and lower Darboux sums for the regular partitions σ_{2n}. Is f integrable?

Solution:

(a) We prove it by induction. If $n = 0$ or 1 the formula holds. Assume that the formula holds for $n - 1$, then

$$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i = n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2} = \frac{n(n+1)}{2}.$$

(b) The regular partition σ_{2n} of the interval $[0, 1]$ is $0, \frac{1}{2n}, \frac{2}{2n}, \ldots, \frac{2n-1}{2n}, \frac{2n}{2n} = 1$. So

$$\underline{S}_{\sigma_{2n}} = \sum_{i=1}^{2n} \left(\inf_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x)\right)\left(\frac{i}{2n} - \frac{i-1}{2n}\right), \quad \overline{S}_{\sigma_{2n}} = \sum_{i=1}^{2n} \left(\sup_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x)\right)\left(\frac{i}{2n} - \frac{i-1}{2n}\right).$$

We observe that $\frac{i}{2n} - \frac{i-1}{2n} = \frac{1}{2n}$ for all i. Also,

$$\inf_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x) = \begin{cases} \frac{i-1}{2n} & \text{if } i \leq n \\ \frac{1}{2} & \text{if } i > n \end{cases}, \qquad \sup_{x \in [\frac{i-1}{2n}, \frac{i}{2n}]} f(x) = \begin{cases} \frac{1}{2} & \text{if } i \leq n \\ \frac{i}{2n} & \text{if } i > n \end{cases}$$

So we compute

$$\begin{aligned}
\underline{S}_{\sigma_{2n}} &= \left(\sum_{i=1}^{n} \frac{i-1}{2n}\cdot\frac{1}{2n}\right) + \left(\sum_{i=n+1}^{2n} \frac{1}{2}\cdot\frac{1}{2n}\right) = \frac{1}{4n^2}\left(\left(\sum_{i=1}^{n} i\right) - \sum_{i=1}^{n} 1\right) + \frac{1}{4n}\sum_{i=n+1}^{2n} 1 \\
&= \frac{1}{4n^2}\left(\frac{n(n+1)}{2} - n\right) + \frac{1}{4} = \frac{3}{8} - \frac{1}{8n} \\
\overline{S}_{\sigma_{2n}} &= \left(\sum_{i=1}^{n} \frac{1}{2}\cdot\frac{1}{2n}\right) + \left(\sum_{i=n+1}^{2n} \frac{i}{2n}\cdot\frac{1}{2n}\right) = \frac{1}{4} + \frac{1}{4n^2}\left(\left(\sum_{i=1}^{2n} i\right) - \left(\sum_{i=1}^{n} i\right)\right) \\
&= \frac{1}{4} + \frac{1}{4n^2}\left(\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2}\right) = \frac{5}{8} + \frac{1}{8n}.
\end{aligned}$$

We observe that the sequences $(\underline{S}_{\sigma_{2n}})$ and $(\overline{S}_{\sigma_{2n}})$ converge to different limits, so we expect that the function is not integrable. But finding two converging sequences of Darboux sums that do not have the same limit is not enough to prove that $\underline{S} \neq \overline{S}$, because the sup and inf are taken over all the partition and partitions like $0 = x_0 < \cdots < x_i = \frac{1}{\sqrt{2}} < \ldots x_m = 1$ cannot be refined by a regular partition because $x_i \notin \mathbb{Q}$.

First method. One way to proceed is by looking at the graph of the function f, which looks like the union of the graphs of the two functions

$$\underline{f}(x) = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ \frac{1}{2}, & x \in [\frac{1}{2}, 1] \end{cases} \qquad \overline{f}(x) = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$

and show that for every partition σ we have that $\underline{S}_\sigma$ is smaller that the area under the graph of $\underline{f}$, and that $\overline{S}_\sigma$ is bigger that the area under the graph of $\overline{f}$, since these two areas are different $\underline{S}$ and $\overline{S}$ cannot be equal.

Second method. Another way to prove that f is not integrable is by comparing arbitrary partitions with the regular partitions that we computed above and compute the Darboux integrals $\underline{S}$ and $\overline{S}$. Recall that if σ, τ are two partitions and σ is a refinement of τ, then $\underline{S}_\sigma \geq \underline{S}_\tau$ and $\overline{S}_\sigma \leq \overline{S}_\tau$. So given an arbitrary partition τ it is enough to make the computations for its refinements, in particular, we can choose a partition σ that refined both τ and σ_{2n} for some n.

Given a partition σ that refines σ_{2n}, we can write σ as a collection of partitions

$$\begin{aligned}
0 = x_{1,0}, \ldots, x_{1,m_1} = \frac{1}{2n} &\quad \text{of} \quad [0, \frac{1}{2n}] \\
&\vdots \\
\frac{i-1}{2n} = x_{i,0}, \ldots, x_{i,m_i} = 1 &\quad \text{of } [\frac{i-1}{2n}, \frac{1}{2n}] \\
&\vdots \\
\frac{2n-1}{2n} = x_{2n,0}, \ldots, x_{2n,m_{2n}} = 1 &\quad \text{of } [\frac{2n-1}{2n}, 1]
\end{aligned}$$

then

$$\begin{aligned}
\underline{S}_\sigma &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m_i} x_{i,j-1}(x_{i,j}-x_{i,j-1})\right) + \left(\sum_{i=n+1}^{2n}\sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j}-x_{i,j-1})\right)\\
&\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{i}{2n}(x_{i,j}-x_{i,j-1})\right) + \left(\sum_{i=n+1}^{2n}\sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j}-x_{i,j-1})\right)\\
&= \left(\sum_{i=1}^{n} \frac{i}{2n}\cdot\frac{1}{2n}\right) + \left(\sum_{i=n+1}^{2n} \frac{1}{2}\cdot\frac{1}{2n}\right) = \underline{S}_{\sigma_{2n}} + \sum_{i=1}^{n}\frac{1}{2n}\cdot\frac{1}{2n} = \underline{S}_{\sigma_{2n}} + \frac{1}{4n}
\end{aligned}$$

So

$$\underline{S} = \sup_\sigma \underline{S}_\sigma \leq \sup_{n\geq 2}\left(\underline{S}_{\sigma_{2n}} + \frac{1}{4n}\right) = \sup_{n\geq 2}\left(\frac{3}{8}+\frac{1}{8n}\right) = \frac{3}{8}+\frac{1}{16} = \frac{7}{16}$$

Similarly,

$$\begin{aligned}
\overline{S}_\sigma &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j}-x_{i,j-1})\right) + \left(\sum_{i=n+1}^{2n}\sum_{j=1}^{m_i} x_{i,j}(x_{i,j}-x_{i,j-1})\right)\\
&\geq \left(\sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{1}{2}(x_{i,j}-x_{i,j-1})\right) + \left(\sum_{i=n+1}^{2n}\sum_{j=1}^{m_i} \frac{i-1}{2n}(x_{i,j}-x_{i,j-1})\right)\\
&= \left(\sum_{i=1}^{n} \frac{1}{2}\cdot\frac{1}{2n}\right) + \left(\sum_{i=n+1}^{2n} \frac{i-1}{2n}\cdot\frac{1}{2n}\right) = \overline{S}_{\sigma_{2n}} - \sum_{i=1}^{n}\frac{1}{2n}\cdot\frac{1}{2n} = \overline{S}_{\sigma_{2n}} - \frac{1}{4n}
\end{aligned}$$

and

$$\overline{S} = \inf_\sigma \overline{S}_\sigma \geq \inf_{n\geq 2}\left(\overline{S}_{\sigma_{2n}} - \frac{1}{4n}\right) = \inf_{n\geq 2}\left(\frac{5}{8}-\frac{1}{8n}\right) = \frac{5}{8}-\frac{1}{16} = \frac{9}{16} > \underline{S}$$

Since $\underline{S} \neq \overline{S}$ we conclude that f is not integrable.

16. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f, g\colon \mathbb{R}\to\mathbb{R}$ be differentiable functions on $\mathbb{R}$ with $g'(x)\neq 0$ for all $x\in\mathbb{R}$.

   (a) If $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$, then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}$.

   (b) If $\lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}$ does not exist, then $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$ does not exist.

Solution:

(a) False. Take for example $f(x) = x+\sin(x)$ and $g(x) = x$. In this example we have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\left(1+\frac{\sin(x)}{x}\right) = 1$ but $\frac{f'(x)}{g'(x)} = 1+\cos(x)$ does not have a limit (One of the hypotheses of L'Hôpital rule is not satisfied).

(b) False. Take the same functions of the previous exercise.

17. Using the definition of convex functions, show that the function $f(x) = \frac{1}{x},\quad x\in]0,+\infty[$ is convex.

Solution: We have

$$
\begin{aligned}
&f(\lambda a+(1-\lambda)b)-(\lambda f(a)+(1-\lambda)f(b))\\
&=\frac{1}{\lambda a+(1-\lambda)b}-\frac{\lambda}{a}-\frac{1-\lambda}{b}\\
&=\frac{ab-b\lambda(\lambda a+(1-\lambda)b)-a(1-\lambda)(\lambda a+(1-\lambda)b)}{(\lambda a+(1-\lambda)b)ab}\\
&=\frac{ab-\lambda^2ab-\lambda(1-\lambda)b^2-\lambda(1-\lambda)a^2-(1-\lambda)^2ab}{(\lambda a+(1-\lambda)b)ab}\\
&=\frac{(1-\lambda^2-(1-\lambda)^2)ab-\lambda(1-\lambda)b^2-\lambda(1-\lambda)a^2}{(\lambda a+(1-\lambda)b)ab}\\
&\leq\frac{(1-\lambda^2+2\lambda-(1-\lambda)^2)ab-\lambda(1-\lambda)b^2-\lambda(1-\lambda)a^2}{(\lambda a+(1-\lambda)b)ab}\\
&=\frac{((1-\lambda)^2-(1-\lambda)^2)ab-\lambda(1-\lambda)b^2-\lambda(1-\lambda)a^2}{(\lambda a+(1-\lambda)b)ab}\\
&=\frac{-\lambda(1-\lambda)b^2-\lambda(1-\lambda)a^2}{(\lambda a+(1-\lambda)b)ab}\leq 0
\end{aligned}
$$

The last inequality holds since the denominator is always positive and the numerator is strictly negative. So the function is convex.

18. Let the function $f(x):[-4,4]\backslash\{2\}$ be defined by

$$f(x)=\frac{x^2}{x+2}$$

then

(a) f attains its maximum at $x=-4$ and its minimum at $x=0$.

(b) f attains its maximum at $x=-4$ and has a local minimum at $x=0$.

(c) f has a local maximum at $x=-4$ and attains its minimum at $x=0$.

(d) f does not have a maximum or a minimum on $[-4,4]\backslash\{2\}$.

Solution: (d) is correct. For the derivative of f we have:

$$f'(x)=\frac{2x(x+2)-x^2}{(x+2)^2}=\frac{x(x+4)}{(x+2)^2}$$

it can be checked that if $x<-4$ then $f'(x)>0$, $-4<x<-2$ or $-2<x<0$ then $f'(x)<0$ and when $x>0$ then $f'(x)>0$. So f has a local maximum at $x=-4$ and a local minimum at $x=0$. To find the global minimum and maximum of f we see that

$$\lim_{x\to-2^-}f(x)=-\infty,\qquad \lim_{x\to-2^+}f(x)=+\infty$$

So f does not have absolute maximum and minimum.

19. Let $a,b\in\mathbb{R}\cup\{\pm\infty\}$, $a<b$. Let $f:]a,b[\to\mathbb{R}$ be a differentiable function. State if the following statements are true or false. If it is true, prove it. If not, give a counter example.

(a) If f is Lipschitz continuous, then f' is bounded.
(b) The function $f(x) = \sqrt{x}$ defined on $]0, +\infty[$ is Lipschitz continuous.
(c) If f is uniformly continuous, then f' is bounded.
(d) If f is uniformly continuous, then it is Lipschitz continuous.

Solution:

(a) True. Let $k \geq 0$ such that f is Lipschitz continuous with Lipschitz constant k. Then for every $x_0 \in]a, b[$ we compute

$$|f'(x_0)| = \lim_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \lim_{x \to x_0} \frac{k|x - x_0|}{|x - x_0|} = k.$$

Note that we can exchange the limit with the absolute value because we know that the limit exists as f is differentiable.

(b) False. Assume that f is Lipschitz continuous with Lipschitz constant k for some $k \in \mathbb{R}^+$. Then we take $x = \frac{1}{n^2}$, $y = \frac{1}{(n+1)^2}$ and get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} = |\sqrt{x} - \sqrt{y}| \leq k|x - y| = k\frac{2n+1}{n^2(n+1)^2}$$

So $k \geq \frac{n(n+1)}{2n+1} \geq \frac{n}{2}$ for every integer $n > 0$. This contradicts the fact that $k \in \mathbb{R}$. Another way to see this is to observe that f is differentiable in $]0, +\infty[$ and show that f' is not bounded, because $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty$.

(c) False. We know that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty[$ by Exercise 5 in Exercise Sheet 9, but it is not Lipschitz continuous by the argument above.

(d) False. It is a consequence of the previous point, because we know that Lipschitz continuity is equivalent to boundedness of the derivative.

20. (a) Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
(b) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 2x^2 + 3x - 1$ Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

Solution:

(a) We prove it by induction. If $n = 1$ the formula holds by direct inspection. Assume that the formula holds for $n - 1 \geq 1$, then

$$\sum_{i=1}^{n} i^2 = n^2 + \sum_{i=1}^{n-1} i^2 = n^2 + \frac{n(n-1)(2n-1)}{6} = \frac{6n^2 + n(2n^2 - 3n + 1)}{6} = \frac{n(2n^2 + 3n + 1)}{6}.$$

(b) The regular partition σ_n of the interval $[0, 1]$ is $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1$. We observe that the function f is increasing because $f'(x) = 4x + 3$ is ≥ 0 on $[0, 1]$. Then we have

$$\underline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(\inf_{x \in [\frac{i-1}{n}, \frac{i}{n}]} f(x) \right) \left(\frac{i}{n} - \frac{i-1}{n} \right) = \sum_{i=1}^{n} \left(2 \left(\frac{i-1}{n} \right)^2 + 3\frac{i-1}{n} - 1 \right) \frac{1}{n}$$

$$= \frac{2}{n^3} \sum_{i=1}^{n} (i-1)^2 + \frac{3}{n^2} \sum_{i=1}^{n} (i-1) - \frac{1}{n} \sum_{i=1}^{n} 1 = \left(\frac{2}{n^3} \sum_{j=1}^{n-1} j^2 \right) + \left(\frac{3}{n^2} \sum_{j=1}^{n-1} j \right) - 1$$

$$= \frac{2}{n^3} \frac{n(n-1)(2n-1)}{6} + \frac{3}{n^2} \frac{n(n-1)}{2} - 1 = \frac{7}{6} - \frac{15}{6n} + \frac{3}{2n^2}$$

$$\overline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(\sup_{x \in [\frac{i-1}{n}, \frac{i}{n}]} f(x) \right) \left(\frac{i}{n} - \frac{i-1}{n} \right) = \sum_{i=1}^{n} \left(2 \left(\frac{i}{n} \right)^2 + 3\frac{i}{n} - 1 \right) \frac{1}{n}$$

$$= \frac{2}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n^2} \sum_{i=1}^{n} i - \frac{1}{n} \sum_{i=1}^{n} 1 = \frac{2}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n^2} \frac{n(n+1)}{2} - 1$$

$$= \frac{7}{6} + \frac{15}{6n} + \frac{3}{2n^2}$$

We observe that $\lim_{n \to +\infty} \underline{S}_{\sigma_n} = \lim_{n \to +\infty} \overline{S}_{\sigma_n} = \frac{7}{6}$. So the function f is integrable.

21. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) If $f(x) = x + e^x$, then $(f^{-1})'(1) = 1 + \frac{1}{e}$.

(b) If f is differentiable on the interval $I \subset \mathbb{R}$, then f' is continuous on I.

Solution:

(a) False.
The formula for the derivative of the inverse function is $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.
Here we have $f'(x) = 1 + e^x$ and $f^{-1}(1) = 0$ since $f(0) = 1$. So

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}.$$

(b) False.
Take for example the function f, $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. This function is differentiable at all $x \neq 0$, as it is the composition of differentiable functions. To check the differentiability at $x = 0$, we claim that $f'(0) = 0$:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$$

So we can write the derivative of f as

$$f(x) = \begin{cases} 2x \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We see that f is differentiable on $]-1, 1[$ (in fact on $\mathbb{R}$) but the derivative is not continuous at $x = 0$.