

Analysis 1 - Exercise Set 11

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Let $a, b \in \mathbb{Z}$, $b > 0$. Show that $\sqrt[b]{x^a} = e^{\frac{a}{b} \log(x)}$ for all real numbers $x > 0$.
- (b) Compute the derivative of the following functions:
 - (i) $f(x) = x^a : \mathbb{R}_+^* \rightarrow \mathbb{R}$, $a \in \mathbb{R}$; show that f is strictly increasing when $a > 0$ and strictly decreasing when $a < 0$;
 - (ii) $f(x) = a^x : \mathbb{R} \rightarrow \mathbb{R}_+^*$, $a \in \mathbb{R}_+^*$; show that f is strictly increasing when $a > 1$ and strictly decreasing when $a < 1$;
 - (iii) $f(x) = \log_a(x) : \mathbb{R}_+^* \rightarrow \mathbb{R}$, $a \in \mathbb{R}_+^*$; show that f is strictly increasing when $a > 1$ and strictly decreasing when $a < 1$;

Solution:

- (a) $\sqrt[b]{x^a}$ is the unique positive number such that $(\sqrt[b]{x^a})^b = x^a$. The number $e^{\frac{a}{b} \log(x)}$ is positive, because $e^y > 0$ for all $y \in \mathbb{R}$. So it suffices to check that

$$(e^{\frac{a}{b} \log(x)})^b = e^{a \log(x)} = (e^{\log(x)})^a = x^a.$$

- (b) (i) $(x^a)' = (e^{a \log(x)})' = \frac{a}{x} e^{a \log(x)} = ax^{a-1}$. From this we see that, since $x > 0$, $(x^a)' > 0$ if $a > 0$ and hence x^a is strictly increasing. Likewise, $(x^a)' < 0$ when $a < 0$ and hence x^a is strictly decreasing.
- (ii) $(a^x)' = (e^{x \log(a)})' = \log(a) e^{x \log(a)} = \log(a) a^x$. Since $\log(a) > 0$ for $a > 1$ and $\log(a) < 0$ for $0 < a < 1$, we see that $(a^x)' > 0$ if $a > 1$, and $(a^x)' < 0$ if $a < 1$. Therefore, a^x is strictly increasing if $a > 1$ and strictly decreasing if $a < 1$.
- (iii) $(\log_a(x))' = \left(\frac{\log(x)}{\log(a)} \right)' = \frac{1}{x \log(a)}$. The monotonicity results follows from that $\log_a(x)$ is the inverse of a^x and that the inverse of a strictly increasing/decreasing function is strictly increasing/decreasing.

2. State if the following are true or false.

- (a) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and has two roots, that is, there exist $x \neq y \in \mathbb{R}$, $f(x) = 0 = f(y)$, then f' has at least one root.
- (b) The function $f(x) = \frac{\sin(x^2-2)}{e^{3x+1} + \sqrt{2}x}$ has a critical value in $] \sqrt{2}, \sqrt{2 + \pi/2} [$.

Solution:

(a) True. If the two roots, a and b , are distinct, up to swapping them we may assume $a < b$. Then we apply Rolle's Theorem to the interval $[a, b]$ to prove that f' has a zero between them.

(b) True. We compute the derivative of $\frac{\sin(x^2-2)}{e^{3x+1}+\sqrt{2x}}$:

$$f'(x) = \frac{2x \cos(x^2 - 2)(e^{3x+1} + \sqrt{2x}) - \sin(x^2 - 2)(3e^{3x+1} + \sqrt{2x^{-1}}/2)}{(e^{3x+1} + \sqrt{2x})^2}.$$

We observe that the domain of f is $[0, +\infty[$; the above computation shows that f is differentiable on $]0, +\infty[$. Furthermore, f' is a continuous function on its domain $]0, +\infty[$. We compute

$$f'(\sqrt{2}) = \frac{2\sqrt{2}}{e^{3\sqrt{2}+1} + \sqrt{2\sqrt{2}}} > 0, \quad f'(\sqrt{2 + \pi/2}) = -\frac{(3e^{3\sqrt{2+\pi/2}+1} + \sqrt{(4+\pi)^{-1}})}{e^{3\sqrt{2+\pi/2}+1} + \sqrt{2\sqrt{2+\pi/2}}} < 0$$

So by the intermediate value theorem applied to the continuous function f' , there is $x_0 \in]\sqrt{2}, \sqrt{2 + \pi/2}[$ such that $f'(x_0) = 0$.

3. Calculate the following limits:

(a) $\lim_{x \rightarrow 0} (1 + \sin(x))^{1/x}$

(Hint: Write $(1 + \sin(x))^{1/x} = e^{\left(\frac{1}{x} \log(1 + \sin(x))\right)}$ and first calculate the limit of the exponent.)

(b) $\lim_{x \rightarrow \sqrt{3}} \frac{x^x - \sqrt{3\sqrt{3}}}{x - \sqrt{3}}$. Remember that $x^x = e^{x \log(x)}$.

Solution:

For calculating these limits we apply the L'Hôpital rule, but we must check the hypothesis in each case:

(a) We have $(1 + \sin(x))^{1/x} = e^{\frac{1}{x} \log(1 + \sin(x))}$. We are first going to calculate the limit of the exponent. Take $f(x) = \log(1 + \sin(x))$ and $g(x) = x$. Then $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$ and $g'(x) = 1 \neq 0$. So

$$\lim_{x \rightarrow 0} \frac{\log(1 + \sin(x))}{x} \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow 0} \frac{\frac{\cos(x)}{1 + \sin(x)}}{1} = 1,$$

Finally, by the continuity of the exponential function we have

$$\lim_{x \rightarrow 0} (1 + \sin(x))^{1/x} = e^1 = e.$$

We may apply L'Hôpital's rule since $(x)' \neq 0 \quad \forall$ intervals I containing 0.

(b) Take $f(x) = x^x - \sqrt{3\sqrt{3}}$ and $g(x) = x - \sqrt{3}$. Note that we can rewrite $f(x) = e^{x \log x} - e^{\sqrt{3} \log \sqrt{3}}$. We have that $\lim_{x \rightarrow \sqrt{3}} f = \lim_{x \rightarrow \sqrt{3}} g = 0$ and that $g'(x) = 1 \neq 0$.

So the L'Hôpital rule applies. We have

$$\begin{aligned}\lim_{x \rightarrow \sqrt{3}} \frac{x^x - \sqrt{3^{\sqrt{3}}}}{x - \sqrt{3}} &= \lim_{x \rightarrow \sqrt{3}} \frac{e^{x \log x} - e^{\sqrt{3} \log \sqrt{3}}}{x - \sqrt{3}} \\ &\stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow \sqrt{3}} \frac{(1 + \log x)e^{x \log x}}{1} \\ &= (1 + \frac{1}{2} \log 3) \sqrt{3^{\sqrt{3}}}\end{aligned}$$

We may apply L'Hôpital's rule since $(x - \sqrt{3})' \neq 0 \quad \forall$ intervals I containing 0.

4. Find the Taylor expansion of order 5 at $x = 0$ of the following functions.

- (a) $f(x) = \sin(x)$
- (b) $f(x) = \log(1 + x)$
- (c) $f(x) = \tan(x)$
- (d) $f(x) = \arccos(x)$
- (e) $f(x) = \sinh(x)$
- (f) $f(x) = \log(\cos(x))$

Solution:

The Taylor's expansion of order 5 around the point a of a function f is given by the

$$\begin{aligned}f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ &\quad + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \varepsilon_5(x)\end{aligned}$$

where $\varepsilon_5(x) = \frac{f^{(6)}(u)}{6!}(x-a)^6$ for a certain u between a and x , we can say that $u \in]a, x[$ if $x > a$ and that $u \in]x, a[$ if $x < a$.

- (a) For $\sin x$ we notice that all the even derivatives are zero at $x = 0$ and odd derivatives are $\sin^{(4k+1)}(x) = \cos(x)$ and that $\cos(0) = 1$ and $\sin^{(4k+3)}(x) = -\cos(x)$ and that $-\cos(0) = -1$. If we plug in all the derivatives in the Taylor formula we get

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \varepsilon_5(x)$$

- (b) For $\log(1 + x)$ we see that the n -th derivative is given by

$$\log^{(n)}(1+x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

So we can now use the Taylor expansion to write

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + x^5 \varepsilon(x)$$

(c) By differentiating $f(x) = \tan(x)$ 5 times we see that

$$\begin{aligned}f(0) &= 0 \\f'(0) &= 1 \\f''(0) &= 0 \\f'''(0) &= 2 \\f^{(4)}(0) &= 0 \\f^{(5)}(0) &= 16\end{aligned}$$

which yields $f(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

(d) By differentiating $f(x) = \arccos(x)$ 5 times we see that

$$\begin{aligned}f(0) &= \frac{\pi}{2} \\f'(0) &= -1 \\f''(0) &= 0 \\f'''(0) &= -1 \\f^{(4)}(0) &= 0 \\f^{(5)}(0) &= -9\end{aligned}$$

which yields $f(x) = \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 \dots$

(e) By differentiating $f(x) = \sinh(x)$ (note that $f^{(2k)} = \sinh(x)$ and $f^{(2k+1)}(x) = \cosh(x)$) 5 times we see that

$$\begin{aligned}f(0) &= 0 \\f'(0) &= 1 \\f''(0) &= 0 \\f'''(0) &= 1 \\f^{(4)}(0) &= 0 \\f^{(5)}(0) &= 1\end{aligned}$$

which yields $f(x) = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots$

(f) Note that $f'(x) = -\tan(x)$. Then we can use (c) to see that

$$\begin{aligned}f(0) &= 0 \\f'(0) &= 0 \\f''(0) &= -1 \\f'''(0) &= 0 \\f^{(4)}(0) &= -2 \\f^{(5)}(0) &= 0\end{aligned}$$

which yields $f(x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 \dots$

5. Use Taylor expansion to find the following limits.

- (a) $\lim_{x \rightarrow 0} \frac{x - \frac{x^3}{6} - \sin(x)}{x^5}$
 (b) $\lim_{x \rightarrow 0} \frac{e^x + \sin(x) - \cos(x) - 2x}{x - \log(1+x)}$
 (c) $\lim_{x \rightarrow 0} \frac{x \sin(\sin(x)) - \sin(x)^2}{x^6}$
 (d) $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1-x} - 1}{\sqrt[4]{1-x} - 1}$

Solution: We need to choose the order of the Taylor expansion such that the indeterminacies in the denominator are eliminated. Since we are only interested in the limit, it is enough to keep higher terms as $(x-a)^n \varepsilon(x)$, where $\lim_{x \rightarrow a} \varepsilon(x) = 0$. In particular, if we consider a Taylor expansion of order n and have an error $\varepsilon_n(x) = \frac{f^{(n+1)}(u)}{n!} (x-a)^{n+1}$, we want to write it as $(x-a)^n \cdot \varepsilon(x)$, where we set $\varepsilon(x) = \frac{f^{(n+1)}(u)}{n!} (x-a)$. Then, as $\frac{f^{(n+1)}(u)}{n!}$ is bounded in a neighborhood of a (we are using that $f^{(n+1)}$ is continuous and u is a value, depending on x and a , that is in between a and x) and $x-a \rightarrow 0$ as $x \rightarrow a$, we get that $\varepsilon(x) \rightarrow 0$ as $x \rightarrow a$.

(a) Since

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x),$$

we have

$$\lim_{x \rightarrow 0} \frac{1}{x^5} \left(x - \frac{x^3}{6} - \sin(x) \right) = \lim_{x \rightarrow 0} \left(-\frac{1}{120} + \varepsilon(x) \right) = -\frac{1}{120},$$

because $\varepsilon(x) \rightarrow 0$ when $x \rightarrow 0$.

(b) We have

$$e^x + \sin(x) - \cos(x) - 2x = 1 + x + \frac{x^2}{2} + x - 1 + \frac{x^2}{2} - 2x + x^2 \varepsilon(x) = x^2 + x^2 \varepsilon(x)$$

and

$$x - \log(1+x) = x - x + \frac{x^2}{2} + x^2 \tilde{\varepsilon}(x) = \frac{x^2}{2} + x^2 \tilde{\varepsilon}(x),$$

where $\varepsilon(x)$ (resp. $\tilde{\varepsilon}(x)$) goes to 0 as $x \rightarrow 0$, as it has the form $C(u(x))x^3$, where $C(x)$ is the derivative of order 3 of $e^x + \sin(x) - \cos(x) - 2x$ (resp. $x - \log(1+x)$). Thus, we have

$$\lim_{x \rightarrow 0} \frac{e^x + \sin(x) - \cos(x) - 2x}{x - \log(1+x)} = \lim_{x \rightarrow 0} \frac{x^2 + x^2 \varepsilon(x)}{\frac{x^2}{2} + x^2 \tilde{\varepsilon}(x)} = \lim_{x \rightarrow 0} \frac{1 + \varepsilon(x)}{\frac{1}{2} + \tilde{\varepsilon}(x)} = 2.$$

(c) We need to choose the order of the Taylor expansion such that the indeterminacies in the denominator are eliminated. Since we are only interested in the limit, it is enough to keep higher terms as $(x-a)^n \varepsilon(x)$, where $\lim_{x \rightarrow a} \varepsilon(x) = 0$.

For the 6-th order Taylor expansion of the numerator, we need to first find the 5-th order Taylor expansion of $\sin(\sin(x))$ and then the 6-th order of $\sin(x)^2$.

Since $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5 \varepsilon(x)$, we have

$$\sin(\sin(x)) = \sin(x) - \frac{\sin(x)^3}{3!} + \frac{\sin(x)^5}{5!} + \underbrace{\sin(x)^5 \varepsilon(\sin(x))}_{=x^5 \varepsilon(x)}. \quad (1)$$

In the computation, we replace $\sin(x)^5 \varepsilon(\sin(x))$ with $x^5 \varepsilon(x)$. We can do this because $\sin(0) = 0$ and $\lim_{x \rightarrow 0} \varepsilon(x) = 0$, we have $\lim_{x \rightarrow 0} \varepsilon(\sin(x)) = 0$ and so $\varepsilon(\sin(x))$ behaves as $\varepsilon(x)$. Then, $\frac{\sin(x)}{x}$ is bounded in the neighborhood of 0 (as its limit as $x \rightarrow 0$ is 1) and indeed $\sin(x)^5 \varepsilon(x) = \frac{\sin(x)^5}{x^5} x^5 \varepsilon(x)$ behaves like $x^5 \varepsilon(x)$.

For the powers of $\sin(x)$ we have

$$\begin{aligned}\sin(x)^2 &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^6 \varepsilon(x)\right)^2 = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + x^6 \varepsilon(x), \\ \sin(x)^3 &= \left(x^2 - \frac{x^4}{3} + x^5 \varepsilon(x)\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)\right) = x^3 - \frac{x^5}{2} + x^5 \varepsilon(x), \\ \sin(x)^5 &= \left(x^2 - \frac{x^4}{3} + x^5 \varepsilon(x)\right) \left(x^3 - \frac{x^5}{2} + x^5 \varepsilon(x)\right) = x^5 + x^5 \varepsilon(x),\end{aligned}$$

So we get

$$\begin{aligned}\sin(\sin(x)) &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{1}{6} \left(x^3 - \frac{x^5}{2}\right) + \frac{x^5}{120} + x^5 \varepsilon(x) \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} + x^5 \varepsilon(x)\end{aligned}$$

And finally

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x \sin(\sin(x)) - \sin(x)^2}{x^6} &= \lim_{x \rightarrow 0} \frac{1}{x^6} \left(x^2 - \frac{x^4}{3} + \frac{x^6}{10} - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + x^6 \varepsilon(x)\right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{18} + \varepsilon(x)\right) = \frac{1}{18}.\end{aligned}$$

Remark: We can directly substitute the expansion of $\sin(x)$ inside equation (1). This way we get

$$\begin{aligned}\sin(\sin(x)) &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)\right) - \frac{1}{6} \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)\right)^3 \\ &\quad + \frac{1}{120} \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)\right)^5 + x^5 \varepsilon(x) \\ &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)\right) - \frac{1}{6} \left(x - \frac{x^3}{6} + x^3 \varepsilon(x)\right)^3 \\ &\quad + \frac{1}{120} (x + x \varepsilon(x))^5 + x^5 \varepsilon(x) \\ &= \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) - \frac{1}{6} \left(x^3 - \frac{3x^5}{6}\right) + \frac{1}{120} x^5 + x^5 \varepsilon(x) \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} + x^5 \varepsilon(x).\end{aligned}$$

(d) We have

$$\frac{\sqrt[3]{1-x} - 1}{\sqrt[4]{1-x} - 1} = \frac{(1 - \frac{1}{3}x + x\varepsilon(x)) - 1}{(1 - \frac{1}{4}x + x\varepsilon(x)) - 1}$$

so the limit is $\frac{4}{3}$

6. For a complex number of the form e^{ix} , $x \in \mathbb{R}$, we defined

$$\cosh(ix) := \frac{e^{ix} + e^{-ix}}{2}, \quad \sinh(ix) := \frac{e^{ix} - e^{-ix}}{2}.$$

- (a) Compute the complex numbers $\cosh(ix)$, $\sinh(ix)$;
 (b) For each of the functions $\cosh(x)$, $\sinh(x)$, $\tanh(x)$, $\coth(x)$ compute the derivative and the domain of the derivative.

Which of these functions are invertible on the domain $\mathbb{R}$? which on $\mathbb{R}_+^*$? Recall that

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{1}{\tanh(x)}.$$

- (c) Compute the derivatives of the inverses of the functions in (b) and their domains.

Solution:

(a)

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \frac{1}{2}(\cos(x) + i\sin(x) + \cos(-x) + i\sin(-x)) = \cos(x)$$

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = \frac{1}{2}(\cos(x) + i\sin(x) - (\cos(-x) + i\sin(-x))) = i\sin(x)$$

- (b) $D(\cosh(x)) = D(\sinh(x)) = \mathbb{R}$. Since $\cosh(x) > 0$ for all $x \in \mathbb{R}$, then $D(\tanh(x)) = \mathbb{R}$. Since $\sinh(x) = 0$ only if $x = 0$, we have $D(\coth(x)) = \mathbb{R} \setminus \{0\}$.

Using the fact that $(e^{ax})' = ae^{ax}$ for all $a \in \mathbb{R}$ we see immediately that

$$\cosh(x)' = \frac{e^x - e^{-x}}{2} = \sinh(x), \quad \sinh(x)' = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

and $D(\cosh(x)') = D(\sinh(x)') = \mathbb{R}$. Since $\sinh(x)' > 0$ for all $x \in \mathbb{R}$, then the function $\sinh(x)$ is strictly increasing and hence invertible on the domain $\mathbb{R}$. We compute $\lim_{x \rightarrow +\infty} \sinh(x) = +\infty$ and $\lim_{x \rightarrow -\infty} \sinh(x) = -\infty$, so $R(\sinh(x)) = \mathbb{R}$ because $\sinh(x)$ is a continuous function. So $R(\sinh(x)) = \mathbb{R}$. We compute $\lim_{x \rightarrow \pm\infty} \cosh(x) = +\infty$, so the function $\cosh(x)$ is not invertible on the domain $\mathbb{R}$, but $\cosh(x)' > 0$ for all $x > 0$, so $\cosh(x)$ is strictly increasing and hence invertible on the domain $\mathbb{R}_+^*$. Also, we observe that $\cosh(x)$ has a minimum in $x = 0$, because it is strictly decreasing on $] -\infty, 0[$ and strictly increasing on $]0, +\infty[$. So $R(\cosh(x)) = [\cosh(0), +\infty[= [1, +\infty[$.

Using the rule of derivative of a quotient, we see that

$$\tanh(x)' = \frac{\sinh(x)' \cosh(x) - \sinh(x) \cosh(x)'}{\cosh(x)^2} = \frac{\cosh(x)^2 - \sinh(x)^2}{\cosh(x)^2} = \frac{1}{\cosh(x)^2}$$

$$\coth(x)' = \frac{\cosh(x)' \sinh(x) - \cosh(x) \sinh(x)'}{\sinh(x)^2} = \frac{\sinh(x)^2 - \cosh(x)^2}{\sinh(x)^2} = -\frac{1}{\sinh(x)^2}$$

Then $D(\tanh(x)') = \mathbb{R}$, $D(\coth(x)') = \mathbb{R} \setminus \{0\}$. Since $\tanh(x)' > 0$ for all $x \in \mathbb{R}$ the function $\tanh(x)$ is strictly increasing, and hence invertible. Since $\coth(x)' < 0$ for all $x \neq 0$, the functions $\coth(x)|_{]-\infty, 0[}$, $\coth(x)|_{]0, +\infty[}$ are both strictly decreasing and hence invertible on their domains. To compute the range of monotone functions

defined on intervals, it suffices to compute the limits at the extremities of the domain. We have

$$\lim_{x \rightarrow -\infty} \tanh(x) = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = -1, \quad \lim_{x \rightarrow +\infty} \tanh(x) = \lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1$$

so $R(\tanh(x)) = [-1, 1]$.

$$\lim_{x \rightarrow -\infty} \coth(x) = \lim_{x \rightarrow -\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = -1, \quad \lim_{x \rightarrow +\infty} \coth(x) = \lim_{x \rightarrow +\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = 1,$$

$$\lim_{x \rightarrow 0^-} \coth(x) = \lim_{x \rightarrow 0^-} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow 0^-} \frac{1 + e^{-2x}}{1 - e^{-2x}} = -\infty,$$

$$\lim_{x \rightarrow 0^+} \coth(x) = \lim_{x \rightarrow 0^+} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \rightarrow 0^+} \frac{1 + e^{-2x}}{1 - e^{-2x}} = +\infty$$

so $R(\coth(x)) =]-\infty, -1[\cup]1, +\infty[$.

- (c) Note that we know the ranges of the hyperbolic functions above, which will be the domains for the inverses. Therefore, we automatically get the domains of the inverses from the ranges in (b).

Consider $y = \cosh(x)$. Then, if $x \geq 0$, we have $\cosh^{-1}(y) = x$. By the rule for the derivative of the inverse function, we have $(\cosh^{-1}(y))' = \frac{1}{(\cosh(\cosh^{-1}(y)))'}$. Thus, we have

$$\begin{aligned} (\cosh^{-1}(y))' &= \frac{1}{(\cosh(\cosh^{-1}(y)))'} \\ &= \frac{1}{\sinh(\cosh^{-1}(y))} \\ &= \frac{1}{\sqrt{\sinh^2(\cosh^{-1}(y))}} \\ &= \frac{1}{\sqrt{\cosh^2(\cosh^{-1}(y)) - 1}} \\ &= \frac{1}{\sqrt{y^2 - 1}}, \end{aligned}$$

where in the second line we used that $\cosh' = \sinh$, in the third line we used that $\cosh^{-1}(y) = x \geq 0$ by assumption and that $\sinh(x) \geq 0$ when $x \geq 0$, which guarantees that $\sinh(x) = \sqrt{\sinh^2(x)}$ when $\sinh(x) \geq 0$. Lastly, in the fourth line we used the hyperbolic trigonometric identity $\cosh^2(t) - \sinh^2(t) = 1$. Recall that the domain of $\cosh^{-1}(y)$ is $[1, \infty)$; thus, the domain of its derivative is $(1, \infty)$.

We apply a similar argument. Notice that $y = \sinh(x)$ is invertible on $\mathbb{R}$, so there is no restriction to write $x = \sinh^{-1}(y)$. Then, we have

$$\begin{aligned} (\sinh^{-1}(y))' &= \frac{1}{(\sinh(\sinh^{-1}(y)))'} \\ &= \frac{1}{\cosh(\sinh^{-1}(y))} \\ &= \frac{1}{\sqrt{\cosh^2(\sinh^{-1}(y))}} \\ &= \frac{1}{\sqrt{\sinh^2(\sinh^{-1}(y)) + 1}} \\ &= \frac{1}{\sqrt{y^2 + 1}}, \end{aligned}$$

where we argued as in the previous step and used that $\cosh(t) = \sqrt{\cosh^2(t)}$ for all t , as $\cosh(t) \geq 1$ for all t . Thus, the derivative is defined on all $\mathbb{R}$.

Applying a similar argument to $\tanh$ we write $y = \tanh(x)$, so $x = \tanh^{-1}(y)$, where $x \in \mathbb{R}$ and $y \in]-1, 1[$. Then, we have

$$\begin{aligned} (\tanh^{-1}(y))' &= \frac{1}{(\tanh(\tanh^{-1}(y)))'} \\ &= \cosh^2(\tanh^{-1}(y)) \\ &= \frac{1}{1 - \tanh^2(\tanh^{-1}(y))} \\ &= \frac{1}{1 - y^2}, \end{aligned}$$

where we used the identity $\operatorname{sech}^2(t) = 1 - \tanh^2(t)$, which can be rearranged as $\cosh^2(t) = \frac{1}{1 - \tanh^2(t)}$. So, the domain of the derivative coincides with the domain of $\tanh^{-1}(y)$, which is $] - 1, 1[$.

Applying a similar argument to $y = \coth(x)$, we write

$$\begin{aligned} (\coth^{-1}(y))' &= \frac{1}{(\coth(\coth^{-1}(y)))'} \\ &= -\sinh^2(\coth^{-1}(y)) \\ &= \frac{1}{1 - \coth^2(\coth^{-1}(y))} \\ &= -\frac{1}{1 - y^2} \\ &= \frac{1}{y^2 - 1}, \end{aligned}$$

where we used the identity $\operatorname{csch}^2(t) = \coth^2(t) - 1$, which can be rearranged as $\sinh^2(t) = \frac{1}{\coth^2(t) - 1}$. So, the domain of the derivative coincides with the domain of $\coth^{-1}(y)$, which is $] - \infty, -1[\cup]1, +\infty[$.

7. For the following functions, find the stationary points and discuss whether they are points at which the function attains a local maximum or minimum.

- (a) $f(x) = x \log^2(x)$ in $]0, +\infty[$
 (b) $f(x) = 2 \sin(x) + \cos(2x)$ in $[-\frac{1}{10}, \frac{1}{15}]$

Solution:

- (a) The derivative is

$$f'(x) = \log(x)(\log(x) + 2)$$

and it is zero at $x = 1$ and $x = e^{-2}$. The derivative is positive in the interval $]0, e^{-2}[$, negative in $]e^{-2}, 1[$ and positive in $]1, +\infty[$, so e^{-2} is a local maximum and 1 is a local minimum. Since $f(1) = 0$ and $f(x) \geq 0$ for all x in the domain, it follows that

the minimum at 1 is also global. On the other hand, the local maximum is not a global maximum, as $\lim_{x \rightarrow +\infty} f(x) = +\infty$.

(b) The derivative is

$$f'(x) = 2 \cos(x) - 2 \sin(2x) = 2 \cos(x)(1 - 2 \sin(2x)).$$

The derivative vanishes only if $\cos(x) = 0$ (i.e., $x = \frac{\pi}{2} + k\pi$) or $\sin(2x) = \frac{1}{2}$ (i.e., $x = \frac{\pi}{12} + k\pi$ or $x = \frac{5\pi}{12} + k\pi$), and no one of these values is in $[-\frac{1}{10}, \frac{1}{15}]$. So, f is strictly increasing on its domain and has no stationary points on its domain. Therefore, as the domain is a closed interval, we have a minimum at $-\frac{1}{10}$ and a maximum at $\frac{1}{15}$.

8. Calculate the following limits:

(a) $\lim_{x \rightarrow +\infty} x (\tanh(x) - 1)$

(b) $\lim_{x \rightarrow +\infty} \frac{e^x}{x^n}$ where $n \in \mathbb{N}$. First find the limit using $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and then using l'Hôpital's rule.

(c) $\lim_{x \rightarrow 0^+} x^n \log(x)$

(d) $\lim_{x \rightarrow \infty} \frac{x^n}{\log(x)}$

Solution:

(a) Here we should apply the L'Hôpital rule several times. For the first time, we have $f(x) = \tanh(x) - 1$ and $g(x) = \frac{1}{x}$. Since $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} g(x) = 0$ and $g'(x) = -\frac{1}{x^2} \neq 0$, the hypothesis is satisfied and we can apply the L'Hôpital rule (the hypothesis for the second and third iterations of L'Hôpital rule will be verified below):

$$\begin{aligned} \lim_{x \rightarrow +\infty} x (\tanh(x) - 1) &= \lim_{x \rightarrow +\infty} \frac{\tanh(x) - 1}{\frac{1}{x}} \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow +\infty} \frac{\frac{1}{\cosh(x)^2}}{-\frac{1}{x^2}} = - \lim_{x \rightarrow +\infty} \frac{x^2}{\cosh(x)^2} \\ &\stackrel{\text{L'Hôpital}}{=} - \lim_{x \rightarrow +\infty} \frac{2x}{\sinh(2x)} \stackrel{\text{L'Hôpital}}{=} - \lim_{x \rightarrow +\infty} \frac{2}{2 \cosh(2x)} = 0. \end{aligned}$$

To apply the L'Hôpital rule a second time we have $\tilde{f}(x) = x^2$ and $\tilde{g}(x) = \cosh(x)^2$ we have $\lim_{x \rightarrow +\infty} \tilde{f}(x) = \lim_{x \rightarrow +\infty} \tilde{g}(x) = +\infty$ and $\tilde{g}'(x) = 2 \sinh(x) \cosh(x) = \sinh(2x) \neq 0$ for $x \neq 0$ (which is the case when $x \rightarrow +\infty$).

Finally for the third time with $\bar{f}(x) = 2x$ and $\bar{g}(x) = \sinh(2x)$ and so $\lim_{x \rightarrow +\infty} \bar{f}(x) = \lim_{x \rightarrow +\infty} \bar{g}(x) = +\infty$ and that $\bar{g}'(x) = 2 \cosh(2x) \neq 0$. And so we can apply the L'Hôpital rule for the third time.

(b) Here we need to apply the L'Hôpital rule n times. We notice that every time we apply the L'Hôpital rule the numerator remains e^x and the denominator will be of the form mx^k for some $m \in \mathbb{N}$ and $0 \geq k \leq n$. So take $f(x) = e^x$ and $g(x) = mx^k$. We have that $\lim_{x \rightarrow +\infty} f = \lim_{x \rightarrow +\infty} g = +\infty$ and that $g'(x) \neq 0$. So we can apply L'Hôpital rule n times. We have

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x^n} = \lim_{x \rightarrow +\infty} \frac{e^x}{nx^{n-1}} \stackrel{\text{L'Hôpital}}{=} \dots \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow +\infty} \frac{e^x}{n!} = +\infty$$

Now using the Taylor series expansion we see that $\frac{e^x}{x^n} = \delta(x) + \sum_{k=n}^{\infty} \frac{x^{k-n}}{k!} = \delta(x) + \sum_{k=0}^{\infty} \frac{x^k}{(k+n)!} \geq \delta(x) + \frac{1}{k!} + \frac{1}{(k+1)!}x \rightarrow +\infty$ when $x \rightarrow +\infty$ where $\delta(x)$ is a function s.t. $\delta(x) \rightarrow 0$ as $x \rightarrow +\infty$.

(c)

$$\lim_{x \rightarrow 0^+} x^n \log(x) = \lim_{x \rightarrow 0^+} \frac{\log(x)}{\frac{1}{x^n}} \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{n}{x^{n+1}}} = 0$$

(d) $\lim_{x \rightarrow +\infty} \frac{x^n}{\log(x)} = \lim_{h \rightarrow 0^+} \frac{1}{-h^n \log(h)} = +\infty$ by using (c).

9. Calculate the derivative f' of the function $f(x) = \log_3(\cosh(x))$ and give the domain of f and f' .

Solution:

We observe that

$$f(x) = \frac{\log(\cosh(x))}{\log(3)}.$$

So $f'(x) = \frac{\sinh(x)}{\log(3) \cosh(x)} = \frac{\tanh(x)}{\log(3)}$ and $D(f) = D(f') = \mathbb{R}$, as $\cosh(x) \geq 1$ for all x .

10. State if the following are true or false.

- (a) The function $f : [0, +\infty[\rightarrow [1, +\infty[$ defined by $f(x) = x^3 - x + e^x$ is invertible.
(b) Let $a, b \in \mathbb{R}$, $a < b$. Given a continuous function $f :]a, b[\rightarrow \mathbb{R}$ which is not monotone, there exists a point $x_0 \in (a, b)$ at which f admits a local minimum.

Solution:

- (a) True. We compute $f'(x) = 3x^2 - 1 + e^x$. We observe that $f'(x) > 0$ for $x > 0$, so the function f is injective because it is strictly increasing. It has a minimum at $x = 0$, $f(0) = 1$. The supremum of the range is $\lim_{x \rightarrow +\infty} f(x) = +\infty$. So f is invertible.
(b) False. Take for example $f(x) = -x^2$, $a = -1$, $b = 1$.

11. Calculate the following limit $\lim_{x \rightarrow 2} \frac{\log(x-1)}{x-2}$.

Solution: Let $f(x) = \log(x-1)$ and $g(x) = x-2$. Then we have $\lim_{x \rightarrow 2} f(x) = 0$, $\lim_{x \rightarrow 2} g(x) = 0$ and $g'(x) = 1 \neq 0$. So the hypothesis of the L'Hôpital rule is satisfied, so we have,

$$\lim_{x \rightarrow 2} \frac{\log(x-1)}{x-2} \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow 2} \frac{\frac{1}{x-1}}{1} = 1.$$

12. Let the function $f :]-\pi/2, \pi/2[\rightarrow \mathbb{R}$ be defined by $f(x) = \log(1 + \sin(x))$. What is the Taylor expansion of order 3 of f at $x = 0$.

- (a) $x - \frac{x^3}{6} + \dots$
- (b) $x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$
- (c) $x - \frac{x^2}{2} + \frac{x^3}{6} + \dots$
- (d) $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

Solution: (c) is correct. We use the following Taylor expansions

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3 \varepsilon(x)$$

$$\sin(x) = x - \frac{x^3}{6} + x^3 \varepsilon(x)$$

So

$$\log(1 + \sin(x)) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2}(x)^2 + \frac{1}{3}(x)^3 + x^3 \varepsilon(x) = x - \frac{x^2}{2} + \frac{x^3}{6} + x^3 \varepsilon(x)$$

Revision Exercises

13. Consider the bijective function $f :]1, \infty[\rightarrow]-\infty, -2[$ defined as $f(x) = \log(x) - 2x$. Then the derivative of the inverse function $f^{-1}(y)$ at $y = -2$ is

- (a) $(f^{-1})'(-2) = -1$
- (b) $(f^{-1})'(-2) = 1$
- (c) $(f^{-1})'(-2) = -\frac{2}{5}$
- (d) $(f^{-1})'(-2) = \frac{2}{5}$

Solution: (a) is correct. Using the formula for the derivative of the inverse function we have

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

First we compute $f'(x)$

$$f'(x) = \frac{1}{x} - 2$$

Then we should find x such that $f(x) = -2$, clearly $x = 1$. By substitution we get $(f^{-1})'(-2) = -1$.

14. For which of the following item you can prove, using the intermediate value theorem, that there exists a c in I such that $f(c) = k$.

- (a) $f(x) = \frac{x^2+8}{x}$, $k = 5$, $I = [1, 3]$
- (b) $f(x) = x^2 + x + 1$, $k = 2$, $I = [-2, 3]$
- (c) $f(x) = \frac{1}{2x-1}$, $k = 0$, $I = [0, 1]$
- (d) $f(x) = \frac{10}{x^2+1}$, $k = 8$, $I = [0, 1]$

Solution: (d) is correct. The function must be continuous on the interval $[a, b]$ and satisfy $f(a) < k < f(b)$ or $f(a) > k > f(b)$.

15. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)$$

then.

- (a) $f'(x) = \cos\left(\frac{1-x^2}{1+x^2}\right) \frac{-4x}{1+x^2}$
- (b) $f'(x) = \cos\left(\frac{1-x^2}{1+x^2}\right) \frac{-4x}{(1+x^2)^2}$
- (c) $f'(x) = \sin\left(\frac{1-x^2}{1+x^2}\right) \frac{-4x}{(1+x^2)^2}$
- (d) $f'(x) = \sin\left(\frac{1-x^2}{1+x^2}\right) \frac{-4x}{1+x^2}$

Solution: (b) is correct.

16. Let the function $f : \mathbb{R} \setminus \{1/2\} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{2x^3 + x^2 + 2x + 1}{2x - 1}$$

then

- (a) f has at least one root in $[-1, 0]$
- (b) f has at least one root in $[0, 1]$
- (c) f has at least two roots in $[-1, 1]$
- (d) f has no roots

Solution: (a) is correct. To use the intermediate value theorem, we need f to be continuous on a given interval, since the function is not defined at $x = 1/2$ so the intermediate value theorem cannot be applied on $[0, 1]$ and $[-1, 1]$. On the interval $[-1, 0]$, f is continuous and we see that $f(-1) > 0$, $f(0) < 0$. The intermediate value theorem suggests that f has a solution in $[-1, 0]$.

17. Let the function $f :]-1, 1[\setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{x \log(1+x)}{\cos(x)-1}$. Let $g :]-1, 1[\rightarrow \mathbb{R}$ be an extension of f that is continuous at 0. Then

- (a) g exist and $g(0) = -2$.
- (b) g exist and $g(0) = 2$.
- (c) g exist and $g(0) = 0$.
- (d) f does not have a continuous extension at 0.

(Hint: Note that $\log(1) = 0$ so $\log(1+x) = \log(1+x) - \log(1)$. Then use the definition of derivative.)

Solution: (a) is correct. We need to find the limit f as $x \rightarrow 0$. We use polynomial expansion to get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \log(1+x)}{\cos(x) - 1} &= \lim_{x \rightarrow 0} \frac{x \log(1+x)}{\cos(x) - 1} \cdot \frac{\cos(x) + 1}{\cos(x) + 1} \\ &= \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} \cdot \frac{x^2}{-\sin^2(x)} \cdot (\cos(x) + 1) \\ &= \left(\lim_{x \rightarrow 0} \frac{\log(1+x) - \log(1)}{x} \right) \cdot \lim_{x \rightarrow 0} \frac{x^2}{-\sin^2(x)} \cdot (\cos(x) + 1) \\ &= \left(\lim_{x \rightarrow 0} \log(1+x)' \right) \cdot (-1) \cdot 2 = -2 \end{aligned}$$

18. Check if the following series are convergent

- (a) $\sum_{n=0}^{+\infty} \frac{n}{(n+1)!}$
- (b) $\sum_{n=0}^{+\infty} \frac{3n+1}{n^2(n+1)^2}$

Solution:

(a) We have $0 \leq \frac{n}{(n+1)!} \leq \frac{1}{n^2}$, so the series converges by the comparison test.

(b) We have $\frac{3n+1}{n^2(n+1)^2} = \frac{1}{n^3} \frac{3n+1}{n+2+\frac{1}{n}}$. The sequence $\frac{3n+1}{n+2+\frac{1}{n}}$ converges to 3, so for n big enough we have

$$0 \leq \frac{3n+1}{n^2(n+1)^2} \leq \frac{4}{n^3}$$

(We could have replaced 4 with any real number strictly bigger than 3). The series converge by the comparison criterion.

19. For what values of $t > 0$ the following series converges? If convergent, what is the limit?

$$\sum_{n=0}^{+\infty} \left(\frac{t}{t+1} \right)^{2n}$$

Solution: The series is a geometric series. Since $\left(\frac{t}{t+1} \right)^2$ is always strictly smaller than 1, the series is always convergent and the limit is

$$\frac{1}{1 - \left(\frac{t}{t+1} \right)^2}.$$