Lecturer: R. Svaldi
Analysis I - (n/a)
17 November 2021
Duration : 1 hour

n/a

n/a

SCIPER: **999999**

Do not turn the page before the start of the exam.
This document is double-sided, contains 8 pages, the last pages possibly blank.
Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- The use of **calculators** or any electronic device is **not permitted** for the entire duration of the exam.
- **Multiple choice** questions are assigned:
  +3 points for a correct answer,
  0 points if you give no answer or more than one,
  −1 points for a single incorrect answer.
- **True/False** questions are assigned:
  +1 points for a correct answer,
  0 points if you give no answer or more than one,
  −1 points for a single incorrect answer.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- Should a question contain a mistake, the teacher may decide to invalidate said question, that is, the score assigned to the question will not count towards the final grade.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien		
choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Part 1: this part of the exam contains 7 multiple choice questions.

For each question, mark the box corresponding to the answer that you wish to select as the correct one.
Each question has a unique correct answer.

Question 1 : Let $(a_n)_{n\geq 0}$ be the sequence $a_n := \dfrac{(n+3)^{1/2} - n^{1/2}}{(n+1)^{-1/2}}$. Then,

- ☐ $\lim\limits_{n\to\infty} a_n = +\infty$
- ☐ $\lim\limits_{n\to\infty} a_n = \frac{3}{2}$
- ☐ $\lim\limits_{n\to\infty} a_n = 3$
- ☐ $\lim\limits_{n\to\infty} a_n = 0$

Question 2 : Let z be the complex number $z := 1 + \sqrt{3}\mathrm{i}$. Then,

- ☐ $|z^6| > 73$
- ☐ $\mathrm{Re}(z^8) > 0$
- ☐ $z^{14} \in \mathbb{R}$
- ☐ $\mathrm{Im}(z^{11}) < 0$

Question 3 : For what values of the parameter $b \in \mathbb{R}$ does the series

$$\sum_{k=1}^{\infty} \left(b + \frac{1}{k}\right)^k$$

converge?

- ☐ $b < 1$
- ☐ $b \leq 1$
- ☐ $b \in \,]-1, 1[$
- ☐ $b \in \,]-1, 1]$

Question 4 : Consider the sets $A, B \subseteq \mathbb{C}$,

$$A := \left\{z \in \mathbb{C} \colon z^2\left(|z| - 2\right) = 0\right\}, \qquad B := \left\{z \in \mathbb{C} \colon \mathrm{Re}\,(z) = 1\right\}.$$

Then,

- ☐ $A \cap B$ is the empty set.
- ☐ $A \cap B$ consists exactly of one point.
- ☐ $A \cap B$ consists exactly of two distinct points.
- ☐ $A \cap B$ contains a line.

Question 5 : Let $(a_n)_{n\geq 1}$ be the sequence $a_n := \dfrac{(5n+1)^n}{n^n 5^n}$. Then,

- ☐ $\lim\limits_{n\to\infty} a_n = 0$
- ☐ $\lim\limits_{n\to\infty} a_n = \mathrm{e}^{\frac{1}{5}}$
- ☐ $\lim\limits_{n\to\infty} a_n = +\infty$
- ☐ $\lim\limits_{n\to\infty} a_n = 5$

Question 6 : Let $A \subset \mathbb{R}$ be the set $A := \left\{x \in \mathbb{R} \colon 0 < \arctan\left(\frac{1}{x}\right) < \frac{\pi}{4}\right\}$. Then,

- ☐ A is not bounded
- ☐ $A = \left]1, \frac{\pi}{2}\right[$
- ☐ $\mathrm{Inf}\, A = \frac{\pi}{2}$
- ☐ $A = \,]0, 1[$

Question 7 :
Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) := x^3$. We define the function $f_1 := f$ and, for any natural number $n \geq 2$, the function $f_n := f \circ f_{n-1}$. Then, for all natural numbers $m \geq 1$,

- ☐ $f_m(x) = x^{(3m)}$
- ☐ $f_m(x) = mx^3$
- ☐ $f_m(x) = x^{(3^m)}$
- ☐ $f_m(x) = (3x)^m$

Part 2: this part of the exam contains 4 true/false questions.

For each question, mark the box next to "TRUE" if you think the statement of the question is correct or mark the box next to "FALSE" if you think the statement of the question is incorrect.

Question 8 : Let $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ be two converging sequences. Assume that $b_n \neq 0$ for all $n \geq 0$. Then, the limit $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ exists.

☐ TRUE ☐ FALSE

Question 9 : Let $f\colon \mathbb{N} \to \mathbb{R}$ be a function such that for all $n \geq 1$, $f(n) > n$. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges.

☐ TRUE ☐ FALSE

Question 10 : Let $A \subset \mathbb{R}$ be a subset of the real numbers. If $\operatorname{Inf} A \in A$ and $\operatorname{Sup} A \in A$, then A is a closed interval.

☐ TRUE ☐ FALSE

Question 11 : For all given $y \in \mathbb{R}$, $y \neq 0$, the equation $z^4 = \mathrm{i}y$, in the indeterminate z, has exactly 4 distinct solutions in $\mathbb{C}$.

☐ TRUE ☐ FALSE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 12: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 *Do not write here.*

Let $\sum_{n=0}^{\infty} x_n$ be an absolutely convergent series.

(a) Prove that

$$\lim_{n\to+\infty} |x_{2020n+1} + x_{2020n+2} + x_{2020n+3} + \cdots + x_{2021n-3} + x_{2021n-2} + x_{2021n-1} + x_{2021n}| = 0.$$

(b) Consider the function $f(x) = \sin(x^5) + x^6$. Show that the series

$$\sum_{n=0}^{\infty} (-1)^n f(x_n) x_n$$

is convergent.