

(a) Let $(s_n)_{n \in \mathbb{N}}$, $s_n := \sum_{n=0}^n x_n$ be the sequence of truncated sums. Notice that, by definition of s_n ,

$$x_{2020n+1} + x_{2020n+2} + x_{2020n+3} \cdots + x_{2021n-3} + x_{2021n-2} + x_{2021n-1} + x_{2021n} = s_{2021n} - s_{2020n}.$$

As the series $\sum_{n=0}^{\infty} x_n$ is absolutely convergent, in particular, by one of the theorems proven in class, it is also convergent. Thus, there exists the limit $l \in \mathbb{R}$ of the series, $\lim_{n \rightarrow \infty} s_n = l$.

We want to show that

$$\lim_{n \rightarrow \infty} |s_{2021n} - s_{2020n}| = 0.$$

But we have seen in the course that

$$\lim_{n \rightarrow \infty} |s_{2021n} - s_{2020n}| = 0$$

if and only if

$$\lim_{n \rightarrow \infty} s_{2021n} - s_{2020n} = 0.$$

On the other hand the sequences $(s_{2021n})_{n \in \mathbb{N}}$, $(s_{2020n})_{n \in \mathbb{N}}$ are subsequences of $(s_n)_{n \in \mathbb{N}}$. As $(s_n)_{n \in \mathbb{N}}$ converges to l , the same must hold for $(s_{2021n})_{n \in \mathbb{N}}$, $(s_{2020n})_{n \in \mathbb{N}}$. Hence, by the addition rules for finite limits,

$$\lim_{n \rightarrow \infty} s_{2021n} - s_{2020n} = \lim_{n \rightarrow \infty} s_{2021n} - \lim_{n \rightarrow \infty} s_{2020n} = l - l = 0,$$

which completes our proof.

Alternative proof.

Let $(s_n)_{n \in \mathbb{N}}$, $s_n := \sum_{n=0}^n x_n$ be the sequence of truncated sums. Notice that, by definition of s_n ,

$$x_{2020n+1} + x_{2020n+2} + x_{2020n+3} \cdots + x_{2021n-3} + x_{2021n-2} + x_{2021n-1} + x_{2021n} = s_{2021n} - s_{2020n}.$$

We want to show that

$$\lim_{n \rightarrow \infty} |s_{2021n} - s_{2020n}| = 0.$$

Hence, to complete the proof, we need to show that for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for every $n \geq n_\epsilon$, $|s_{2021n} - s_{2020n}| \leq \epsilon$. Let us fix $\epsilon > 0$.

As the series $\sum_{n=0}^{\infty} x_n$ is absolutely convergent, in particular, by one of the theorems proven in class, it is also convergent. Thus, there exists the limit $l \in \mathbb{R}$ of the series, $\lim_{n \rightarrow \infty} s_n = l$. Another theorem proved in class shows that if (s_n) converges, then it is a Cauchy sequence.

As (s_n) is Cauchy, there exists $N' \in \mathbb{N}$ such that, if $m, m' \geq N'$, then $|s_m - s_{m'}| \leq \epsilon$. Hence,

Then, to conclude, it suffices to have $2020n \geq N'$. So, we may take $n_\epsilon = \lfloor \frac{N'}{2020} \rfloor$

- (b) As the series $\sum_{n=0}^{\infty} x_n$ is convergent, then, one of the theorems from the lectures implies that the sequence (x_n) converges to 0. Thus, the sequence $(y_n), y_n := x_n^6$ converges to 0 as well, since, by the theorems on algebraic operations and finite limits,

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} x_n^6 = \left(\lim_{n \rightarrow \infty} x_n \right)^6 = 0^6 = 0.$$

In particular, the sequence (y_n) is bounded. Similarly, $-1 \leq \sin(x_n^5) \leq 1$ for all n . As the sum of two bounded quantities is bounded, we have that $f(x_n)$ is bounded, that is, there exists a positive real number $M > 0$ such that $|f(x_n)| \leq M$ for all n .

Now, the above observations imply that for all $n \in \mathbb{N}$, $|(-1)^n f(x_n) x_n| \leq M |x_n|$. Now, since $\sum_{n=0}^{\infty} x_n$ converges absolutely, the series $\sum_{n=0}^{\infty} |x_n|$ converges. Similarly, as M is a constant, also $\sum_{n=0}^{\infty} M |x_n|$ converges. Then, we conclude by the series version of the Squeeze theorem.