MATH 101 (en)– Analysis I (English)
Notes for the course given in Fall 2021

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1 Proofs

The means to explore analysis from a mathematical viewpoint within this course will be mathematical proofs. Part of the goal of the course will be for you to learn how to prove mathematical statements via mathematical proofs.

There are two main types of proof that we will encounter:

- **Constructive proof**: an argument in which, starting from certain hypotheses/assumptions, one tries to explicitly construct a mathematical object or to explicitly show that a certain mathematical property hold for a mathematical object;
- **Proof by contradiction**: an argument in which we assume that the conclusion that we are trying to reach does not hold and we show that such assumption, together with our starting hypotheses leads to a contradiction.

You have probably already encountered many constructive proofs; on the other hand, the reader may be encountering proofs by contradiction for the first time. So, let us start by giving a classical example of proof by contradiction.

Before we explain our first example, let us recall that the set of rational numbers is the set of numbers of the form $\frac{a}{b}$, with a, b integers, $b \neq 0$, where the following identification between different fractions holds: for any non-zero integer c,

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}.$$

We shall start by showing a classical argument by contradiction. For the time being we shall assume that we know how to construct the real numbers, and that we know that $\sqrt{3}$, that is, the positive solution to the equation $X^2 - 3 = 0$, is a real number. For a more detailed discussion about the real numbers, we refer the reader to Section 2.

Proposition 1.1. *The real number $\sqrt{3}$ is not a rational number.*

We are going to use a proof by contradiction; that is, we are going to assume that $\sqrt{3}$ is rational and we are going to derive, by means of logical implications, a contradiction to some other fact that we already know or to some other fact that is implied by the assumed rationality of $\sqrt{3}$.

Let us recall here that a natural number p is *prime* if and only if the only natural numbers that divide p are 1 and p itself.

Exercise 1.2. Prove that the following two properties for a natural number p are equivalent:

- p is prime;
- if a, b are natural numbers and p divides ab, then either p divides a or p divides b.

Proof of Proposition 1.1. Assume that $\sqrt{3}$ is rational. Thus, we may write

$$\sqrt{3} = \frac{a}{b} \tag{1.2.a}$$

for some integers a and $b \neq 0$. As $\sqrt{3} > 0$, a and b should have the same sign. If they are both negative, by multiplying both by -1 we may assume that they are positive. Hence, we will assume that a, b are both positive integers.
Furthermore, by dividing both a, b by their greatest commond divisor $\gcd(a, b)$[1], we may assume

[1] Let us recall here the Fundamental Theorem of Arithmetic: any natural number n can be written uniquely as a product of powers of the prime numbers: namely, $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \cdots \cdot p_n^{k_n}$, where $p_1, \ldots, p_k$ are distinct prime numbers and $k_1, \ldots, k_n$ are natural numbers > 0. For example, $36 = 4 \cdot 9 = 2^2 \cdot 3^2$. In view of this, given two natural numbers a, b, then $\gcd(a, b)$ is defined by writing it as a product $\gcd(a, b) = q_1^{j_1} \cdot q_2^{j_2} \cdot \ldots q_n^{j_n}$ where the q_i are primes that divide both a and b and j_i is the maximal natural number such that $q_i^{j_1}$ divides both a and b.

that a and b are relatively prime, that is, they do not share any prime factors. Multiplying both sides of (1.2.a) by b, then, since $b \neq 0$,

$$b\sqrt{3} = a. \tag{1.2.b}$$

Squaring both sides of (1.2.b) yields

$$b^2 \cdot 3 = a^2. \tag{1.2.c}$$

Hence, as 3 divides the left hand side of (1.2.c), 3 must divide the right hand side, too. Thus,

$$a = 3r. \tag{1.2.d}$$

Substituting the relation (1.2.d) into equation (1.2.c), we obtain that

$$b^2 \cdot 3 = (3r)^2 = 9r^2$$

Hence, $b^2 = 3r^2$, which implies that $3|(b^2)$. We write $x|y$, with x, y integers to mean that x divides y. Again, as 3 is prime, then, since $3|b^2$,

$$3|b, \tag{1.2.e}$$

But, (1.2.d)-(1.2.e) together contradict the relatively prime assumption on a and b. Thus, we obtained a contradiction with our original assumption, so that $\sqrt{3}$ is not a rational number. $\square$

Remark 1.3. The proof of Proposition 1.1 is a nice example of a proof by contradiction. On the other hand, it does not tell us much about the nature of $\sqrt{3}$.
What is $\sqrt{3}$? Is it a real number? How can we define real numbers? What notable properties do those have? We will get back to these questions in Section 2.2-2.4.

We can generalize the above proof to any prime number $p \in \mathbb{N}$.

Exercise 1.4. Imitate the proof of Proposition 1.1, to show that for every prime number $p \in \mathbb{N}$, $\sqrt{p}$ is not rational.

In particular, Exercise 1.4 implies that also $\sqrt{2} \notin \mathbb{Q}$.

As easy as it may seem at a first glance to find and write mathematical proofs, one ought to be extremely careful: it is indeed very easy to write wrong proofs! This is often do to that the fact that one may assume something wrong in the course of a proof: if the premise of an implication is false, then anything can follow from it.

Example 1.5. Here is an example of an (incorrect) proof showing that 1 is the largest natural number, a fact that is clearly false, since $2 > 1$ and $2 \in \mathbb{N}$.

Claim. 1 is the largest integer.

WRONG PROOF. Let l be the largest integer.
Then $l \geq l^2$, so that $l - l^2 = l(1 - l) \geq 0$. Hence, there are two possibilities for $l(1 - l) \geq 0$:

a) either $l < 0$ and $1 - l \leq 0$; or,

b) $l \geq 0$ and $1 - l \geq 0$.

As 0 is an integer, we must be in case b), so that $l \geq 0$ and $l \leq 1$. Hence $l = 1$. $\square$

This claim cannot possibly be true: in fact, 2 is definitely an integer and $2 > 1$. Even better, the set of integeral numbers is not *bounded from above*[2], that is, there is no real number C such that $z \leq C$ for all $z \in \mathbb{Z}$.
What went wrong in the above proof? All the algebraic manipulations that we made following the first line of the proof appear to be correct. [Go back and check that!!] Thus, the issue must be contained in the (absurd) assumption we made in the first sentence:

Let l be the largest integer.

In fact, as we just explained, there cannot be a largest element in the set of integers: in fact, given an integer l, then $l+1$ is also an integer and $l+1 > l$, which clearly shows that the above assumption was unreasonable.

Analysis is mostly focused on the study of real and complex numbers[3] and their properties. Even more generally, analysis is concerned with studying (or analyzing, hence the name Analysis) functions defined over the real (alternatively, over the complex numbers) with values in the real numbers (alternatively, over the complex numbers) and their important properties[4]. In order to carry out such analysis, we will often need to deal with infinity. Roughly speaking, we will often be interested in understanding numbers/functions from the point of view of an infinitely small or at an infinitely large viewpoint. Our main goal will be to provide a framework to be able to treat in a formal mathematical way all the different aspects of infinity in the realm of real/complex numbers. To make a slightly better sense of this statement, you may try to think (and formalize) of how to define the speed of a particle moving linearly on a rod, at a given time t.

How should we define the real numbers? Even more importantly, how can we represent them numerically? Intuitively, we have been taught that real numbers are those numbers that we can represent numerically by writing down a decimal expansion, for example,

$$\begin{aligned}\sqrt{2} =&1.4142135623730950488016887242096980785696718753769480731766797379907\\&324784621070388503875343276415727350138462309122970249248360\dots\ .\end{aligned}$$

As it suggested from this example, it may be the case that when we try to represent certain real numbers, we have to account for an infinite decimal part[5] of the expansion, that is, there is an infinite sequence of digits to the right of the decimal dot ".". Hence, we may at first tempted to adopt the following definition of the set of real numbers:
The real numbers are all those numbers that we can represent with a decimal expansion whose integral part (the digits to the left of ".") can be written using a finite number of digits (chosen in the set $\{0, 1, 2, \dots, 9\}$*), whereas its decimal part (the digits to the right of ".") is any infinite sequence of digits (as above, chosen in the set* $\{0, 1, 2, \dots, 9\}$*).* While this may seem, at first, as an intuitively fine definition for the real numbers, it actually hides some subtleties.

Here we illustrate one of the main subtleties of this definition: namely, we show that, in the above definition, we certainly have to be careful. We show that there is non unique correspondence between a real number and its decimal expansion. An example is given by the following proposition, which also provides a great basic example of how we deal with infinity in Analysis.

[2]We will give a formal definition of what being bounded from above means later, cf. Definition 2.8.

[3]See Section 3 for the definition and basic properties of complex numbers.

[4]Some of the most important classes of functions that we will encounter are those of continuous, differentiable, integrable, analytic functions, but there are many more other possible classes of functions that are heavily studied in analysis

[5]The decimal part of the expansion is that part of the expansion that lays on the right hand side of the point ".". For example, the decimal part of the expansion of 41369.57693 is the sequence 57693. The integral part of the decimal expansion is instead that part of the expansion that lays on the left hand side of the point ".". The integral part of 41369.57693 is 41369. The integral part always has finite length, that is, it can be written using a finite number of digits.

Proposition 1.6. $0.\bar{9} = 1$

By $0.\bar{9}$ we denote the real number whose decimal representation is given by an infinite sequence of 9 in the decimal part, $0.999999\ldots$.

Proof. We give two proofs none of which is completely correct, at least as far as our current definition and knowledge of the real numbers go. Nevertheless, we carefully explain what the issues are in each case; we also explain how these issues will be clarified and taken care of during this course.

(1) First an elementary proof:

$$9 \cdot 0.\bar{9} = (10-1) \cdot 0.\bar{9} = 10 \cdot 0.\bar{9} - 1 \cdot 0.\bar{9} = 9.\bar{9} - 0.\bar{9} = 9$$

So, $0.\bar{9}$ is a solution of the equation $9X - 9 = 0$; the only solution to this equation is clearly $X = 1$, thus, $0.\bar{9} = 1$.

At first sight, this proof is definitely a reasonable one from the point of view of the algebraic manipulations that we carried out. However, we assumed that we know what $0.\bar{9}$ is. Moreover, we also assumed that we can algebraically manipulate $0.\bar{9}$ as usual, despite the fact that it has an infinite decimal expansion. None of these facts are that clear if you think about it, as we have not really defined what the properties of numbers like $0.\bar{9}$ are.

So, what kind of number is $0.\bar{9}$? What are its properties? For example, what algebraic manipulations are we allowed to make with it?

(2) Analysis provides us with a precise definition of $0.\bar{9}$

$$0.\bar{9} := \sum_{i=1}^{\infty} \frac{9}{10^i}.$$

On the hand, what kind of mathematical object is $\sum_{i=1}^{\infty} \frac{9}{10^i}$? This is a series and we will study series in detail in Section 4. By definition,

$$\sum_{i=1}^{\infty} \frac{9}{10^i} := \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{9}{10^i} \right).$$

We have yet to learn a precise definition of lim, thus, we cannot quite continue in a precise way from here, nevertheless we continue the argument for completeness. If you are not comfortable with it now, it is completely OK, just skip this part of the proof.
However, before we proceed, we need to show an identity for the sum of elements in a geometric series[6].

Claim. Let $a \in \mathbb{R}$, $a \neq 1$. Then,

$$a + a^2 + \cdots + a^n = \frac{a - a^{n+1}}{1-a}. \tag{1.6.f}$$

Proof of the Claim. To prove this equality, we just multiply the left side by $1 - a$ to obtain:

$$\begin{aligned}(a + a^2 + \cdots + a^n)(1-a) &= a - a \cdot a + a^2 - a^2 \cdot a + a^3 - \ldots \\ &\quad - a^{n-1} \cdot a + a^n - a^n \cdot a = a - a^{n+1}\end{aligned}$$

This shows that (1.6.f) indeed holds, since to obtain the form of the equation in the statement of the claim , it suffieces to . □

[6]A geometric series is a series whose elements are of the form ca^q, for $c, a \in \mathbb{R}$ and $q \in \mathbb{N}$. This will be explicitly defined when we introduce series, later; hence, do not worry about this definition for now.

And then we can proceed showing the statement:

$$\sum_{i=1}^{\infty} \frac{9}{10^i} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i} = 9 \cdot \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{10^i} \right) =$$
$$9 \cdot \lim_{n \to \infty} \left(\frac{\frac{1}{10} - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} \right) = 9 \cdot \frac{\frac{1}{10} - \lim_{n \to \infty} \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} =$$
$$9 \cdot \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 9 \frac{1}{9} = 1.$$

$\square$

In Section 2 and in the following one, we will introduce all the necessary tools, definitions, notations and conventions to answer all of the questions that were raised in these first few pages.

2 Basic notions

2.1 Sets

A *set* S is a collection of objects called elements. If a is an element of S, we say that a belongs to S or that S contains a, and we write $a \in S$. If an element a is not in S, we then write $a \notin S$. If the elements $a, b, c, d, \dots$ form the set S, we write $S = \{a, b, c, d, \dots\}$. We can also define a set simply by specifying that its elements are given by some condition, and we write

$$S := \{s \mid s \text{ satisfies some condition}\}.$$

Notation 2.1. The symbol := indicates that we are identifying the object on the LHS (left hand side) of ":=" with the object on the RHS (right hand side) of ":=". You can read it as "defined as".

Example 2.2. The set $S = \{0, 1, 2, 3, 4, 5\}$ of natural numbers that are at most 5 can be defined as follows

$$S := \{n \mid n \text{ is a natural number and } n \leq 5\}.$$

A set T is said to be a *subset* of a set S if any element of T is also an element of S. If T is a subset of S, we denote it by writing $T \subseteq S$. Given a set S, one can always define a subset $T \subset S$, $T := \{s \in S | \text{" condition"}\}$, that is, S' is the set formed by those elements of S that satisfy the given condition.

Example 2.3. The subset $2\mathbb{N}$ of $\mathbb{N}$ of even natural numbers can be defined as

$$2\mathbb{N} := \{n \in \mathbb{N} \mid 2 \text{ divides } n\}.$$

If $T \subseteq S$, it may happen that there are elements of S which are not contained in T. In this case we say that T is a *strict subset* of S, or that T is *strictly included/contained* in S. When we want to stress that we know that a subset T of a set S is strictly included in S we shall write $T \subsetneq S$.

Example 2.4. $2\mathbb{N} \subsetneq \mathbb{N}$ since $1 \notin 2\mathbb{N}$.

If we just write $T \subseteq S$, we mean that T is a subset of S that may be equal to S, but we are not making any particular statement about whether or not T is a strict subset of S. Hence, in the previous Example 2.4, we may have also used the notation $2\mathbb{N} \subseteq \mathbb{N}$ and that would have been correct. To write that a set T is <u>not</u> a subset of a set S, we write $T \not\subseteq S$.

We will consider the standard operations between sets, such as intersection, union, taking the complent. More precisely, given two subsets U, V, we define:

$$\begin{aligned} \textbf{Intersection:} &\quad U \cap V := \{x \mid x \in U \text{ and } x \in V\}; \\ \textbf{Union:} &\quad U \cup V := \{x \mid x \in U \text{ or } x \in V\}; \\ \textbf{Complement:} &\quad U \setminus V := \{x \mid x \in U \text{ and } x \notin V\}. \end{aligned}$$

Exercise 2.5. Given sets E, F and D prove that the following relations hold:

Commutativity: $E \cap F = F \cap E$ and $E \cup F = F \cup E$;

Associativity: $D \cap (E \cap F) = (D \cap E) \cap F$ and $D \cup (E \cup F) = (D \cup E) \cup F$;

Distributivity: $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$ and $D \cup (E \cap F) = (D \cup E) \cap (D \cup F)$;

De Morgan laws: $(E \cap F)^c = E^c \cup F^c$ and$(E \cup F)^c = E^c \cap F^c$.

2.2 Number sets

There are a few important sets that we are going to work with all along this course:

(1) $\emptyset$: the empty set; it is the set which has no elements, $\emptyset := \{\ \}$.

Exercise 2.6. Show that for any set S, $\emptyset \subseteq S$.

(2) $\mathbb{N}$: the set of natural numbers, $\mathbb{N} := \{0, 1, 2, 3, 4, 5, 6, \dots\}$.
$\mathbb{N}$ is well ordered, that is, all its subsets contain a smallest element. We will prove that later in Proposition 2.34.

(3) $\mathbb{Z}$: the set of integral numbers[7], $\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$.

(4) $\mathbb{Q}$: the set of rational numbers, $\mathbb{Q} := \{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \setminus \{0\}\}$, where we impose the following identification between fractions

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}, \quad \text{for } c \in \mathbb{Z} \setminus \{0\}.$$

(5) $\mathbb{R}$: the set of real numbers. It is not easy to actually construct it and there are some subtleties in trying to define real numbers by means of their decimal representation, as we have already understood from Proposition 1.6.

Remark 2.7. In this course, we will not attempt to provide a rigorous construction of the set of real numbers $\mathbb{R}$, although there are many equivalent constructions. If you are curious, you can click here to find out more about these constructions. Instead of going through the construction of $\mathbb{R}$ in the course, we proceed to list here certain properties that uniquely define $\mathbb{R}$ [we also do not prove such uniqueness, but, please, believe it] and we will assume them going forward:

(1) $\mathbb{Q} \subseteq \mathbb{R}$;

(2) $\mathbb{R}$ is an *ordered field*

- the word *field* refers to the fact that addition, substraction, multiplication are all well-defined operation within $\mathbb{R}$; moreover, these operations respect commutativity, associativity and distributivity properties and for all $x \in \mathbb{R}$, $x \neq 0$ it is possible to defined a multiplicative inverse x^{-1} such that $x \cdot x^{-1} = 1$;
- the world *ordered* refers to the fact that given two elements $x, y \in \mathbb{R}$ we can always decide whether $x < y$, or $x > y$, or $x = y$; moreover, this comparison is also compatible with the operations that make $\mathbb{R}$ into a field.

(3) $\mathbb{R}$ satisfies the Infimum Axiom 2.22, that will be introduced in next section.

The following inclusions hold among the sets just defined:

$$\emptyset \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}.$$

To justify these inclusions:

- $\emptyset \subsetneq \mathbb{N}$: $\mathbb{N}$ is non-empty. For example, $0 \in \mathbb{N}$.
- $\mathbb{N} \subsetneq \mathbb{Z}$: an integral number can also be negative, for example, $-1 \in \mathbb{Z}$, while natural number are always non-negative; thus $\mathbb{Z} \ni -1 \notin \mathbb{N}$.
- $\mathbb{Z} \subsetneq \mathbb{Q}$: $\frac{1}{2} \in \mathbb{Q}$, but $\frac{1}{2} \notin \mathbb{Z}$.
- $\mathbb{Q} \subsetneq \mathbb{R}$: we saw in Proposition 2.38 that $\sqrt{3} \notin \mathbb{Q}$; we will prove formally in Section 2.4.1 that $\sqrt{3} \in \mathbb{R}$.

[7] We will often call an integral number an "integer".

2.2.1 Half lines, intervals, balls

We introduce here further notation regarding the real numbers and some special classes of subsets that we will be using all throughout the course.

(1) Invertible real numbers: $\mathbb{R}^* := \{x \in \mathbb{R} \mid x \neq 0\}$.

(2) Closed half lines: $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}_- := \{x \in \mathbb{R} \mid x \leq 0\}$.
At times, these are also denoted by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$, respectively.

(3) Open half lines: $\mathbb{R}^*_+ := \{x \in \mathbb{R} \mid x > 0\}$, $\mathbb{R}^*_- := \{x \in \mathbb{R} \mid x < 0\}$.
At times, these are also denoted by $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$, respectively.

We use the analogous definitions also for the sets

$$\begin{gathered}\mathbb{N}^*, \mathbb{Z}^*, \mathbb{Q}^*,\\ \mathbb{N}_+, \mathbb{Q}_+, \mathbb{Z}_+,\\ \mathbb{N}_-, \mathbb{Q}_-, \mathbb{Z}_-,\\ \mathbb{N}^*_+, \mathbb{Q}^*_+, \mathbb{Z}^*_+,\\ \mathbb{N}^*_-, \mathbb{Q}^*_-, \mathbb{Z}^*_-.\end{gathered}$$

(4) Bounded intervals: if $a < b$ are real numbers, we define

$$\begin{aligned}\text{Open bounded interval: } &]a,b[:= \{x \in \mathbb{R} \mid a < x < b\}.\\ \text{Closed bounded interval: } & [a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}.\\ \text{Half-open bounded interval: } & \begin{cases}]a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}.\\ [a,b[:= \{x \in \mathbb{R} \mid a \leq x < b\}.\end{cases}\end{aligned}$$

If $a = b$, then $[a,b] = [a,a] = \{a\}$. When we say that a subset I is a bounded interval of $\mathbb{R}$ of extreme $a < b$, we mean that I may be either one of

$$[a,b],\ [a,b[,\ ]a,b],\ ]a,b[.$$

(5) Open balls: let $a, \delta \in \mathbb{R}$, $\delta > 0$; we define the *open ball* $B(a,\delta) \subseteq \mathbb{R}$ of radius δ and center a as

$$B(a,\delta) :=]a-\delta, a+\delta[.$$

(6) Closed balls: let $a, \delta \in \mathbb{R}$, $\delta \geq 0$; we define the *closed ball* $\overline{B(a,\delta)} \subseteq \mathbb{R}$ of radius δ and center a as

$$\overline{B(a,\delta)} := [a-\delta, a+\delta].$$

When $\delta = 0$, then $B(a,0) = \{a\}$.

2.2.2 Extended real numbers

The extended real line is the set

$$\overline{\mathbb{R}} := \{-\infty, +\infty\} \cup \mathbb{R}.$$

The symbol $+\infty$ (resp. $-\infty$) is called "plus infinity" (resp. "minus infinity"). In this course $\pm\infty$ *shall not be treated as numbers*: they are just symbols indicating two elements of the extended real line $\bar{\mathbb{R}}$. That means that we will not try to make sense of algebraic operations involving $\pm\infty$; thus, be very careful not to treat those as numbers. If you think carefully a bit, you can see that it is hard to coherently define for example the result of the addition

$$+\infty + (-\infty).$$

Later in the course we will use extensively these symbols. For the time being, we just want to use them to define the following subsets of $\mathbb{R}$. Let $a \in \mathbb{R}$, then

$$\begin{aligned} \text{Open unbounded intervals:} &\quad]a, +\infty[:= \{x \in \mathbb{R} | x > a\},\ ]-\infty, a[:= \{x \in \mathbb{R} | x < a\}. \\ \text{Closed unbounded intervals:} &\quad [a, +\infty[:= \{x \in \mathbb{R} | x \geq a\},\ ]-\infty, a] := \{x \in \mathbb{R} | x \leq a\}. \end{aligned}$$

Finally

$$]-\infty, +\infty[:= \mathbb{R}.$$

These sets are also called open/closed half lines, or open/closed unbounded intervals, or open/closed extended intervals, where open/closed is determined by whether or not a belongs to the set.

So, from now on, when we say that a subset I of $\mathbb{R}$ is an interval, we will mean that I has one of the following forms:

- $[a, b],\ ]a, b[,\ ]a, b],\ [a, b[, \quad a, b \in \mathbb{R}, \quad a < b$;
- $[a, +\infty[,\ ]a, +\infty[,\ ]-\infty, a],\ ]-\infty, a[, \quad a \in \mathbb{R}$;
- $]-\infty, +\infty[= \mathbb{R}$.

2.3 Bounds

We now start entering the realm of modern (and rigorous) analysis.
We start by defining some important properties of subset of $\mathbb{R}$.

2.3.1 Basic definitions, properties, and results.

Definition 2.8. Let $S \subseteq$ be a non-empty subset of $\mathbb{R}$.

(1) A real number $a \in \mathbb{R}$ is an *upper* (resp. *lower*) bound for S if $s \leq a$ (resp. $s \geq a$) holds for all $s \in S$.

(2) If S has an upper (resp. a lower) bound then S is said to be *bounded from above* (resp. *bounded from below*).

(3) The set S is said to be *bounded* if it is bounded both from above and below.

For a set $S \subseteq \mathbb{R}$ in general upper and lower bounds are not unique.

Example 2.9. (1) The set $\mathbb{N} \subset \mathbb{R}$ is bounded from below, since $\forall n \in \mathbb{N}$, $n \geq 0$; in particular, 0 is a lower bound. In fact, any negative real number is also a lower bound for $\mathbb{N}$.
On the other hand, $\mathbb{N}$ is not bounded. While this fact may appear intuitively clear, it is not immediately clear how to prove it formally. Can you find a proof using only the concepts and tools that we have introduced so far in the course? The answer is no, at this time of the course. For a formal proof of the unboundedness of $\mathbb{N}$, we shall need Archimedes' property for $\mathbb{R}$, see Proposition 2.30.

(2) $\mathbb{Z}$ is neither bounded from above nor from below. In fact, it cannot be bounded from above since $\mathbb{N} \subseteq \mathbb{Z}$. It is also not bounded from below: if a lower bound $l \in \mathbb{R}$ existed for $\mathbb{Z}$, then $-l$ would be an upper bound for $\mathbb{N}$, which we saw above does not hold. [Prove this assertion in detail!].

(3) The set $S := \{n^2 | n \in \mathbb{Z}\}$ is bounded from below: in fact, $\forall n \in \mathbb{N}$, $n^2 \geq 0$, thus 0 is a lower bound. On the other hand, it is not bounded. In fact, assume for the sake of contradiction that S were bounded from above, i.e., that there exists $u \in \mathbb{R}$ and $u \geq s$, $\forall s \in S$. Since for any $n \in \mathbb{N}$, $n^2 \geq n$, then it would follow that $u > n$, for all $n \in \mathbb{N}$, but this contradicts part (1).

(4) The set $S := \{n^3 | n \in \mathbb{Z}\}$ is neither bounded from above nor from below. [Prove it! The proof is similar to that in part (2).]

(5) The set $S := \{\sin(n^2) | n \in \mathbb{Z}\}$ is bounded since for all $x \in \mathbb{R}$, $-1 \leq \sin x \leq 1$. Examples of possible lower bounds are -5 and -13; example of possible upper bounds are 1 and 27. As $\sin x \in [-1, 1]$, then it is certainly true that

- any real number y such that $y \geq 1$ is an upper bound for S, while
- any real number y such that $y \leq -1$ is a lower bound for S.

(6) Let $S := [3, 5[= \{x \in \mathbb{R} \mid 3 \leq x < 5\}$. Then, 5 is an upper bound for S since for any element x of S, $x < 5$. Moreover, if c is a real number and $c > 5$, then c is also an upper bound for S, since $c > 5 > x$ for all $x \in S$.
The same reasoning shows that 3 is a lower bound for S and that for any real number d such that $d < 3$, then d is a lower bound for S as well.
(It is left to you to prove that in this example you will obtain the exact same conclusions if instead of considering the interval $[3, 5[$, you considered any of the intervals $[3, 5],]3, 5],]3, 5[$.)

Using the discussion of the above examples, we summarize here some of the main properties of upper and lower bounds.

Proposition 2.10. *Let $S \subset \mathbb{R}$ be a non-empty set. Let $c \in \mathbb{R}$.*

(1) If u is an upper bound for S, then for any $d \geq u$, d is also an upper bound for S.

(2) If l is a lower bound for S, then for any $e \leq l$, e is also a lower bound for S.

(3) If $T \subseteq S$ is a non-empty subset and c is a lower (resp. an upper) bound for S, then c is also a lower (resp. an upper) bound for T.

(4) If $T \subseteq S$ is a non-empty subset and T is not bounded from above (resp. from below), then also S is is not bounded from above (resp. from below).

(5) If S is a bounded interval of extremes $a < b$, then the set of lower bounds (resp. of upper bounds) of S is given by

$$]-\infty, a] \quad (\textit{resp. } [b, +\infty]).$$

(6) If $S := [b, +\infty[$ or $S :=]b, +\infty[$, $b \in \mathbb{R}$, then the set of lower bounds of S is given by $]-\infty, b]$.

(7) If $S :=]-\infty, a]$ or $S :=]-\infty, a[$, $a \in \mathbb{R}$, then the set of upper bounds of S is given by $[a, +\infty]$.

Proof. (1) Let u be an upper bound for S. Then $\forall s \in S$, $u \geq s$. If $d \geq u$, then $\forall s \in S$, $d \geq u \geq s$, in particular, $d \geq s$, which shows the desired property.

(2) Analogous to (1) and left as an exercise (see the sheet from Week 2).

(3) If c is a lower bound for S, then $c \leq s$ for all element $s \in S$. Since $T \subseteq S$, this means that any element $t \in T$ is also an element of S. Hence, a fortiori, the inequality $c \leq s$, $\forall s \in S$ implies also that $c \leq t$, $\forall t \in T$.
The case of an upper bound is analogous, it suffices to change the verse of the inequalities.

(4) Since T is not bounded from above, this means that $\forall u \in \mathbb{R}$, there exists an element $x_u \in T$ (which will depend in general from the real number u we fix) such that $x_u > u$. As $T \subseteq S$, then $x_u \in S$, hence $\forall u \in \mathbb{R}$, there exists an element $x_u \in S$ such that $x_u > u$ and u cannot be an upper bound for S. As this holds $\forall u \in \mathbb{R}$, then also S is not bounded from above.
The case of T not bounded from below is analogous, it suffices to change the verse of the inequalities.

(5) Let us assume that $S :=]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$. The other cases are similar – it is left to you to prove that in you will obtain the exact same conclusions if instead of considering the interval $]a, b]$, you considered any of the intervals $[a, b], [a, b[,]a, b]$.
Then, a is a lower bound for S, since for all $s \in S$, $a < s$. Also for any real number $d < a$, d is also a lower bound for S, since $d < a < s$, for all $s \in S$. Similarly, b is an upper bound for S, since $\forall s \in S$, $s \leq b$, by definition. Thus, for any real number $c > b$, then $c > b \geq s$, $\forall x \in S$ and c is an upper bound for S. Then, part (1) implies that any element of the half line $[b, +\infty[$ (resp. $]-\infty, a]$) is an upper bound (resp. lower bound) for S. To conclude we need to show that no real number $c > a$ (resp. $d < b$) is a lower bound (resp. an upper bound) of S. To show this, it suffices to show that there exists an element $m \in S$ such that $m < c$. Since $c > a$, then $a < a + \frac{c-a}{2} < c$. If $a + \frac{c-a}{2} \in S$, it suffices to take $m := a + \frac{c-a}{2}$. If $a + \frac{c-a}{2} \notin S$, then $a + \frac{c-a}{2} > b$ then $c > b$, and it suffices to take $m := b$.

(6) Analogous to the proof of (5).

□

We have just seen that upper/lower bounds of a set S are never unique, when some exist. Moreover, if S is an interval of extremes $a < b$, then a is a lower bound and b is an upper bound. We may be tempted to ask whether in general there exists upper lower bounds of a set $S \subseteq \mathbb{R}$ that are element of S itself and what we can say in that case. In general, this is not always true but nonetheless upper/lower bounds of S that are in S are very special elements of S.

Definition 2.11. Let $S \subseteq \mathbb{R}$ be a non-empty set.

(1) The maximum of S is a real number $M \in S$ which is also an upper bound for S.

(2) The minimum of S is a real number $m \in S$ which is also a lower bound for S.

In Definition 2.11, we used the determinative article "the" to intriduce maximum and minimum of a set of real numbers. This suggests that they should both be uniquely determined. This is indeed the content of the next exercise.

Proposition 2.12. *Let S be a non-empty subset of $\mathbb{R}$. If $\max S$ (resp. $\min S$) exists, then it is unique.*

Notation 2.13. For $S \subseteq \mathbb{R}$, we denote the maximum (resp. the minimum) of S by $\max S$ (resp. $\min S$).

Proof. Suppose, for the sake of contradiction, that a maximum of S exists and it is not unique. Then there are at least two distinct numbers $n, n' \in \mathbb{R}$ which are both a maximum for S. As

n, n' are distinct, i.e., $n \neq n'$, we can assume that $n < n'$. As n' is a maximum, then $n' \in S$. But as n is also a maximum, in particular, n is also an upper bound, i.e., $n \geq s$, $\forall s \in S$; hence, also $n \geq n'$, which is in contradiction with our assumption above that $n' > n$.
You can apply a similar argument for the uniqueness of the minimum. $\square$

Example 2.14. (1) Let us define $S :=]1, 2[= \{x \in \mathbb{R} \mid 1 < x < 2\}$. Then S does not have minimum or maximum.
In fact, if $u \in \mathbb{R}$ is an upper bound for S, then, by definition, $u \geq x$, $\forall x \in]1, 2[$, which implies that $u \geq 2$. Hence $u \notin]1, 2[$.
Analogously, if $l \in \mathbb{R}$ is a lower bound for S, then, by definition, $l \leq x$, $\forall x \in]1, 2[$, which implies that $l \leq 1$. Hence $l \notin]1, 2[$.

(2) $S := [1, 2]$ has both a minimum and a maximum.
$\min S = 1$, since $1 \in S$ and $1 \leq s$, $\forall s \in S$, so that 1 is also a lower bound for S.
$\max S = 2$, since $2 \in S$ and $2 \geq s$, $\forall s \in S$, so that 2 is also an upper bound for S.

(3) Let $a < b$ be real numbers. $S :=]a, b]$ has maximum but no minimum.
$\max S = b$, since $b \in S$ and $b \geq s$, $\forall s \in S$, so that b is also an upper bound for S.
$\min S$, since any lower bound for S is $\leq a$, hence there is no lower bound that is contained in S.

The above examples suggest that it should not be hard to understand when an interval S admits a maximum or a minimum. Indeed, the following characterization is an immediate consequence of Definition 2.11 and of Proposition 2.10

Proposition 2.15. *Let $S \subseteq \mathbb{R}$ be a bounded interval of extremes $a < b$.*

(1) The maximum of S exists if and only if $b \in S$. In this case, $\max S = b$.

(2) The minimum of S exists if and only if $a \in S$. In this case, $\min S = b$.

When S is not an interval, it may be more complicated to understand whether a maximum/minimum exists.

Example 2.16. (1) Take $S := \left\{\frac{n-1}{n} \mid n \in \mathbb{Z}^*_+\right\}$. Then S has a minimum but it does not have a maximum.
Indeed, $\min S = 0$, since $0 = \frac{1-1}{1} \in S$ and $\frac{n-1}{n} \geq 0$, $\forall n \in \mathbb{Z}^*_+$, so that 0 is a lower bound which belongs to S. However, S does not have a maximum. To see this, let $l \in \mathbb{R}$, then:

(i) assume that $l < 1$. Then a natural number n satisfies $n > \frac{1}{1-l}$ if and only if $1 - \frac{1}{n} = \frac{n-1}{n} > 1 - (1 - l) = l$. then $1 - \frac{1}{n} = \frac{n-1}{n} > 1 - (1 - l) = 1$; Thus, l cannot be an upper bound for S, hence a fortiori it cannot be a maximum either.

(ii) on the other hand, if $a \geq 1$, then $l \notin S$, so no such l can be a maximum for S.

One can actually show that the upper bounds of S are exactly the real numbers ≥ 1; indeed, it is easy to show that any $l \geq 1$ is an upper bound for S, since $1 - \frac{1}{n} \leq 1 \leq a$, for all $n \in \mathbb{Z}^*_+$. On the other hand (i) above shows that no real number $l < 1$ can be an upper bound for S. Hence, 1 is the least of all possible upper bounds for S.

Example 2.16.3 above, suggests that we might need a new notion generalizing the concept of maximum/minimum. In that example, 1 is very close to being the maximum of $S := \left\{\frac{n-1}{n} \mid n \in \mathbb{Z}^*_+\right\}$, as it is the least of all possible upper bounds. On the other hand, 1 cannot be the maximum of S as $1 \notin S$. This phenomenon motivates the next definition.

Definition 2.17. Let $S \subseteq \mathbb{R}$ be a non-empty subset.

(1) If the set U of all upper bounds of S is non-empty and U admits a minimum $u \in U$, then we call u the *supremum* of S.

(2) If the set L of all lower bounds of S is non-empty and L admits a maximum $l \in L$, then we call l the *supremum* of S.

Remark 2.18. Let $S \subseteq \mathbb{R}$ be a non-empty subset.
If the set U of all upper bounds of S is empty, then S is not bounded from above, cf. Definition 2.8. In this case, then the supremum of S does not exist, by the above definition.
Similarly, if the set L of all lower bounds of S is empty, then S is not bounded from below, cf. Definition 2.8. In this case, then the infimum of S does not exist, by the above definition.

As in the case of maximum/minimum, the use of the determinative article in Definition 2.17 suggests that, when they exist, the supremum/infimum of a non-empty subset of $\mathbb{R}$ should be unique.

Proposition 2.19. *Let S be a non-empty subset of $\mathbb{R}$. If $\sup S$ (resp. $\inf S$) exists, then it is unique.*

Notation 2.20. For $S \subseteq \mathbb{R}$, we denote the supremum (resp. the infimum) of S by $\sup S$ (resp. $\inf S$), when those exist as real number.
If S is not bounded from above, we write $\sup S = +\infty$. If S is not bounded from below, we write $\inf S = -\infty$.

Proof. By definition, if the supremum of S exists, it is the minimum of the set

$$U := \{u \in \mathbb{R} \mid u \text{ is an upper bound for } S\}.$$

As the maximum of a set is unique when it exists, cf. 2.12, then the conclusion follows at once. You can apply a similar argument for the uniqueness of the minimum. $\square$

Example 2.21. (1) Let $S := \left\{ \frac{n-1}{n} \middle| n \in \mathbb{Z}_+^* \right\}$. Then, $\sup S = 1$, cf. Example 2.16.3.

(2) Take $S := \{n^3 | n \in \mathbb{Z}\}$. Then, S is unbounded. Thus, $\inf S, \sup S$ do not exist.

(3) If S is a bounded interval of extremes $a < b$, then

$$\sup S = b, \qquad \inf S = a.$$

Indeed, we saw in Proposition 2.10 that the set of lower (resp. upper) bounds of S is $]-\infty, a]$ (resp. $[b, +\infty[$).

(4) Similarly, if $S := [a, +\infty[$ or $S := [a, +\infty$, $a \in \mathbb{R}$ then $\inf S = a$, while $\sup S$ does not exit since S is not bounded from below.

(5) If $S :=]-\infty, b]$ or $S :=]-\infty, b[$, $b \in \mathbb{R}$, then $\inf S = a$, while $\sup S$ does not exit since S is not bounded from below.

How do we know whether the supremum or infimum of a non-empty subset $S \subseteq \mathbb{R}$ exist as real numbers? We saw in Remark 2.18 that a necessary condition for the existence of the supremum (resp. infimum) of S is that S be bounded from above (resp. below).

On the other hand, if, for example, S is bounded from above (resp. below), then we know that the set U (resp. L) of all upper (resp. lower) bounds of S is non-empty. Hence, it is legitimate to ask if U (resp. L), when non-empty, admits a least (resp. largest) element.

The existence of the largest of all possible lower bounds (resp. of the least of all possible upper bounds) is one of the features of the construction of the real numbers. As we have already mentioned that we are not going to explain the construction of $\mathbb{R}$, we will assume the existence of such elements. Indeed, it suffices to assume the following axiom, which then implies the full existence of infima and suprema, cf. Corollary 2.26.

Axiom 2.22. [Infimum axiom] Each non-empty subset S of $\mathbb{R}_+^*$ admits an infimum (which is a real number).

Remark 2.23. In Mathematics, an axiom is a statement that we are going to assume to be true, without requiring for it a formal proof. When we introduce an axiom, we are free to use the properties stated in the axiom, without requiring a proof for them, and we can use those to derive other mathematical properties of the objects that we are studying.

The property stated in the Infimum Axiom is a very important one. In a sense, which we will try to make more precise when we introduce sequences of real numbers, this property says that $\mathbb{R}$ does not contain any gaps. While at this time, this is a rather nebulous statement, let us at least show that this axiom does not necessarily hold for all the number sets that we have introduced so far, cf. Section 2.2: indeed, it is possible to show that the infimum axioms does not necessarily hold for $\mathbb{Q}$, for example, cf. Example 2.24 below. Hence, the Infimum Axiom is indeed an axiom stating a (very relevant) property that is indeed peculiar to the real numbers and, as such, in this course we actually utilize it to characterize the real numbers, again, cf. Remark 2.7.

Example 2.24. Let $S :=]\sqrt{3}, 5[\cap \mathbb{Q}$.[8] Then $S \subseteq \mathbb{R}_+^*$ and the Infimum Axiom implies that $\inf S$ exists in the real numbers. We will show in Example 2.46 that $\inf S = \sqrt{3}$. In particular, the set of lower bounds of S coincides with the real numbers $\leq \sqrt{3}$.
Since S, by its very definition, is also a subset of $\mathbb{Q}$, we may wonder whether it possible to find a largest rational number l among the rational numbers which are lower bounds for S. Such $l \in \mathbb{Q}$ would then be an infimum for S among the rational numbers. By the above observation, we know that if such l existed, then $l < \sqrt{3}$, since $\sqrt{3} \notin \mathbb{Q}$, cf Proposition 2.38, and l is certainly a lower bound for l. But then, Proposition 2.44 shows that there exists a rational number m such that $l < m < \sqrt{3}$. As $m < \sqrt{3}$, then we know that m is also a lower bound for S. This is clearly a contradiction, as $m \in \mathbb{Q}$ nad is a lower bound for S, while we had assumed that l was the largest of all lower bounds of S that are rational. Hence, the infimum of S cannot exist in $\mathbb{Q}$.

Axiom 2.22 requires that we work with subsets of $\mathbb{R}_+^*$ to be guaranteeed to find their infimum. But, in general, we can find the infimum also for sets that are not necessarily contained in $\mathbb{R}_+^*$, as long as we have some lower bounds.

Example 2.25. The infimum of a set $S \subseteq \mathbb{R}$ can exist even when $S \not\subseteq \mathbb{R}_+^*$. For example, let $S := \{x \in \mathbb{R} \mid x > -\sqrt{17}\}$. As S contains -1, for example, then $S \not\subseteq \mathbb{R}_+^*$. On the other hand, by Proposition 2.10.6, the set of lower bounds of S is given by $]-\infty, -\sqrt{17}]$. Hence, $\inf S = -\sqrt{17}$.

Using the Infimum Axiom 2.22, we can actually prove that the infimum (resp. the supremum) exists for any subset $S \subseteq \mathbb{R}$ which is bounded from below (resp. from above).

Corollary 2.26. *Let $S \subseteq \mathbb{R}$ be a non-empty set.*

(1) If S is bounded from below, then S admits an infimum.

(2) If S is bounded from above, then S admits a supremum.

Proof. (1) As S is bounded from below, there exists a lower bound $l \in \mathbb{R}$ for S, that is, $l \leq s$, for all $s \in S$. We can rewrite the previous inequality as

$$s - l \geq 0, \quad \forall s \in S. \tag{2.26.a}$$

[8] See Section 2.4.1 for a formal proof that $\sqrt{3}$ is actually a real number.

Let $W \subseteq \mathbb{R}$ be the subset obtained by translating the elements of S by $-l+1$,

$$W := \{s - l + 1 \mid s \in S\}.$$

Why did we choose to translate the elements of S by $-l+1$? The reason is that $W \subseteq \mathbb{R}^*_+$: in fact, by (2.26.a), $s-l+1 \geq 1 > 0$, for all $s \in S$.[9] As $W \subseteq \mathbb{R}^*_+$, the Infimum Axiom 2.22 implies that $\inf W$ exists, call it $a := \inf W$. Then a is the largest lower bound for the set W.

How can we use a to compute $\inf S$? To construct W, we translated all elements of S by $-l+1$. If we translate the elements of W back by $l-1$, then we undo what we did before and we recover S. So, what happens if we translate a by $l-1$ as well? The number we obtain by this translation should be the largest lower bound for S, as addition is compatible with the order relation. Let us verify this.

Let $a' := a + l - 1$. Then $a' \leq w + l - 1$ for any element $w \in W$. As any $w \in W$ is of the form $w = s - l + 1$ for some $s \in S$, then $w + l - 1 = s$. Hence, $a' \leq s$ for all $s \in S$ and a' is a lower bound for S. If a' is not the largest lower bound for S, then there is a real number $b' > a'$ which is a lower bound for S. But then $b' - l + 1 > a = a' - l + 1$ and $b' - l + 1$ would be a lower bound for W [prove it!]. But this is a contradiction, since $a = \inf W$.

(2) The details are left to the reader. Here is a sketch.
Let $S' \subseteq \mathbb{R}$ be the set constructed by flipping the sign of the elements of S,

$$S' := \{-x \mid x \in S\}.$$

Since S is bounded from above, then S' is bounded from below. [Prove this!] Then by part (1), $\inf S'$ exists. It is left to you to show that $\sup S = -\inf S'$.

□

We have seen the definition of infimum/supremum and minimum/maximum. Both the infimum (resp. supremum) and minimum (resp. maxima) of a set S, provided that they exist, are lower bounds (resp. upper bounds) for S. Can we be more precise about what is the relationship among these notions?

Example 2.27. Let $S := [3, 5[\subseteq \mathbb{R}$. Then, $\min S = 3 = \inf S$. On the other hand, $\max S$ does not exist as $\sup S = 5$ is the least upper bound and $5 \notin S$; hence no upper bound of S is contained in S, as any element of S is < 5.

The example above seems to suggest that, at least for intervals, if the minimum (resp. maximum) of an interval exists, then it should coincide with the infimum (resp. the supremum) of the interval. This property actually holds for any non-empty subset $S \subset \mathbb{R}$, as long as the minimum (resp. maximum) of S exists.

Proposition 2.28. *Let $S \subseteq \mathbb{R}$ a non-empty set.*

(1) If $\min S$ exists, then $\min S = \inf S$.

(2) If $\max S$ exists, then $\max S = \sup S$.

Proof. We prove (1), whereas (2) is left as an exercise. As $\min S$ exists, then S is bounded from below, since $\min S$ is in particular a lower bound, cf. Definition 2.11. Hence, $\inf S$ exists, by Corollary 2.26. Then $\inf S \geq \min S$ since $\inf S$ is the largest of all lower bounds. On the other hand, $\min S \in S$, and $\inf S \leq s$, for all $s \in S$. In particular, $\inf S \leq \min S$. Thus, $\inf S \leq \min S$ and $\inf S \geq \min S$, which implies that $\inf S = \min S$. □

[9] We could have choosen to translate by $-l + c$, for any $c > 0$. Hence the choice of $c = 1$ was arbitrary, but I needed to choose something explicit, so I went for 1.

2.3.2 Archimedean property of $\mathbb{R}$

As we have already mentioned, given any two real numbers x, y we can always compare them, that is, we can decide whether either $x = y$, or $x < y$ or $x > y$. On the other hand, whenever it makes sense, for example, if x, y are both non-negative real numbers with $x < y$, we may ask a more general question: namely, we may ask whether, by taking multiples of x, we can eventually construct a real number $nx > y$.

Example 2.29. Let $y = \pi^{20}$ and let $x = 1$. We want to find a natural number n such that $nx = n \cdot 1 = n$ is $> \pi^{20}$. If we write π^{20} in its decimal representation,

$$\begin{aligned}\pi^{20} =& 8769956796.08269947475225559370389706606411444719543724342098426 0\\ & 518412390435479909902349851866735983156956048648923727056 66\ldots \quad .\end{aligned}$$ [10]

Then if we take $n = 8769956797$, that is, n is equal to the integral part of the decimal representation of $\pi^{20} + 1$, then $n = n \cdot 1 = nx > \pi^{20} = y$.

When we discussed real numbers at the start of the course, we saw that perhaps it is not an ideal method that of relying on their decimal representation. After all, it is not even clear that we can compute effectively the decimal representation of any real number. (Have you ever thought about how computers are able to calculate decimal representations of irrational numbers? If you are curious about that, you may want to take a look here). We said that in this course, we should rather try to prove properties of the real numbers by relying on their intrinsic mathematical properties, and by using mathematical proofs as the tools to connect properties and discover new one.

The interesting fact, is that we can actually prove that the conclusion of Example 2.29 holds, in full generality, for any pair of positive numbers x, y.

Proposition 2.30 (Archimedeand property of $\mathbb{R}$)**.** *Let x, y be real numbers, with $x > 0$, $y \geq 0$. Then there exists $n \in \mathbb{N}^*$ such that $nx > y$.*

Proof. If $y = 0$, then take $n = 1$. Then $nx = 1 \cdot x = x > 0$ and we are done.
Let us now assume that $y > 0$. We make a proof by contradiction. Let us assume that

$$\forall\, n \in \mathbb{N}, \quad nx \leq y. \tag{2.30.b}$$

Let $S \subseteq \mathbb{R}$ be the set

$$S := \{nx \mid n \in \mathbb{N}\}.$$

Then S is non-empty as $x \in S$, and S is bounded from above, as y is an upper bound by (2.30.b). Hence, by Corollary 2.26 $\sup S$ exists and $(n+1)x \leq \sup S$ for all $n \in \mathbb{N}$. Thus, $nx \leq \sup S - x$ for all $n \in \mathbb{N}$, that is, $s \leq \sup S - x$, for all $s \in S$. But this implies that $\sup S - x$ is an upper bound for S, too. As $\sup S - x < \sup S$, since $x > 0$, this gives a contradiction to the fact that $\sup S$ is the supremum of S, i.e., the smallest upper bound for S. $\square$

Corollary 2.31. *Let $y \in \mathbb{R}_+$. Then there exists $n \in \mathbb{N}^*$ such that $n > y$.*

Proof. It is enough to apply Proposition 2.30 to y, taking $x = 1$. $\square$

2.3.3 An alternative definition for infimum/supremum

Let $S \subset \mathbb{R}$ be a non-empty set. We have seen in Section 2.3.1 that the infimum and supremum of S are unique, when they exist. Moreover, as the infimum (resp. supremum) of S is the largest (resp. the smallest) lower bound (resp. upper bound) of S, then whenever we take a number c larger than $\inf S$ (resp. smaller than $\sup S$), we must be able to find an element $s \in S$ contained between $\inf S$ and c (resp. between c and $\sup S$), that is, $\inf S \leq s < c$ (resp.

$c < d \le \sup S$).
Using this reasoning, we can characterize the infimum (resp. supremum) of S in the following alternative way.

Proposition 2.32. *Let $S \subset \mathbb{R}$ be a non-empty set.*

(1) A real number u is the supremum of S if and only if

  (i) u is an upper bound for S, and

  (ii) for all $\varepsilon > 0$, there is $s_\varepsilon \in S$, such that $s_\varepsilon > u - \varepsilon$.

(2) A real number l is the infimum of S if and only if

  (i') l is an lower bound for S, and

  (ii') for all $\varepsilon > 0$, there is $s'_\varepsilon \in S$, such that $s'_\varepsilon < l + \varepsilon$.

The criterion just introduced is very useful in practice when trying to prove that a certain real number is the infimum/supremum of a given subset of the real numbers.

Example 2.33. Let $S := \{1 - \frac{1}{n} \mid n \in \mathbb{Z}^*_+\}$. We show that $\sup S = 1$ using Proposition 2.32.1. To this end, we must verify that 1 satisfies both properties:

(i) 1 is an upper bound for S, and

(ii) for all $\varepsilon > 0$, there is $s_\varepsilon \in S$, such that $s_\varepsilon > 1 - \varepsilon$.

Since $1 \ge 1 - \frac{1}{n}$, for all $n \in \mathbb{N}^*$, then, by definition, of upper bound, 1 is an upper bound for S; thus, property (i) is satisfied.
To verify (ii), let, for example, $\epsilon = \frac{3}{17}$; then we have to show that there exists an element s_ϵ of S such that

$$1 - \frac{3}{17} < s_\epsilon < 1.$$

(The second inequality comes for free from the fact that 1 is an upper bound for S). If we take $s_\epsilon = 1 - \frac{1}{17}$, then $s_\epsilon \in S$, and since $\frac{1}{17} < \frac{3}{17}$

$$1 - \frac{3}{17} < 1 - \frac{1}{17} < 1$$

which is what we wanted.
To make the proof more general, we have to fix a positive real number ϵ (this could be any positive real number, but we are thinking that we have fixed one specific value for ϵ). Again, we have to find an element $s_\epsilon \in S$ (this element that we construct will depend on the initial choice of ϵ, that is why we denote it as s_ϵ, to remind ourselves about the dependence from ϵ) such that $1 - \epsilon < s_\epsilon$.
If $\epsilon > 1$ then $1 - \epsilon < 0$, hence we can just take $s_\epsilon = 0 = 1 - \frac{1}{1} \in S$. If $\epsilon \le 1$, then $1 - \epsilon \in [0, 1[$. How can find find $n \in \mathbb{N}$ such that $1 - \epsilon < 1 - \frac{1}{n}$? The inequality $1 - \epsilon < 1 - \frac{1}{n}$ is equivalent to the inequality $n > \frac{1}{\epsilon}$ [Check that!]. As $\epsilon > 0$, also $\frac{1}{\epsilon} > 0$. Hence, by Corollary 2.31 we can find a natural number k such that $k > \frac{1}{\epsilon}$. Then $1 - \epsilon < 1 - \frac{1}{k}$, so that we can take $s_\epsilon := 1 - \frac{1}{k}$.

Proof of Proposition 2.32. We show part (1). The proof of part (2) is completely analogous and is left as an exercise for the reader.
We first prove the implication

$$l = \inf S \implies l \text{ satisfies properties (i) and (ii) in Proposition 2.32.}$$

Let $l = \inf S$. As $\inf S$ is the largest of all lower bounds for S, by Proposition 2.32.1, in particular l is a lower bound for S; thus, l satisfy said property. As $\inf S$ is the largest lower

bound, by itw definition, then if we take any $\epsilon > 0$, $l + \epsilon$ cannot be a lower bound for S. This means that there exists an element of S (which will depend on the choice of ϵ in general, cf. Example 2.33), call it s_ϵ, such that $s_\epsilon < l + \epsilon$, which shows that l satisfies also property (ii) of Proposition 2.32.
We then prove the other implication:

$$l \text{ satisfies properties (i) and (ii) in Proposition 2.32} \implies l = \inf S.$$

Let us assume, by contradiction, that $l \neq \inf S$. Since by property (i) l is a lower bound, the assumption that $l \neq \inf S$ means that l is not the largest lower bound. Hence, there exists $l' \in \mathbb{R}$, $l' > l$ and l' is a lower bound for S. In particular,

$$\text{for all } s \in S,\ s \geq l'. \tag{2.33.c}$$

Take $\epsilon := l' - l > 0 \implies l + \epsilon = l'$. Then (2.33.c) implies that no element of S is $< l + \epsilon$. But, this is in contradiction with property (ii) of Proposition 2.32 which we assumed to begin with. □

2.3.4 Infimum and supremum for subsets of $\mathbb{Z}$

When we defined the natural numbers in Section 2.2 we mentioned that any subset of $\mathbb{N}$ has a minimum. We have now all the tools to prove this statement, which will be one of our standard tools for the duration of the course.

Proposition 2.34. *Let $S \subseteq \mathbb{R}$ be a non-empty set of natural numbers. Then,* $\inf S = \min S$.

What is the important information contained in the statement of the above proposition? As $S \subseteq \mathbb{N}$, S is bounded from below. Hence, the Infimum Axiom 2.22 implies that $\inf S$ exists. On the other hand, we know from Proposition 2.28 that if the minimum of S exists, then it must always coincide with $\inf S$. Hence, the important bit of information contained in Proposition 2.34 is that the minimum of any set $S \subseteq \mathbb{N}$ indeed exists, a property that we had already mentioned in Section 2.2.

Example 2.35. Let

$$S := \{x \in \mathbb{R} \mid x \in \mathbb{N}^* \text{ and } x \text{ is divisible by at least 5 distinct prime numbers}\}.$$

Then, by definition, S is a set of natural numbers and certainly $1, 2, 3, 5$ are not elements of S; even better, no prime number $p \in \mathbb{N}$ is an element of S. On the other hand, Proposition 2.34 implies that S has a minimum.
How can we compute $\min S$? That is, what is the minimum natural number that is divisible by 5 distinct prime numbers? As any natural number can be written essentially uniquely as a product of prime numbers, then $\min S$ is the product of the 5 smallest prime numbers. The first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots.$$

Hence, $\min S = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$.

Proof of Proposition 2.34. Let $d := \inf S$, which exists by Corollary 2.26, since S is bounded from below. We have to show that $d \in S$.
Assume by contradiction that $d \notin S$. Then, as $\inf S$ is the largest lower bound of S, for each $\varepsilon > 0$, $d + \varepsilon$ is not a lower bound. Hence:

$$\text{for all } \varepsilon > 0, \text{ there is } s_\varepsilon \in S, \text{ such that } s_\varepsilon < d + \varepsilon. \tag{2.35.d}$$

Apply (2.35.d) with $\varepsilon' := \frac{1}{2}$. This yields an element $s_{\varepsilon'}$ of S such that

$$d < s_{\varepsilon'} < d + \varepsilon' = d + \frac{1}{2}$$

Apply then again the above property of S, but now for $\varepsilon'' := s_{\varepsilon'} - d > 0$. Then, we can find $s_{\varepsilon''} \in S$ such that

$$d < s_{\varepsilon''} < d + \varepsilon'' = s_{\varepsilon'} < d + \varepsilon' = d + \frac{1}{2}.$$

In particular, $0 < s_{\varepsilon'} - s_{\varepsilon''} < d + \frac{1}{2} - d = \frac{1}{2}$. This gives a contradiction, since $s_{\varepsilon'}, s_{\varepsilon''} \in \mathbb{N}$ and the distance between two different natural numbers is always at least 1 one from. Hence, our initial assumption that $d \notin S$ must be false, so that $d \in S$. $\square$

Exercise 2.36. Let $S \subseteq \mathbb{R}$ a subset of the integers.

(1) If S is bounded from below, then $\min S = \inf S$.

(2) If S is bounded from above, then $\max S = \sup S$.

[*Hint*: for (1), let a be a lower bound for S; then $a > [a] - 1$ is an integer $> a$. Consider the set $S' := \{s - [a] + 1 \mid s \in S\} \subseteq \mathbb{N}$ and try to imitate the proof of Corollary 2.26.
For (2), define the set $S'' := \{-x \mid x \in S\}$ and then imitate the proof of Corollary 2.26 and use (1) to prove (2).]

2.4 Rational numbers vs real numbers

2.4.1 $\sqrt{3}$ is a real number

We have seen that $\sqrt{3}$ is not a rational number, cf. Proposition 1.1.

Question 2.37. Why is $\sqrt{3}$ a real number?

We are going to show that using the Axiom 2.22, we can formally show that there exists a positive real number x satisfying the equation $x^2 = 3$. By, its own definition, then $x = \sqrt{3}$. To this end, let us consider $S := \{x \in \mathbb{R} \mid x^2 \leq 3\}$. First of all, S is a non-empty subset of $\mathbb{R}$, since $1 \in S$. Moreover, S is bounded: in fact, 3 is an upper bound and -3 is a lower bound for S. [Prove it! Remember that for real numbers $x > y > 0$, then $x^2 > y^2 > 0$.] As S is bounded then by Corollary 2.26 both the infimum and the supremum of S exists. As $1 \in S$, then the supremum of S is ≥ 1, in particular it is > 0. We will show that $\sup S = \sqrt{3}$.

Proposition 2.38. *Let $S \subseteq \mathbb{R}$ be the subset*

$$S := \{x \in \mathbb{R} \mid x^2 \leq 3\}.$$

Then $\inf S < 0 < \sup S$ *and* $(\sup S)^2 = 3 = (\inf S)^2$. *Thus,* $\sup S = \sqrt{3}$, $\inf S = -\sqrt{3}$.

Proof. We have already shown above that $\inf S$ and $\sup S$ exist. Moreover, as $\pm 1 \in S$, then it follows at once that $\inf S < -1 < 0 < 1 < \sup S$. Hence, if $(\sup S)^2 = 3 = (\inf S)^2$, then the above chain of inequalities implies that $\sup S = \sqrt{3}$, $\inf S = -\sqrt{3}$.
We now show that $(\sup S)^2 = 3$. The verification for $\inf S$ is analogous.
Let us assume, for the sake of contradiction, that $(\sup S)^2 \neq 3$ and let us show that we obtain a contradiction. We have 2 possible cases:

$$\begin{cases} (\sup S)^2 > 3, \\ (\sup S)^2 < 3 \end{cases}.$$

Case 1: Assume $(\sup S)^2 > 3$.
We shall show that there exists a sufficiently large $n \in \mathbb{N}$ such that $\sup S - \frac{1}{n}$ is an upper bound

for S. This immediately yields the desired contradiction, since $\sup S - \frac{1}{n} < \sup S$ and $\sup S$ is by definition the least of all upper bounds.
As $\sup S > 1$, then $\sup S - \frac{1}{n} > 0$ for all $n \in \mathbb{N}^*$. Hence to show that for some $n \in \mathbb{N}^*$, $\sup S - \frac{1}{n}$ is an upper bound for S, it suffices to show that $(\sup S - \frac{1}{n})^2 > 3$, since for $x > 0$, $x < \sup S - \frac{1}{n}$ if and only if $x^2 < (\sup S - \frac{1}{n})^2$. But

$$(\sup S - \frac{1}{n})^2 = (\sup S)^2 + \frac{1}{n^2} - \frac{2\sup S}{n} > (\sup S)^2 - \frac{2\sup S}{n}.$$

Hence, it suffices to show that we can find $n \in \mathbb{N}$ large enough such that $(\sup S)^2 - \frac{2\sup S}{n} > 3$. Let us denote by $d := (\sup S)^2 - 3$ which is a positive real number. But then, finding $n \in \mathbb{N}^*$ such that $(\sup S)^2 - \frac{2\sup S}{n} > 3$ is equivalent to finding $n \in \mathbb{N}^*$ such that $\frac{2\sup S}{n} < d$, and the last inequality is equivalent to $n > \frac{d}{2\sup S}$, since $\sup S > 0$. The existence of $n \in \mathbb{N}^*$ such that $n > \frac{d}{2\sup S}$ is guaranteed by the archimidean property, Proposition 2.30. This concludes the proof in Case 1.

Case 2: Assume $(\sup S)^2 < 3$.
We shall show that there exists $n \in \mathbb{N}^*$ such that $(\sup S + \frac{1}{n})^2 < 3$. As $\sup S + \frac{1}{n} > \sup S > 0$, this implies that $\sup S + \frac{1}{n} \in S$ which will yield the desired contradiction, since $\sup S$ must be an upper bound of S. Let d' be the positive real number $d' := 3 - (\sup S)^2$. Then since

$$(\sup S + \frac{1}{n})^2 = (\sup S)^2 + \frac{1}{n^2} + \frac{2\sup S}{n} < (\sup S)^2 + \frac{1}{n} + \frac{2\sup S}{n},$$

it suffices to show that there exists $n \in \mathbb{N}^*$ such that $(\sup S)^2 + \frac{1}{n} + \frac{2\sup S}{n} < 3$. But this is equivalent to finding $n \in \mathbb{N}^*$ such that $n > \frac{d}{1+2\sup S}$. The existence of one such $n \in \mathbb{N}^*$ is again guaranteed by the archimidean property of $\mathbb{R}$, cf. Proposition 2.30. $\square$

2.4.2 Integral part

Let x be a real number. According to Exercise 2.36, the set $S := \{n \in \mathbb{N} \mid n \leq x\}$ has a maximum, since it is bounded from above. Call $m := \max S$. Then $m + 1$ is not in S, as m is the largest element of S. We call the integer m the *integral part* of x and we denote it by $[x]$.

Definition 2.39. Let $x \in \mathbb{R}$.

(1) The round-down $\lfloor x \rfloor$ of x is the largest integer that is $\leq x$.

(2) The round-up $\lceil x \rceil$ of x is the least integer that is $\geq x$.

(3) The integral part $[x]$ of x is defined as

$$[x] := \begin{cases} \lfloor x \rfloor & \text{for } x \geq 0, \\ \lceil x \rceil & \text{for } x < 0. \end{cases}$$

We can also define the fractional part of x.

Definition 2.40. Let x be a real number. Then the *fractional part* $\{x\}$ of x is defined as

$$\{x\} := x - [x].$$

Exercise 2.41. For all $x \in \mathbb{R}$,

(1) $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$;

(2) $\lceil x \rceil - 1 < x \leq \lceil x \rceil$;

(3) $[x] = -[-x]$;

(4) $\{x\} \in]-1, 1[$ and $\{x\} = -\{-x\}$

(5) $x = [x] + \{x\}$;

(6) $x \in \mathbb{Z}$ if and only if $x = \lfloor x \rfloor = \lceil x \rceil = [x]$.

Example 2.42. (1) $[-4] = -4$ and hence $\{-4\} = 0$. In general, if $z \in \mathbb{Z}$, then $[z] = z$, $\{z\} = 0$.

(2) Considering the number $x = \pi^2 + \pi$,

$$\pi^2 + \pi = 13.01119705467915185729713438315565401951086880661589644738829396852786122870541424162980822906066929980617400028730545072486619 2\ldots$$

Hence, $[\pi^2 + \pi] = 13$, and $\{\pi^2 + \pi\} = \pi^2 + \pi - 13$ – not a number that we can fully write down with decimals.

(3) For rational numbers, things are a bit easier. For example, $\left[-\frac{3}{2}\right] = -1$ and $\left\{-\frac{3}{2}\right\} = -\frac{1}{2}$.

(4) Roughly speaking, when we write a real number x by means of its decimal representation, then the integral part $[x]$ (as its name suggests) stands for the integral number whose digits are left of the "." dividing integral and decimal part, while $\{x\}$ stands for the real number in $]-1, 1[$ whose digits are right of the "." dividing integral and decimal part: for example, $[7.\overline{8324123}] = 7$, $\{7.\overline{8324123}\} = 0.\overline{8324123}$.

2.4.3 Rational numbers are dense in $\mathbb{R}$

We have already observed that $\mathbb{Q} \subsetneq \mathbb{R}$. It would be nice to have some more information about how rational numbers are distributed among real numbers. For example, we may ask if we can find rational numbers between two arbitrary real numbers.

Example 2.43. For example, is there a rational number c, such that $0 < c < \pi$? The left inequality, that is, $0 < c$, is an easy one to guarantee. It suffices to choose c to be a positive rational number. But, how do we guarantee that the inequality on right holds as well? Well, as soon as c is positive, $c < \pi$ is equivalent to $\frac{1}{c} > \frac{1}{\pi}$. So, if one chooses $\frac{1}{c}$ to be any integer that is larger than $\frac{1}{\pi}$ we are fine. For example, we can choose

$$\frac{1}{c} = \left[\frac{1}{\pi}\right] + 1 \quad \text{that is,} \quad c = \left(\left[\frac{1}{\pi}\right] + 1\right)^{-1}.$$

It is not too hard to show that the above example can be extended in more generality to any two real numbers.

Proposition 2.44. *If $a < b$ are real numbers, then there is a rational number c, such that $a < c < b$.*

We can summarize the property stated in Proposition 2.44 by saying that "rational numbers are arbitrarily close to any real number". In more precise mathematical terms, we refer to the property stated in Proposition 2.44 above by saying that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 2.45. Let us consider

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432764157273501384623091229702492483 60\ldots\ .$$

We know that $\sqrt{2}$ is not a rational number. Then, how can we show that rational numbers are arbitrarily close to $\sqrt{2}$? We could try to approximate $\sqrt{2}$ by means of rational numbers.
So, for example, what is a rational number that is close within $\frac{1}{10}$ of $\sqrt{2}$? Proposition 2.44 tells us that such approximation certainly exists, as it guarantees that we can find a rational number c such that $\sqrt{2} - \frac{1}{10} < c < \sqrt{2}$. But, in practice, how can we find such c? Using the decimal expansion of $\sqrt{2}$ above, we can immediately notice that

$$\begin{aligned}\sqrt{2} - 1.4 =& 0.01421356237309504880168872420969807856967187537694807317667973799 0\\ & 7324784621070388503875343276415727350138462309122970249248360\ldots\ .\end{aligned}$$

Hence, $\sqrt{2} - \frac{1}{10} < 1.4 < \sqrt{2}$.
In the same way, if we want to approximate $\sqrt{2}$ up to $\frac{1}{10000}$ with rational, we can search for a rational number c' such $\sqrt{2} - \frac{1}{10000} < c' < \sqrt{2}$. As before, by taking $c' = 1.41421$ we obtain that

$$\begin{aligned}\sqrt{2} - 1.41421 =& 0.00000356237309504880168872420969807856967187537694807317667 9\\ & 7379907324784621070388503875343276415727350138462309\cdots < \frac{1}{10000}.\end{aligned}$$

In the same way, if we want to approximate $\sqrt{2}$ within $\frac{1}{10^n}$, then it is enough to take the rational number whose decimal representation is given by taking that of $\sqrt{2}$ and truncating it after the n-th decimal digit.

Proof. Let us start with a simple case of our proof.

Easy case; we assume $a = 0$:
We have $\left\lfloor \frac{1}{b} \right\rfloor + 1 > \frac{1}{b}$ and $\left\lfloor \frac{1}{b} \right\rfloor + 1$ is a positive integer. We conclude that

$$0 < \frac{1}{\left\lfloor \frac{1}{b} \right\rfloor + 1} < b\,,$$

so we can take $c = \frac{1}{\left\lfloor \frac{1}{b} \right\rfloor + 1}$.

General case:
Let us define the number $n := \left\lfloor \frac{1}{b-a} \right\rfloor + 1$. Then,

$$\begin{aligned} n = \left\lfloor \frac{1}{b-a} \right\rfloor + 1 \Rightarrow n > \frac{1}{b-a} \Rightarrow \frac{1}{n} < b - a \\ a = \frac{an}{n} < \frac{\lfloor an \rfloor + 1}{n} \leq \frac{an+1}{n} = a + \frac{1}{n} < a + b - a = b\end{aligned}$$

Furthermore, $\frac{\lfloor an \rfloor + 1}{n}$ is a rational number. Hence, to conclude it suffices to take $c = \frac{\lfloor an \rfloor + 1}{n}$.
[This is not the unique rational number between a and b, it is just one example of a rational number between a and b.] $\square$

Example 2.46. This is a continuation of Example 2.24. We can finally prove that for $S :=]\sqrt{3}, 5[\cap \mathbb{Q}$ then the infimum of S in $\mathbb{R}$ is $\sqrt{3}$.
By definition of S, any element of S is $> \sqrt{3} \implies \sqrt{3}$ is a lower bound.
Let us assume by contradiction that that $\sqrt{3}$ is not the infimum of $S \implies \sqrt{3} < \inf S < 5$ and by Proposition 2.44, there exists a rational number c such that $\sqrt{3} < c < \inf S < 5$. But then $c \in S$ since $c \in \mathbb{Q}$ and $c \in]\sqrt{3}, 5[$, and $c < \inf S$, which provides a contradiction. Hence $\inf S = \sqrt{3}$.

2.4.4 Irrational numbers are dense in $\mathbb{R}$

The same property of density in $\mathbb{R}$ that we showed holds for $\mathbb{Q}$, in the previous section, holds also for the complement $\mathbb{R} \setminus \mathbb{Q}$ of $\mathbb{Q}$ in $\mathbb{R}$. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational numbers.*

Proposition 2.47. *If $a < b$ are real numbers, then there is $c \in \mathbb{R} \setminus \mathbb{Q}$, such that $a < c < b$.*

The set $\mathbb{R} \setminus \mathbb{Q}$ of real numbers which are not rational is called the set of *irrational numbers.*

Remark 2.48. Let us recall that if $f \in \mathbb{Q}^*$ and $g \in \mathbb{R}^* \setminus \mathbb{Q}$, then $fg \in R^* \setminus \mathbb{Q}$.

Proof. Apply Proposition 2.44 to $\frac{a}{\sqrt{3}} < \frac{b}{\sqrt{3}}$. This yields a rational number d such that $\frac{a}{\sqrt{3}} < d < \frac{b}{\sqrt{3}}$. Additionally we can assume that $d \neq 0$: indeed, if $d = 0$ then it suffices to replace d by the rational number that one can obtain by applying Proposition 2.44 to 0 and $\frac{b}{\sqrt{3}}$. Hence,

$$a < \sqrt{3}d < b \text{ and } d \neq 0.$$

It remains to show that $\sqrt{3}d$ is irrationalbut this follows at once from Remark 2.48. □

2.5 *Absolute value*

Definition 2.49. If $x \in \mathbb{R}$, then the *absolute value* $|x|$ of x is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Example 2.50. $|3| = 3$, $|-5| = 5$, $|-\pi| = \pi$, $|0| = 0$ and $|5| = 5$.

It is useful to remember the graph of the absolute value function,see Figure 1.

../Images/graph_abs_function.png

Figure 1: The graph of $f(x) = |x|$

Another way to define the absolute value $|x|$ of $x \in \mathbb{R}$ is to define it as the distance between x and 0 on the real line.

2.5.1 Properties of the absolute value

How does the absolute value $|x|$ of a real number x compare to x itself, in relation to the usual ordering on $\mathbb{R}$?

Example 2.51. $-\left|-\sqrt{3}\right| \leq -\sqrt{3}$ and $|-\sqrt{3}| \geq -\sqrt{3}$.

The inequalities in the above example hold for any real number: that is, for $x \in \mathbb{R}$

$$-|x| \leq x \leq |x|. \tag{2.51.a}$$

The absolute value behaves well with respect to the multiplication.

Example 2.52. $|5 \cdot (-3)| = |-15| = 15 = 5 \cdot 3 = |5| \cdot |-3|$. Similarly,

$$\left|(-\sqrt{2}) \cdot (-4)\right| = \left|4\sqrt{2}\right| = 4\sqrt{2} = \sqrt{2} \cdot 4 = \left|-\sqrt{2}\right| \cdot |-4|.$$

We can generalize Example 2.52: indeed, for all $x, y \in \mathbb{R}$

$$|x| \cdot |y| = |x \cdot y|.$$

To prove this, you can just list all possible combinations for the signs of x, y (that is, "positive"–"positive"; "positive"– "negative"; "negative"–"negative") and prove the equality in each case. Analogously, in the case of division, for $x, y \in \mathbb{R}$, $y \neq 0$, we have that

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

Example 2.53. $\left|\frac{5}{-4}\right| = \frac{|5|}{|-4|}$.

The absolute value is also needed to relate powers and roots.

Example 2.54. $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$ and $\sqrt{(7)^2} = \sqrt{49} = 7 = |7|$.

In general, for $x \in \mathbb{R}$, $\sqrt{x^2} = |x|$. This can be generalized to any n-th root of the n-th power of a real number when n is an even natural number.

2.5.2 Triangular inequality

While we have seen that the absolute value is compatible with multiplication,that is, the absolute value of a product of two terms is equal to the product of the absolute values of the terms, the same does not hold for addition.

Example 2.55. $|(-3) + 2| = |-1| = 1 \neq |-3| + |5| = 8$. To be more precise, $|(-3) + 2| = 1 < 8 = |-3| + |5|$.

So, while it is clear from the above example the the absolute value of a sum of two real numbers is not necessarily equal to the sum of their absolute values, perhaps we may hope to still be able to say something. What the second part of the example suggests is that Is this a general property of the absolute value over the real numbers?

Indeed, it is. A deep property of the absolute value is the so-called triangle inequality, whose name is rooted in geometric considerations that we already clear at the times of Euclid.

Question 2.56. Can you draw a triangle with sides of length 1, 4, and 600?

I do not think so. On the other hand, it is possible to draw a triangle whose sides have length 3,4 and 6 (give it a try, you might need a compass).

What kind of constraints should we place on The reason is that for every triangle, the sum of the length of two edges is always bigger then the length of the third edge. This implies a triangle inequality for the absolute value, we will understand better the relation with triangles when dealing with complex numbers, let us give a couple of examples now:

$$|3 + (-7)| \leq |3| + |-7|$$

and

$$|(-5) + (-4)| \leq |-5| + |-4|$$

In general, we can prove the following.

Proposition 2.57 (Triangle inequality). *For all* $x, y \in \mathbb{R}$

$$|x+y| \leq |x| + |y|.$$

Proof. Recall that $x \leq |x|$ and $y \leq |y|$. So, if $x+y \geq 0$, then $|x+y| = x+y \leq |x|+|y|$.

Similarly, $x \geq -|x|$ and $y \geq -|y|$. So, if $x+y \leq 0$, then $|x+y| = -x-y \leq |x|+|y|$. $\square$

Exercise 2.58. Prove that for any $x, y \in \mathbb{R}$

$$|x-y| \geq ||x|-|y||$$

3 Complex numbers

When we work over the real numbers, we will be often working with functions of the form $f\colon \mathbb{R} \to \mathbb{R}$. We will be intersted in understanding and studying the properties (e.g., derivatives, integrals, monotonicity) of certain classes of functions (e.g., continuous, differnetiable, integrable functions). Oftentimes, we will also be interested in understanding if and when a function $f\colon \mathbb{R} \to \mathbb{R}$ attains a specific value $c \in bR$. Let us give an example.

Example 3.1. Imagine that we are observing a particle moving along a linear rod. We can model the linear rod with the real line. We would like to keep track of how the particle moves as a function of time. Hence, we can think of the position of the particle as a function $p\colon \mathbb{R} \to \mathbb{R}$ defined as follows

$$p(t) := \text{position of the particle along the line at time } t.$$

We can assume that at time $t = 0$ (the starting time of our observation) the particle is placed at the origin. Let us assume that we also know that at time $t = 0$ the particle is moving with velocity v[11]. If no outer forces act on the particle, then the velocity of the particle stays constant and the position can be easily written in terms of time in the form $p(t) = v \cdot t$.
Let us assume instead that we know that there is there is a force acting on the particle and that force applies a (constant) deceleration to the particle of magnitude a directed in the opposite verse than that of the velocity. In this casem then the position of the particle is given by $p(t) = -\frac{1}{2}at^2 + vt$. Hence, if we wanted to know whether at a certain point in time the particle passes at a fixed point $c \in \mathbb{R}$ on the rod, we have to solve the equation

$$p(t) = c$$

which we can rewrite as

$$-\frac{1}{2}at^2 + vt - c = 0 \quad \text{or, equivalently} \quad at^2 - 2vt + 2c = 0,$$

where the second equality follows from the first by flipping the signs and multiplying the first equation by 2. We rewrite the above equation in the form

$$aT^2 - 2vT + 2c = 0, \tag{3.1.a}$$

where a, v, c are fixed real numbers, while the unknown that we are trying to compute is given by T. As you probably already know, the above equation admits the following two real solutions

$$t_1 = \frac{2v + \sqrt{4v^2 - 8ac}}{2a}, \quad t_2 = \frac{2v - \sqrt{4v^2 - 8ac}}{2a},$$

provided that the quantity $4v^2 - 8ac \geq 0$ (since the square root of a real number is well defined only for non-negative real numbers). If $4v^2 - 8ac < 0$, then we cannot possibly find any real solution to (3.1.a)

How do we remedy the lack of solutions for polynomial equations in the real numbers?

Polynomials are a big and relatively simple class of functions that appear rather naturally in many contexts. Hence, it would be nice to know that we can always find solutions to polynomial equations. On the other hand, the above example tells us that this is not possible, if we just work with real numbers. The solution to this problem is a classic piece of mathematical wisdom. When you are lacking a tool, why not invent it yourself? This is the idea behind the definition of the complex numbers that we now proceed to explain.

[11] Here v could have both positive or negative sign, meaning that the particle is moving in the direction of the positive real numbers or in the direction of the negative ones, along the linear rod.

3.1 Definition

As discussed in the previous section, one obstruction to finding real solutions already for quadratic equations is the lack, within the real numbers, of the square root for negative real numbers.

To define the complex numbers, we introduce a new number i called *the imaginary unit.* The number i is the square root of -1, that is, i satisfies the property

$$i^2 = -1. \tag{3.1.a}$$

or, equivalently, i is a solution of the equation $T^2 + 1 = 0$ in the indeterminate T.

The introduction of the imaginary unit i can be compared, in terms of the philosophical leap that it required for its ideation, to the introduction of 0, or of the negative numbers. It is remarkable that the equation $x^2 = -1$ has no solution in the set of real numbers, but two distinct solutions in the set of complex numbers, namely i and $-i$.

The complex numbers can be intuitively defined as all those numbers that can be created by using the real numbers and the usual operations $(+, -, \cdot, /)$, together with i, keeping in mind the relation in (3.1.a). Let us give a more formal definition of the complex numbers.

Definition 3.2. (1) A *complex number* is an expression of the form $x + yi$, where x, y are real numbers, and i is the imaginary unit defined above.

(2) The set of complex numbers is denoted by $\mathbb{C}$.

Thus,

$$\mathbb{C} := \{x + yi \mid x, y \in \mathbb{R},\ i^2 = -1\}.$$

Often elements of $\mathbb{C}$ are denoted with the letter z.

Definition 3.3. Let $z = x + iy$ be a complex number.

(1) The real part $\mathrm{Re}(z)$ of z is the real number x.

(2) The imaginary part $\mathrm{Im}(z)$ of z is the real number y.

We will write $z = x + yi$ when we want to remind ourselves the real and imaginary part of z.

Remark 3.4. When we write a complex number z whether we write it in the form $x + yi$, $x, y \in \mathbb{R}$, or in the form $x + iy$,both representations stand for the same complex number, as the imaginary unit i commutes with all real numbers; that is,

$$s \cdot i = i \cdot s, \quad \forall s \in \mathbb{R}.$$

Considering the notation for complex numbers introduced in Definition 3.2, in the form $x + yi$, taking $y = 0$ and letting x vary in $\mathbb{R}$, we immediately obtain that $\mathbb{R} \subseteq \mathbb{C}$. As $i \notin \mathbb{R}$, by the definition of i, cf. (3.1.a), then we can be even more precise and write $\mathbb{R} \subsetneq \mathbb{C}$.

Example 3.5. (1) The real numbers 0, 3, and $-\pi$ are complex numbers.

(2) Other examples of complex numbers are $5 - i$, $3i$, $-2i$ and $\frac{1}{2} + \sqrt{2}i$.

(3) $\mathrm{Re}(5 + 3i) = 5$, $\mathrm{Im}(5 + 3i) = 3$; $\mathrm{Re}(-3i) = 0$, $\mathrm{Im}(-3i) = -3$.

Complex numbers are not ordered: it makes no sense to ask if a complex number is bigger than another; in particular, it does not make sense to ask if a complex number is positive or negative

3.2 Operations between complex numbers

We can add and multiply complex numbers using the standard formal properties of addition and multiplication, always remembering that $i^2 = -1$.

Example 3.6. (1) $(5+3i)+(2-i) = (2+5)+(3-1)i = 7+2i$. In general:

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i.$$

(2) $(1-2i)(3+4i) = 3 - 6i + 4i - 8i^2 = 3 - 6i + 4i + 8 = 11 - 2i$. In general:

$$(x_1 + y_1 i) \cdot (x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i.$$

In the previous section we defined complex numbers as those numbers that we can write in the form $x + yi$, with $x, y \in \mathbb{R}$. In particular, it follows from our definition that any complex number $z \in \mathbb{C}$ is completely determined by its real and imaginary part. Hence, we could think of parametrizing all complex numbers by means of their real and imaginary part. This is indeed possible, as shown in Figure 2. We identify the set of complex numbers with the points in the Cartesian plane, which we will in this case rename the *complex plane*.

`../Images/complex-plane1.png`

Figure 2: The complex plane.

Thus, thus for a complex number of the form $z = x + yi$, we will use the real part x (resp. the imaginary part y) as the cartesian coordinates of z in the complex plane. Then, the line $\{y = 0\}$ in the complex plane is automatically identified with the set of real numbers within the complex numbers. For this reason, this line is called the *real axis*. The line $\{x = 0\}$ in the complex plane identifies instead with the set of complex numbers whose real part is 0. Numbers of this form are called *purely imaginary* numbers. For this reason, the line $\{x = 0\}$ is called the *imaginary axis*.

Using this representation complex numbers become vectors, and the sum of complex numbers is equal to the sum of vectors, as in Figure 3. Moreover, multiplication of a complex number z by a positive real number $r > 0$ corresponds to scaling the length of the vector representing z by the factor r.

Definition 3.7. The *modulus* (or, *absolute value*) $|z|$ of a complex number $z \in \mathbb{C}$ is its distance from the origin in the complex plane. It is computed using the Pythagorean Theorem in terms of the real and imaginary part of z:

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}.$$

Images/sum_complex.png

Figure 3: Sum of complex numbers as vectors

Example 3.8. (1) $|3-2i| = \sqrt{3^2+2^2} = \sqrt{25} = 5$,

(2) $|-3i| = \sqrt{3^2+0^2} = 3$

Using the representation of a complex number $z \in \mathbb{C}$ as $z = x + yi$, then the formula for the modulus $|z|$ of z can be written as

$$|z| = |x + yi| = \sqrt{x^2+y^2}.$$

As we can represent the addition of complex numbers as addition of the corresponding vectors, we can derive from this the classical triangle inequality

$$\forall z, w \in \mathbb{C}, \qquad |z+w| \le |z| + |w| \tag{3.8.a}$$

cf. Figure 4.
With reference to the picture, we can compose a triangle using the vector connecting the origin to z_1 (corresponding to the side of the triangle in the picture of length C), the vector connecting the origin to z_1+z_2 (corresponding to the side of length AC in the picture), and the translation of the vector connecting the origin to z_2, where we have moved the starting point of the vector to z_1 (this corresponds to the side of length B in the picture). The classical triangle inequality tell us that $A \le B+C$. But given the way we constructed the triangle, this inequality translates to

$$|z+w| \le |z| + |w|. \tag{3.8.b}$$

Definition 3.9. The conjugate $\overline{z}$ of a complex number $z = x + yi$ is defined as the complex number $\overline{z} = \overline{x+iy} := x - iy$.

Hence, the conjugate of z is simply obtained by changing the sign of the imaginary part of z. It is important to understand that geometrically in the complex plane this corresponds to reflection across the real line.

Example 3.10. $\overline{3-4i} = 3+4i$, $\overline{3i} = -3i$, $\overline{1} = 1$.

Images/tr_ineq.png

Figure 4: Triangle inequality.

Conjugation is compatible with all operations by explicit computation: namely,for $z_1, z_2 \in \mathbb{C}$, $z_3 \in \mathbb{C}^*$,

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\
\overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2}, \\
\overline{\left(\frac{z_1}{z_3}\right)} &= \frac{\overline{z_1}}{\overline{z_3}}.
\end{aligned}$$

To verify the formulas above, it suffices to write all numbers involved as $x + iy$ and expand all the expressions obtained.

Similarly, we can use conjugation also to compute the modulus of a complex number:

$$z\overline{z} = (x + iy)\overline{(x + iy)} = (x + iy)(x - iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2 = |z|^2$$

Hence, if $z \neq 0$, we can use the formula above to show that any such $z \in \mathbb{C}$ has a multiplicative inverse, that is, z^{-1} exists[12] and moreover it can be computed as

$$z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}. \tag{3.10.c}$$

[12] By z^{-1} we denote the (unique) complex number that $z \cdot z^{-1} = 1 = z^{-1} \cdot z$.

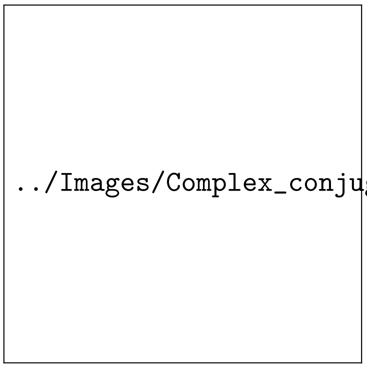

Figure 5: Conjugate of a complex number.

We can use the above formula to better understand division between two complex numbers. Given two complex numbers z and w, with $w \neq 0$, we would like explicitly write $\frac{z}{w}$ in the form $x + yi$.

Example 3.11. We can try to turn the denominator of the fraction into a real number by multiplying with the conjugate of w, both above and below.

$$\frac{2-3i}{5+i} = \frac{(2-3i)(5-i)}{(5+i)(5-i)} = \frac{7-17i}{26} = \frac{7}{26} - \frac{17}{26}i$$

In fact, we can write down a general formula using (3.10.c):

$$\frac{z}{w} = \frac{z\overline{w}}{\overline{w} \cdot w} = \frac{z\overline{w}}{|w|^2}.$$

Example 3.12. Here is another example.

$$\frac{1}{3-\sqrt{3}i} = \frac{3+\sqrt{3}i}{12} = \frac{1}{4} + \frac{\sqrt{3}}{4}i, \text{ or}$$
$$\frac{i}{1-i} = \frac{i(1-i)}{2} = \frac{1}{2} + \frac{1}{2}i$$

We also have the following relation between conjugation, real part and imaginary part

$$\operatorname{Re}(z) = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad \operatorname{Im}(z) = \frac{1}{2i}(z - \overline{z}).$$

3.3 Polar form

We can associate to every non-zero complex number $z \in \mathbb{C}$ an angle, called *the argument* or *the phase* of z, and denoted $\arg z$, in the following way. In the complex plane, we take the the angle formed by the half line $\mathbb{R}_+$ of the non-negative real numbers and the half-line L_z spanned by the vector connecting the origin to z. For example, in Figure 6, the angle $\arg z$ has been denoted with ϕ. The argument $\arg z$ is then defined as the angle between $\mathbb{R}_+$ and L_z, moving in the anti-clockwise direction.

Example 3.13. $\arg 3 = 0$; $\arg i = \frac{\pi}{2}$; $\arg \frac{\sqrt{2}}{2}(1+i) = \frac{\pi}{4}$.

Take now a non-zero complex number z, we have seen that its distance from the origin is $|z|$. Let ϕ be its argument. The number $\frac{z}{|z|}$ has distance 1 from the origin, so it lies on the trigonometric (or, unit) circle

$$\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\} = \{x + yi \in \mathbb{C} \mid x, y \in \mathbb{R}, \text{ and } x^2 + y^2 = 1\}.$$

Hence, the real part of $\frac{z}{|z|}$ (resp. the imaginary part of z) is just $\cos(\phi)$ (resp. $\sin(\phi)$), where ϕ is the angle (measured in radiants) formed by the positive part of the real axis and the half line passing through the origin and the point z on the complex plane, cf. Figure 6.

../Images/complex_angle.png

Figure 6

Thus, under this assumptions, we conclude that

$$z = |z|(\cos(\phi) + \sin(\phi)i). \tag{3.13.a}$$

The expression of a complex number $z \in \mathbb{C}$ given in (3.13.a) is called the *polar form* of z. It is a very important and useful way to represent complex numbers, as we will see below. Conversely, when we write a complex number z in the form $x + iy$, we say that we are using the Cartesian form, or Cartesian representation. Let us note that because of the presence of cos and sin, one can add any multiple of 2π to the argument on the right hand side.

Example 3.14. The polar form of $1 + i$ is $\sqrt{2}\left(\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})i\right)$

The multiplication of complex numbers becomes simple if we use the polar form and we use some well-known trigonometric identities.

Example 3.15. Let ϕ and ψ be two numbers. Then

$$\begin{aligned}
&(5(\cos(\phi) + \sin(\phi)i))(3(\cos(\psi) + \sin(\psi)i)) = \\
=&15(\cos(\phi)\cos(\psi) - \sin(\phi)\sin(\psi)) + (\cos(\phi)\sin(\psi) + \sin(\phi)\cos(\psi))i = \\
=&15(\cos(\phi + \psi) + \sin(\phi + \psi)i),
\end{aligned}$$

where we have used the addition formulas for sine and cosine

$$\begin{aligned}
\cos(\phi + \psi) &= \cos(\phi)\cos(\psi) - \sin(\phi)\sin(\psi), \\
\sin(\phi + \psi) &= \cos(\phi)\sin(\psi) + \sin(\phi)\cos(\psi).
\end{aligned} \tag{3.15.b}$$

Thus, the example above can be immediately generalized to show that for two non-zero complex numbers $z_1, z_2 \in \mathbb{C}$, then $\arg z_1 \cdot z_2 = \arg z_1 + \arg z_2$, while we already saw that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$,

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot (\cos(\arg z_1 + \arg z_2) + \sin(\arg z_1 + \arg z_2)i). \tag{3.15.c}$$

Thus, when we multiply two non-zero complex numbers, the modulus of the product is the product of the moduli and the argument of the product is the sum of the arguments!

Example 3.16. $\left|\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| = 1$, and $\arg\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{\pi}{3}$. Thus,

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{2017} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right).$$

because $1^{2017} = 1$, so the absolute values does not change; then $2017 = 336 \cdot 6 + 1$, so $2017 \cdot \frac{\pi}{3} = 336 \cdot 2\pi + \frac{\pi}{3}$, so also the argument does not change.

The above example shows that the polar form is really useful to compute, for example, powers of complex numbers.

We can also use the polar form to divide complex numbers. As with multiplication when the moduli (plural of the modulus) multiplied and the arguments added up, with division, we have to do the inverse. That is, the absolute value of a fraction is the fraction of the absolute values and its argument is just the difference of the arguments:

$$\frac{z}{w} = \frac{|z|(\cos(\phi) + \sin(\phi)i)}{|w|(\cos(\psi) + \sin(\phi)i)} = \frac{|z|}{|w|}(\cos(\phi - \psi) + \sin(\phi - \psi)i).$$

Example 3.17. Let $z \in \mathbb{C}$ be given in polar form by

$$z := 3\left(\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)\right)$$

Then the inverse of z is

$$\begin{aligned} z^{-1} =& \frac{1}{3}\left(\cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right)\right) \\ =& \frac{1}{3}\left(\cos\left(2\pi - \frac{2\pi}{7}\right) + i\sin\left(2\pi - \frac{2\pi}{7}\right)\right) \\ =& \frac{1}{3}\left(\cos\left(\frac{12\pi}{7}\right) + i\sin\left(\frac{12\pi}{7}\right)\right). \end{aligned}$$

3.4 Euler formula

We can write the polar form of a non-zero complex number z in an even more compact form.

Definition 3.18 (Euler's formula)**.** Let ϕ be a real number. We define

$$e^{i\phi} := \cos(\phi) + i\sin(\phi) \tag{3.18.a}$$

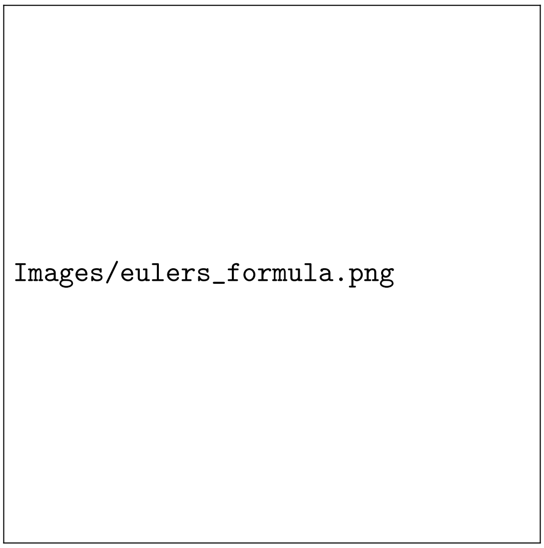

Figure 7: Euler's formula

We will treat the Euler formula above as a formal definition, a shorten notation to describe the points on the unitary circle. At this point, we have not developed the tools to actually discuss the mathematics behind this formula, as we have not defined exponentiation for complex numbers. So, for now, just think about it as a shortcut for the part of the polar form depending on the argument.

As an immediate consequence of Definition 3.18, we have the following properties.

Proposition 3.19. *Let $\phi, \psi \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then,*

(1) $e^{i\phi} \cdot e^{i\psi} = e^{i(\phi+\psi)}$*;*

(2) $e^{i\phi+2k\pi} = e^{i\phi}$.

Proof. (1) Use the trigonometric formulas in (3.15.b).

(2) As we measure angles in radiants, this is a simple consequence of the 2π-periodicity of the sine and cosine functions.

□

We have mentioned above that we can use Euler's formula to write the polar form of z in a more compact form than the one introduce in (3.13.a). Indeed, in view of Definition 3.18, we can rewrite the polar form of z as

$$z = |z|(\cos(\phi) + \sin(\phi)i) = |z|e^{i\phi}.$$

Example 3.20. Let $z = 1 + i$. Then

$$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}e^{i\frac{\pi}{4}}.$$

We can rewrite the formula for multiplication of polar forms, (3.15.c), as

$$zw = \left(|z|e^{i\phi}\right)\left(|w|e^{i\psi}\right) = |z||w|e^{i(\phi+\psi)}. \tag{3.20.b}$$

3.5 Finding solutions of equations with complex coefficients

Example 3.21. The usual formula to compute solutions of quadratic equations works in the complex numbers exactly as it worked in the real numbers. [The proof is the same as usual, verify that!]
For example, the solutions of the equation

$$T^2 + 2T + 3 = 0$$

in the indeterminate T are given by

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$$

More in general, given a quadratic equation of the form

$$aT^2 + bT + c = 0, \quad a, b, c \in \mathbb{C}, \quad a \neq 0,$$

in the indeterminate T, then this equation always admits exactly two complex solutions (that may possibly coincide) The two solutions are computed as

$$t_1 = \frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad t_2 = \frac{b - \sqrt{b^2 - 4ac}}{2a}.$$

In the above expression, what do we mean when we write $\sqrt{b^2 - 4ac}$, with $a, b, c \in \mathbb{C}$? By definition of square root as the inverse operation to taking the second power of a given number, $\sqrt{b^2 - 4ac}$ denotes those complex numbers $t \in \mathbb{C}$ satisfying $t^2 = b^2 - 4ac$. There are indeed two distinct such numbers in $\mathbb{C}$ that satisfy this equation, when $b^2 - 4ac \neq 0$: given a complex number t is a root of the equation $W^2 - (b^2 - 4ac) = 0$ (in the indeterminate W), also $-t$ will be a root of that same equation. Hence we can rewrite the formulas for the two solutions as

$$t_1 = \frac{b+t}{2a}, \quad t_2 = \frac{b-t}{2a}, \quad \text{where } t^2 = b^2 - 4ac.$$

As we work over the complex numbers, it is not hard to see – and we shall see it below – that we can always solve the equation $t^2 = b^2 - 4ac$, where t is the unknown and $a, b, c \in \mathbb{C}$ are fixed constants. This is not true in $\mathbb{R}$, where the equation $T^2 = -1$ does not have solutions since for any real number t, $t^2 \geq 0$.

The main importance of complex numbers is that any polynomial equation has a solution among the complex numbers. This result is called the *fundamental theorem of algebra.*

Let us recall that given a polynomial $p(T)$ in the indeterminate T, a root of $p(T)$ is number c such that $p(c) = 0$.

Theorem 3.22 (Fundamental Theorem of Algebra)**.** *Let $p(T)$ be a polynomial in the indeterminate T with complex coefficients. Then, there exists a root $t \in \mathbb{C}$ of $p(T)$.*

Let us also recall that if t_1 is a root of a polynomial $p(T)$, then we can factor $p(T)$ as a product

$$p(T) = (T - t_1) \cdot q(T)$$

where $q(T)$ is another polynomial with complex coefficients and the degree $\deg q(T)$ of $q(T)$ satisfies $\deg q(T) = \deg p(T) - 1$. But then, in turn, Theorem 3.22 implies that also $q(T)$ possesses a complex root t_2, $q(t_2) = 0$. In turn, then

$$q(T) = (T - t_2) \cdot r(T)$$

where $r(T)$ is another polynomial with complex coefficients and $\deg r(T) = \deg q(T) - 1$. Moreover

$$p(T) = (T - t_1) \cdot (T - t_2) \cdot r(T).$$

Iterating, we obtain that we can factor any degree d polynomial $p(T)$ into d linear factors, that is, there d complex numbers $t_1, \ldots, t_d$ (some of them may happen to coincide) such that

$$p(T) = (T - t_1)(T - t_2)(T - t_3) \ldots (T - t_{d-1})(T - t_d). \tag{3.22.a}$$

The *multiplicity* of a root t of $p(T)$ is equal to the number of elements of the set of roots $\{t_1, \ldots, t_d\}$ that are equal to t.

Example 3.23. Let $p(T) = T^3 + (-1 + 2i)T^2 + (-1 - 2i)T + 1$. Then $p(1) = 1 + (-1 + 2i) - (1 + 2i) + 1 = 0$ and

$$p(T) = (T - 1)(T^2 + 2iT - 1) = (T - 1)(T - i)^2.$$

Hence, the roots of $p(T)$ are 1 and i, where the multiplicity of i is 2.

The Fundamental Theorem of Algebra, Theorem 3.22, is equivalent to the statement that any polynomial $p(T)$ with complex coefficients admits a factorization into linear polynomials as in (3.22.a).

We have already discussed at length how the equation $T^2 = -1$ has no solutions in $\mathbb{R}$, whereas it has two distinct solutions in the complex numbers, $z = i$ and $z = -i$. Thus, this implies that the Fundamental Theorem of Algebra cannot possibly work over $\mathbb{R}$. Nonetheless, we can still give derive some nice properties.

Theorem 3.24 (Fundamental Theorem of Algebra over $\mathbb{R}$)**.** *Let $p(T)$ be a polynomial in the indeterminate T with real coefficients.*

(1) If the degree $\deg p(T)$ of $p(T)$ is odd, then there exists a real root $t \in \mathbb{R}$ of $p(T)$.

(2) In general, if $t \in \mathbb{C}$ is a complex root of $p(T)$, then also $\bar{t} \in \mathbb{C}$ is a complex root. Moreover, the multiplicity of t and $\bar{t}$ coincide as roots of $p(T)$.

3.5.1 Solving complex equations

We are going to learn how to solve some slightly more general equations, thanks to the polar form.

Problem 3.25. For $a \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$, how many solutions does the equation $T^n = a$ in the indeterminate T has? Can we compute them all?

Using what we discussed in the first part of Section 3.5, it is immediate to see that the equations $T^n = a$ has exactly n solutions in the complex numbers.

Example 3.26. When an equation is simple, we can try to solve it directly, by computing the real and imaginary parts of a solutions separately.
For example, let us consider the equation in the indeterminate T

$$T^2 = i$$

We are searching for a complex number of the form $X + iY$, where X, Y are indeterminates with real values, such that $(X + iY)^2 - i = 0$. We can rewrite this equation in the form $(X^2 - Y^2) + i(2XY) - i = 0$ which yields two distinct equations with real coefficients, by separating real and imaginary part:

$$\begin{cases} X^2 - Y^2 = 0 \\ 2XY = 1 \end{cases} .$$

From the first equation we deduce that $X = \pm Y$. We can substitute the two relations $X = Y$ and $X = -Y$ into the second equation above to obtain:

$$\begin{aligned} X = Y &\Longrightarrow X \cdot (X) = \frac{1}{2} \Longrightarrow X = \pm\frac{\sqrt{2}}{2} \\ X = -Y &\Longrightarrow X \cdot (-X) = \frac{1}{2} \Longrightarrow X^2 = -\frac{1}{2} \quad , \end{aligned}$$

and this last equation cannot be solved in $\mathbb{R}$. Thus, the roots of the equation $T^2 - i = 0$ are

$$\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \quad \text{and} -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

To check that indeed those are solution of the equations, it suffices to compute $\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 - i$ and $\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^2 - i$.

The method that we have just seen is a rather inefficient one for solving Problem 3.25 as it requires us to resolve two equations instead of one, and these two equations will have real coefficients and we have to find real solutions for those (which, a priori, we do not know if it is always possible).

Instead, we are going to take a slightly different approach, which is more natural given that we are working in the field of complex numbers: the main idea is to write a in exponential form and do some reasoning with respect to both its modulus and its argument.

Hence, writing $a \in \mathbb{C}$ in its polar form $a = Re^{i\phi}$, where R is a positive real number, and ϕ is an angle expressed in radiants, we can instead try to find all solutions to the equation

$$T^n = Re^{i\phi} \tag{3.26.b}$$

where T is the unknown and n is a fixed positive integer. In particular, we shall assume that $\phi \in [0, 2\pi)$. All solutions of (3.26.b) are of the form

$$t = \sqrt[n]{R}e^{i\psi},$$

where ψ is an angle such that $e^{in\psi} = e^{i\phi}$. Since R is a positive real number, then $\sqrt[n]{R}$ is well defined and it is in turn positive and real, which means that $\sqrt[n]{R}e^{i\psi}$ is the polar form of a complex number (uniquely determined by R and ψ).

How do we compute all possible choices that we have for ϕ? Since $e^{in\psi} = e^{i\phi}$, we must have that

$$n\psi = \phi + 2k\pi, \quad \text{for some } k \in \mathbb{Z},$$

by Proposition 3.19. Thus, $\psi = \frac{\phi}{n} + \frac{2k\pi}{n}$, where k is an integer between 0 and $n-1$, so that the above equation has always exactly n distinct solutions.

Example 3.27. The equation

$$z^2 = 3e^{i\frac{\pi}{5}}$$

has two solutions:

$$z = \sqrt{3}e^{i\frac{\pi}{10}} \quad \text{and} \quad z = \sqrt{3}e^{i\left(\frac{\pi}{10}+\pi\right)}$$

The equation

$$z^3 = 27e^{i\frac{\pi}{7}}$$

has three solutions:

$$z = 3e^{i\frac{\pi}{21}}, \quad z = 3e^{i\left(\frac{\pi}{21}+\frac{2\pi}{3}\right)}, \quad \text{and} \quad z = 3e^{i\left(\frac{\pi}{21}+\frac{4\pi}{3}\right)}.$$

4 Sequences

Definition 4.1. A sequence is a function $x : \mathbb{N} \to \mathbb{R}$.

Traditionally, we denote the value of the function x at $n \in \mathbb{N}$ by x_n, that is, $x_n := x(n)$. We denote instead by (x_n) the whole sequence.

Let us start by looking at a few simple examples of sequences.

Example 4.2. (1) Let us fix a real number $C \in \mathbb{R}$. Then the constant sequence of value C is the sequence (x_n) defined as follows

$$x_n := C \quad \forall n \in \mathbb{N}.$$

(2) *Arithmetic progression*: let a, b be real numbers; we define sequence (x_n) by

$$x_0 := a,\ x_1 := a + b,\ x_2 := a + 2b,\ \dots,\ x_n := a + nb,\ \dots.$$

We call the type of sequence just constructed an arithmetic progression. For example, the arithmetic progression given by $a = 1$ and $b = 2$ is $x_0 = 1, x_1 = 3, x_2 = 5, \dots$; this particular arithmetic progression takes up as values all the positive odd numbers.

(3) *Geometric progression*: let a, q be real numbers; we define a sequence (x_n) by

$$x_0 := a,\ x_1 := aq,\ x_2 := aq^2,\ \dots,\ x_n := aq^n,\ \dots.$$

We call the type of sequence just constructed an geometric progression. For example, the geometric progression given by $a = 2$ and $b = \frac{4}{5}$ is

$$x_0 = 2,\ x_1 = 2 \cdot \frac{4}{5} = \frac{8}{5},\ x_2 = 2 \cdot \left(\frac{4}{5}\right)^2 = \frac{32}{25},\ x_3 = 2 \cdot \left(\frac{4}{5}\right)^3 = \frac{128}{125}, \dots.$$

(4) Let (x_n) be the sequence defined by $x_n := (-1)^n$. Then the sequence only takes two values:

$$x_n = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Notation 4.3. At times, it may happen that the terms of a sequence (x_n) are not defined for all natural number values of the index n. For example, the sequence (x_n) defined as

$$x_n := \frac{1}{n}$$

is only well-defined when $n \neq 0$.

In discussing sequences (and their limits, or lack thereof), we will mostly be concerned with properties of a sequence which are eventually true. That means that we will look for properties of a sequence (x_n) that hold starting from a certain index $l \in \mathbb{N}$ and then holds also for all the indices $> l$. Hence, what will matter for us is that all terms of a sequence (x_n) are defined for all values of n greater or equal of a given natural number $l \in \mathbb{N}$.

Hence, when we want to highlight that a sequence the terms of a sequence (x_n) are defined for all $n \geq l \in \mathbb{N}$, we will write

$$(x_n)_{n \geq l}$$

When we can take $l = 0$, we will also write $(x_n)_{n \in \mathbb{N}}$. When we omit the subscript $n \geq l$, i.e., when we write (x_n), we simply are not specifying what is the initial index starting from which the sequence is defined.

Similarly to what we did for the case of subset of $\mathbb{R}$, we would like to define the concept of boundedness, boundedness from above/below also in the case of sequences. To this end, it suffices to notice that given a sequence $(x_n)_{n\geq l}$, then it uniquely defines a subset $S \subset \mathbb{R}$ given by all the values that the sequence takes,

$$S := \{x_n \mid n \in \mathbb{N},\ x \geq l\}. \tag{4.3.a}$$

We can then use S to make sense of the concept of boundedness for a sequence, as follows.

Definition 4.4. Let $(x_n)_{n\geq l}$ be a sequence. We say that $(x_n)_{n\geq l}$ is $\begin{cases} \text{bounded from above,} \\ \text{bounded from below,} \\ \text{bounded,} \end{cases}$

if the set $\{x_n \mid n \in \mathbb{N},\ x \geq l\}$ of values of the sequence is $\begin{cases} \text{bounded from above,} \\ \text{bounded from below,} \\ \text{bounded,} \end{cases}$ respectively.

It is an immediate consequence of Definition 2.8 that a sequence $(x_n)_{n\geq l}$ is bounded if and only if it is both bounded from above and below.

Remark 4.5. Let $(x_n)_{n\geq l}$ be a sequence. Then $(x_n)_{n\geq l}$ is bounded if and only if there exists a positive real number C such that the set of values of the sequence is a subset of the interval $[-C, C]$. In particular, $(x_n)_{n\geq l}$ is bounded if and only the sequence $(y_n)_{n\geq l}$, defined by $y_n := |x_n|$ is bounded, too.

Example 4.6. (1) Let $(x_n)_{n\in\mathbb{N}}$ be the constant sequence of value C.
The set of value of this sequence is the singleton set $\{C\} \subset \mathbb{R}$.

(2) Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_n := (-1)^n$, cf. Example 4.2.4. Then the set of values of this sequence is $\{x_n \in \mathbb{R} \mid n \in \mathbb{N}\}$ and it coincides with set $\{-1, 1\} \subset \mathbb{R}$. As S is a finite subset of $\mathbb{R}$, it follows that it is bounded and possesses both maximum and minimum, 1 and -1, respectively.

(3) Let $(x_n)_{n\in\mathbb{N}}$ be an arithmetic progression with $a = 0, b = 2$. Then

$$\{x_n \mid n \in \mathbb{N}\} = \{2n \mid n \in \mathbb{N}\}$$

where the latter is the set of even numbers. In particular, (x_n) is not bounded.

We have also the following definitions focusing on the behavior of a sequence (x_n) in the terms both of ordering of the indices of the sequence, which vary in $\mathbb{N}$, and of the ordering of the values of the sequence, which instead vary in $\mathbb{R}$.

Definition 4.7. Let $(x_n)_{n\geq l}$ be a sequence.

(1) We say that

$$(x_n)_{n\geq l} \text{ is } \begin{cases} \textit{increasing} \\ \textit{strictly increasing} \\ \textit{decreasing} \\ \textit{strictly decreasing} \end{cases} \quad \text{if for each } n \in \mathbb{N},\ n \geq l, \begin{cases} x_n \leq x_{n+1} \\ x_n < x_{n+1} \\ x_n \geq x_{n+1} \\ x_n > x_{n+1} \end{cases}.$$

(2) We say that

$$(x_n)_{n\geq l} \text{ is } \begin{cases} \textit{monotone,} \\ \textit{strictly monotone,} \end{cases} \quad \text{if } (x_n) \text{ is } \begin{cases} \textit{increasing or decreasing} \\ \textit{strictly increasing or strictly decreasing} \end{cases}.$$

Example 4.8. (1) Let $C \in \mathbb{R}$ and let $(x_n)_{n\in\mathbb{N}}$ be constant sequence of value C. Then $\{x_n \mid n \in \mathbb{N}\} = \{C\}$. Hence $(x_n)_{n\in\mathbb{N}}$ is bounded.

(2) Let $(x_n)_{n\in\mathbb{N}}$ be an arithmetic progression of the form $x_n := a + nb$, $a, b \in \mathbb{R}$. Then,

  (i) the sequence is constant sequence of value a if and only if $b = 0$;
  (ii) the sequence is increasing if and only if $b \geq 0$: indeed, $x_{n+1} = x_n + b$. The sequence is strictly increasing if and only if $b > 0$;
  (iii) analogously, the sequence is decreasing if and only if $b \leq 0$. It strictly decreasing if and only if $b < 0$;
  (iv) the sequence is bounded from below if and only if $b \geq 0$: indeed, in that case, we already know that $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$, thus, $x_n \geq x_0$ $\forall n \in \mathbb{N}$ and x_0 is a lower bound for the set of values of the sequence;
  (v) the sequence is bounded from above if and only if $b \leq 0$: indeed, in that case, we already know that $x_{n+1} \leq x_n$, $\forall n \in \mathbb{N}$, thus, $x_n \leq x_0$ $\forall n \in \mathbb{N}$ and x_0 is an upper bound for the set of values of the sequence;
  (vi) the sequence is bounded if and only if $b = 0$: indeed, $(x_n)_{n\in\mathbb{N}}$ is bounded if and only if it is both bounded from above and below. But that is possible if and only if $b = 0$.

(3) Let $(x_n)_{n\in\mathbb{N}}$ be an arithmetic progression of the form $x_n := aq^n$, $a, q \in \mathbb{R}$. Then,

  (i) if $a = 0$ or $q = 0$, $x_n = 0$, for all $n \in \mathbb{N}$;
  (ii) if $q = 1$, $x_n = a$, for all $n \in \mathbb{N}$;

  Hence, in both these cases, $(x_n)_{n\in\mathbb{N}}$ is a constant sequence. We will assume that $a \neq 0$, $q \neq 1$.

  (iii) $q = -1$, then $x_n = (-1)^n a$. This sequence is bounded but not monotone;
  (iv) if $q = \frac{1}{2}$, then $x_n = \frac{a}{2^n}$. This sequence is strictly decreasing and bounded;
  (v) if $q = -\frac{1}{2}$, then $x_n = \frac{(-1)^n a}{(2)^n}$. This sequence is bounded but not monotone;
  (vi) if $q = 2$, then $x_n = 2^n$. This sequence is strictly increasing and bounded from below;
  (vii) $q = -2$, then $x_n = (-2)^n$. This sequence is neither bounded nor monotone.

  We will analyze in general for what values of a and q the sequence (x_n) is bounded in Examples 4.14 and 4.20.

(4) The sequence $(x_n)_{n\geq 1}$ defined by $x_n := 5 - \frac{1}{n}$ is strictly increasing. In fact, for all $n \in \mathbb{N}^*$, $\frac{1}{n} > \frac{1}{n+1}$, hence $x_{n+1} > x_n$.

4.1 Recursive sequences

We say that a sequence (x_n) is *recursive* if the n-th term of the sequence x_n is defined by a formula $f(x_{n-1}, \dots, x_{n-j})$ which depends on the previous terms $x_{n-1}, \dots, x_{n-j}$ of the sequence, for some fixed integer $j > 0$ – here, j does not depend on n. We also require that the formula $f(\dots)$ is fixed, i.e. it does not depend on n. For this definition to make sense, we will also have to fix the values of $x_0, x_1, \dots, x_{j-1}$ as those cannot be established using the formula otherwise.

Notation 4.9. We will denote a recursive sequence (x_n) defined by $x_n := f(x_{n-1}, \dots, x_{n-j})$ and with assighned initial values $c_0, c_1, c_2, \dots, c_{j-1}$ with the following notation

$$\begin{cases} x_n = f(x_{n-1}, \dots, x_{n-j}) \\ x_0 = c_0 \\ x_1 = c_1 \\ x_2 = c_2 \\ \vdots \\ x_{j-1} = c_{j-1} \end{cases}$$

Example 4.10. Let us recall the *Fibonacci sequence* $(x_n)_{n\in\mathbb{N}}$:

$$\begin{cases} x_n = x_{n-1} + x_{n-2} \\ x_0 = 1 = x_1. \end{cases}$$

Then, $x_2 = 2$, $x_3 = 3$, $x_4 = 5$, $x_5 = 8$, $\dots$ and (x_n) is strictly increasing as $x_k > 0$, $\forall k \in \mathbb{N}$. [Prove this claim!]

Example 4.11. We can define arithmetic and geometric progressions as recursive sequences.

(1) An arithemetic sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := a + bn$, $a, b \in \mathbb{R}$, can be defined recursively as

$$\begin{cases} x_n = x_{n-1} + b \\ x_0 = a. \end{cases}$$

(2) A geometric sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := aq^n$, $a, q \in \mathbb{R}$, can be defined recursively as

$$\begin{cases} x_n = qx_{n-1} \\ x_0 = a. \end{cases}$$

Example 4.12. Let us consider the following recursive sequence $(x_n)_{n\in\mathbb{N}}$

$$\begin{cases} x_n = x_{n-1} + (-1)^n n^2, \\ x_0 = 0. \end{cases} \tag{4.12.a}$$

Equivalently, $x_n = \sum_{i=0}^{n} (-1)^i i^2$. What can we say about this sequence? For example, is it bounded (resp. bounded from above or from below)? The answer to the above question can be given using *induction* which we will now introduce.

4.2 Induction

Induction is a method of proving a property $P(k)$ which depends on a parameter k which varies among the natural numbers that are greater or equal than a fixed natural number $C \in \mathbb{N}$. More precisely, we want to be able to prove infinitely many different statements – all the versions of property $P(k)$, when $k \geq C$ in $\mathbb{N}$; hence, we want to find a method that allows us to prove all of these statements at once, without having to do infinitely many verifications (one for each value of k).

To prove that property $P(k)$ holds when $k \geq C \in \mathbb{N}$ and $k \in \mathbb{N}$, we can try to use the following 2-step recipe, known as a *proof by induction*:

(1) we first show that $P(k)$ holds for $k = C$ – this is called the *starting step* of a proof by induction;

(2) we then proceed to show that $P(k)$ holds for a given value $k = n \in \mathbb{N}$ (where n here is to be treated as an unspecified number), under the assumption that we already know that $P(k)$ holds all choices of k starting from C and up to $n-1$. This second step is called the *inductive step* of a proof by induction. The assumption that $P(k)$ statement holds for $k = C, C+1, C+2, \ldots, n-2, n-1$ is called the *inductive hypothesis.*

Hence, we can think of

Example 4.13. We continue to work with the sequence $(x_n)_{n\in\mathbb{N}}$ defined in Example 4.12. We will prove by induction the following claim related to this sequence.

Claim. For the recursive sequence $(x_n)_{n\in\mathbb{N}}$ defined in (4.12.a), the even elements of the sequence satisfy the following equality:

$$x_{2k} = (2k+1)k, \qquad \forall k \in \mathbb{N}.$$

Hence, the property $P(k)$ that we want to prove by induction is the following

$$P(k) \,:\, \text{“}x_{2k} = (2k+1)k\text{”}$$

and k is any natural number, i.e., we have to prove that $P(k)$ holds for all values of $k \in \mathbb{N}$.

Proof of the Claim. We prove that $P(k)$ holds by induction on $k \geq 0$.

- *Starting Step:* we need to show that $P(0)$ holds.
  Tat means that we need to show that the equality $x_{2k} = (2k+1)k$ holds when we take $k = 0$. But, $x_0 = 0$ and $(2 \cdot 0 + 1) \cdot 0 = 0$, hence, indeed, $x_{2k} = (2k+1)k$.
  The starting step is proven.

- *Inductive Step:* We will now assume that property $P(k)$ holds for all values $0 \leq k < n$, that is, we assume that we know already that for all $0 \leq k < n$, $x_{2k} = (2k+1)k$ and we will show that $P(k)$ holds for $k = n$, i.e., we will show that $x_{2n} = (2n+1)n$. Thus,

$$\begin{aligned} x_{2n} = {} & \underbrace{x_{2n-1} + (2n)^2}_{\text{recursive formula applied to } x_{2n}} = \underbrace{x_{2(n-1)} - (2n-1)^2}_{\text{recursive formula applied to } x_{2n-1}} + (2n)^2 \\ &= x_{2(n-1)} - \underbrace{((2n)^2 - 4m + 1)}_{=(2n-1)^2} + (2n)^2 = x_{2(n-1)} + 4n - 1 \\ &= \underbrace{(2n-1)(n-1)}_{\text{inductive hypothesis: } x_{2(n-1)} = (2(n-1)+1)(n-1)} + 4n - 1 = 2n^2 - 3n + 1 + 4 - 1 \\ &= 2n^2 + n = (2n+1)n. \end{aligned}$$

  Hence we have shown that $P(k)$ holds for $k = n$, which concludes the proof of the inductive step and, thus, the whole proof by induction of our claim.

$\square$

The claim implies that, as $x_{2k} = (2k+1)k$, then $(x_n)_{n\in\mathbb{N}}$ is not bounded from above: in fact, for any real number b, we can find $k_b \in \mathbb{N}$ such that

$$(2k_b + 1)k_b = 2(k_b)^2 + k_b \geq (k_b)^2 > b.$$

In fact,

$$(k_b)^2 > b \quad \text{if and only if} \quad k_b > \sqrt{|b|}, \tag{4.13.a}$$

and the Archimedean property Corollary 2.31 shows that the second inequality in (4.13.a) is indeed satisfied for some $k_b \in \mathbb{N}$ – it suffices to take $k_b = [\sqrt{|b|}] + 1$. Thus, any given $b \in \mathbb{R}$ cannot be an upper bound for the set of values of the sequence, since for $m > k_b$, $x_{2m} > x_{2k_b} > b$. We can also prove that the sequence is not bounded from below, one can also show that [prove it, by induction again!]

$$x_{2k+1} = x_{2k} - (2k+1)^2 = (2k+1)k - (2k+1)^2 = -(2k+1)(k+1).$$

Hence, one can use a similar argument as before to show that $(x_n)_{n\in\mathbb{N}}$ is also not bounded from below.

4.3 Bernoulli inequality and (non-)boundedness of geometric sequences

Example 4.14. Let $(x_n)_{n\in\mathbb{N}}$ be a geometric progression, that is, $x_n := aq^n$ for some real numbers a and q. If either $a = 0$ or $|q| \leq 1$, then the sequence is bounded: more precisely,

(1) for $a = 0$ or $q = 0, 1$, the sequence is a constant sequence, cf. Example 4.8.3;

(2) if instead $a \neq 0$ and $|q| \leq 1$ then $|x_n| \leq |a|$, for all $n \in \mathbb{N}$.

We show that (2) holds for x_n by induction on $n \in \mathbb{N}$. Indeed:

- *Starting Step:* for $n = 0$, $x_0 = aq^0 = a$, hence $|x_0| = |a|$.
- *Inductive Step:* assuming that $|x_j| \leq |a|$, for all natural numbers $j < n$ then we need to prove that also $x_n \leq |a|$. But then,

$$|x_n| = |aq^n| = |aq^{n-1}||q| = |x_{n-1}||q| \leq a \cdot 1 = a.$$

What can we say in regards to the boundedness of a geometric progression $x_n = aq^n$, when $a \neq 0$ and $|q| > 1$? In this section we will show that, when $a \neq 0$ and $|q| > 1$, then the sequence is unbounded. In order to do that, we need to show that

$$\forall C \in \mathbb{R},\ \exists n_C \in \mathbb{R},\ \text{such that } |x_{n_C}| \geq C,$$

which is to say that there are no upper or lower bounds for the set of values of the sequence (x_n). Equivalently, we need to show that

$$\forall C \in \mathbb{R},\ \exists n_C \in \mathbb{R},\ \text{such that } |q^{n_C}| \geq \frac{C}{|a|}.$$

We saw in Example 4.11 that we can define a geometric sequence recursively. In view of that and of the fact that we are assuming $|q| > 1$, then, as $x_n = qx_{n-1}$ it immediately follows that $|x_n| > |x_{n-1}|$. Even better, we can inductively compute that $|x_{n+l}| > q^l|x_n|$, for any $l \in \mathbb{N}$. Hence the absolute value of x_n is increasing indefinitely with n. Is this enough to prove the unboundedness of a geometric sequence with $|q| > 1$? We will answer this question in the course of this section.

Example 4.15. While one may be tempted to think that an increasing sequence must eventually be unbounded, let us show that this is not always the case.

Let $(x_n)_{n\geq 1}$ be the sequence defined as $x_n := 5 - \frac{1}{n}$. We have already seen in Example 4.2 that x_n is strictly increasing. On the other hand, $0 < x_n < 5$ which implies that (x_n) is bounded. Hence, being strictly monotone does not suffice to imply boundedness of a sequence as this example very simply illustrates.

Before we continue in our analysis of geometric sequences, we introduce the following result that will be useful in proving that aq^n is unbounded when $a \neq 0, |q| > 1$.

Proposition 4.16 (Bernoulli's inequality)**.** *Let q be a positive real number satisfying $q > 1$. Then $q^n \geq 1 + n(q-1)$.*

To prove Proposition 4.16, we first need to introduce a few new mathematical tools and results. The first is the concept of binomial coefficient.

Definition 4.17. If $0 \leq k \leq n$ are natural numbers, then $\binom{n}{k}$ is defined as

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot \dots \cdot 1}.$$

Here the symbol $n!$ for $n \in \mathbb{N}$ is the factorial notation, that is, $n!$ is the product of the first n natural numbers (starting from 1):

$$n! = 1 \cdot 2 \cdot 3 \cdot 3 \cdot \dots \cdot (n-2) \cdot (n-1) \cdot n.$$

We also define $0! := 1$. The number $n!$ can be recursively defined, for $n \geq 1$, by the recurrence

$$\begin{cases} (n+1)! = (n+1) \cdot n! \\ 0! = 1. \end{cases}$$

Remark 4.18. Given natural numbers $0 \leq k \leq n$, then the natural number $\binom{n}{k}$ is equal to the number of possible ways one can choose a subset of unordered[13] k elements from a set of n elements. You can find an explanation of this fact here.

One can show, using induction, the following properties of binomial coefficients.

Proposition 4.19. *Let n, k be natural numbers and let x, y be real numbers. Then,*

(1) For $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

(2) For $1 \leq k \leq n-1$,

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

(3) (Binomial formula) For any $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Proof. This is an exercise in the exercise sheet for Week 4. □

We are now ready to fully prove Bernoulli's inequality.

Proof of Proposition 4.16. The inequality is an actual equality when $n = 0, 1$. Then, we can assume that $n \geq 2$. Let us apply the binomial formula, Proposition 4.19; then,

$$\begin{aligned} q^n = (1 + (q-1))^n &= \sum_{i=0}^{n} \binom{n}{i} (q-1)^i \cdot 1^{n-i} \\ &= \binom{n}{0} (q-1)^0 + \binom{n}{1} (q-1)^1 + \sum_{i=2}^{n} \binom{n}{i} (q-1)^i \cdot 1^{n-i} \\ &= 1 + n(q-1) + \sum_{i=2}^{n} \binom{n}{i} (q-1)^i \cdot 1^{n-i}. \end{aligned}$$

[13] By unordered we mean that we do not distinguish the order in which the k elements are chosen.

As $q > 1$, then $(q-1)^i > 0$, for all $i \in \mathbb{N}^*$. Thus, $\sum_{i=2}^{n} \binom{n}{i} (q-1)^i \cdot 1^{n-i} > 0$ and

$$\begin{aligned} q^n = (1 + (q-1))^n = & \sum_{i=0}^{n} \binom{n}{i} (q-1)^i \cdot 1^{n-i} \\ = & 1 + n(q-1) + \sum_{i=2}^{n} \binom{n}{i} (q-1)^i \cdot 1^{n-i} > 1 + n(q-1). \end{aligned}$$

□

Example 4.20. Let $(x_n)_{n \in \mathbb{N}}$ be a geometric progression for some real numbers a and q, $x_n := aq^n$. Assume that $|q| > 1$ and $a \neq 0$.
Under these assumptions, Bernoulli's inequality, Proposition 4.16, implies that (x_n) is not bounded. In fact,

$$|aq^n| = \underbrace{|a||q|^n \geq |a|(1 + n(|q|-1))}_{\text{Bernoulli's inequality}}$$

We can turn the latter expression into a sequence (y_n), that is, $y_n := |a|(1 + n(|q|-1))$. The sequence (y_n) is not bounded since, for a fixed positive real number $b \in \mathbb{R}_+$,

$$|a|(1 + n(|q|-1)) \leq b \quad \iff \quad n \leq \frac{\frac{b}{|a|} - 1}{|q| - 1},$$

which does not hold for $n \geq \left[\frac{\frac{b}{|a|} - 1}{|q| - 1} \right] + 1$. So, no b can be an upper bound for $|x_n|$.

One can show similarly:

(1) $(x_n)_{n \in \mathbb{N}}$ is bounded if and only if $|q| \leq 1$ or $a = 0$;

(2) $(x_n)_{n \in \mathbb{N}}$ is increasing if and only if $\begin{cases} q \geq 1 \text{ and } a \geq 0, \text{ or} \\ 0 \leq q \leq 1 \text{ and } a \leq 0. \end{cases}$;

(3) $(x_n)_{n \in \mathbb{N}}$ is strictly increasing if and only if $\begin{cases} q > 1 \text{ and } a > 0, \text{ or} \\ 0 < q < 1 \text{ and } a < 0 . \end{cases}$

(4) $(x_n)_{n \in \mathbb{N}}$ is decreasing if and only if $\begin{cases} 0 \leq q \leq 1 \text{ and } a \geq 0, \text{ or} \\ q \geq 1 \text{ and } a \leq 0. \end{cases}$

(5) $(x_n)_{n \in \mathbb{N}}$ is strictly decreasing if and only if $\begin{cases} 0 < q < 1 \text{ and } a > 0, \text{ or} \\ q > 1 \text{ and } a < 0 . \end{cases}$

(6) $(x_n)_{n \in \mathbb{N}}$ is bounded from above if and only if $|q| \leq 1$ or $q > 1$ and $a \leq 0$;

(7) $(x_n)_{n \in \mathbb{N}}$ is bounded from below if and only if $|q| \leq 1$ or $q > 1$ and $a \geq 0$.

4.4 Limit of a sequence

Definition 4.21. Let $(x_n)_{n \geq l}$ be a sequence.

(1) We say that $(x_n)_{n \geq l}$ *converges* (or is *convergent*) to a number $y \in \mathbb{R}$, if for each $\varepsilon \in \mathbb{R}_+^*$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \text{ such that } n \geq n_\varepsilon, \quad \text{then} \quad |x_n - y| \leq \varepsilon.$$

(2) If $(x_n)_{n \geq l}$ does not converge to any $y \in \mathbb{R}$ then we say that it is not convergent.

If the number $y \in \mathbb{R}$ defined above exists then y is called the *limit* of the sequence (x_n). Once again, as we are talking about *the* limit of a sequence, this is only possible if the limit is unique, when it exists. That is indeed the case.

Proposition 4.22. *If a sequence $(x_n)_{n\geq l}$ converges, then its limit is unique.*

Proof. Let us assume by contradiction that $(x_n)_{n\geq l}$ admits two distinct limits $t_1 \neq t_2 \in \mathbb{R}$. Then, for each $0 < \varepsilon \in \mathbb{R}$ there are $n_\varepsilon, n'_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, then

$$|t_1 - x_n| \leq \varepsilon,$$

and for all $n \geq n'_\varepsilon$, then

$$|t_2 - x_n| \leq \varepsilon.$$

So, if we take $n_\varepsilon := \max\{n_\varepsilon, n'_\varepsilon\}$, then both of the above inequalities hold for all integers $n \geq n_\varepsilon$. In particular, for such n, we have

$$\underbrace{|t_1 - t_2| \leq |t_1 - x_n| + |x_n - t_2|}_{\text{triangle inequality}} \leq \varepsilon + \varepsilon = 2\varepsilon$$

Since, this holds for all $0 < \varepsilon \in \mathbb{R}$, we obtain that $t_1 = t_2$. $\square$

Notation 4.23. When the limit $y \in \mathbb{R}$ or a sequence $(x_n)_{n\geq l}$ exists, we denote that by $\lim\limits_{n\to\infty} x_n = y$. Alternatively, we also write $x_n \xrightarrow{n\to\infty} y$.

Example 4.24. The sequence $(x_n)_{n\geq 1}$, defined as $x_n := 1 - \frac{1}{\sqrt{n}}$, is convergent.
Indeed, $\lim\limits_{n\to\infty} \left(1 - \dfrac{1}{\sqrt{n}}\right) = 1$. To verify this claim, for any fixed $\varepsilon \in \mathbb{R}^*_+$ we have to find an index $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n \geq n_\varepsilon, \quad \left|1 - \frac{1}{\sqrt{n}} - 1\right| \leq \varepsilon.$$

On the other hand,

$$\left|1 - \frac{1}{\sqrt{n}} - 1\right| = \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}.$$

Hence, it suffices to show that there exists an index $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$, then $\frac{1}{\sqrt{n}} < \varepsilon$. The latter inequality is equivalent to the inequality $\sqrt{n} > \frac{1}{\varepsilon}$, which in turn is equivalent to the inequality $n > \frac{1}{\varepsilon^2}$. Hence, for any fixed $\varepsilon \in \mathbb{R}^*_+$, we have to find an index $n_\epsilon \in \mathbb{N}$ such that

$$\forall n \geq n_\varepsilon, \quad n > \frac{1}{\varepsilon^2}.$$

Thus, for a fixed $\varepsilon > 0$, it suffices to take $n_\varepsilon := \left[\frac{1}{\varepsilon^2}\right] + 1$.

Example 4.25. Let us introduce an example of a non-converging sequence.
Let us consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := (-1)^n$. Indeed, if (x_n) was convergent with limit y, then we could apply Definition 4.21 with $\varepsilon := \frac{1}{2}$ and find $n_{\frac{1}{2}} \in \mathbb{N}$ such that for all integers $n \geq n_{\frac{1}{2}}$, $|x_n - y| < \frac{1}{2}$. In particular, if $n' \geq n_{\frac{1}{2}}$ is any other integer, then we would have:

$$|x_{n'+1} - x_{n'}| = \underbrace{|x_n - y + y - x_{n'}| \leq |x_n - y| + |y - x_{n'}|}_{\text{triangle inequality}} < \frac{1}{2} + \frac{1}{2} = 1$$

However, in our sequence $|x_{n'+1} - x_{n'}| = |1 - (-1)| = 2 > 1$ which prompts a contradiction. Thus, this sequence cannot converge to any limit $y \in \mathbb{R}$.

Remark 4.26. In fact, the argument used in Example 4.25 shows that if $(x_n)_{n\geq l}$ is a convergent sequence, then for all $0 < \varepsilon \in \mathbb{R}$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n, n' \geq n_\varepsilon$, $|x_n - x_{n'}| < 2\varepsilon$. [Verify this fact using again the triangle inequality!] We will see that this observation can be formalized into the notion of Cauchy sequence, see Section 4.10.

Also, using a similar argument as in Example 4.25 above, we can show the following result.

Proposition 4.27. *Let $(x_n)_{n\geq l}$ be a sequence. If $(x_n)_{n\geq l}$ is convergent, then it is bounded.*

Proof. Set $y := \lim_{n\to\infty} x_n$. Applying Definition 4.21 with $\varepsilon := 1$, then there exists $n_1 \in \mathbb{N}$, such that for all integers $n \geq n_1$, $|x_n - y| \leq 1$. That is, for all integers $n \geq n_1$,

$$-1 + y < x_n < 1 + y, \quad \text{and} \quad |x_n| < \max(|-1+y|, |1+y|). \tag{4.27.a}$$

Let us define

$$R := \max\{|x_l|, |x_{l+1}|, |x_{l+2}| \ldots, |x_{n_1-3}|, |x_{n_1-2}|, |x_{n_1-1}|, |y+1|, |y-1|\}.$$

We claim that R is an upper bound and $-R$ is a lower bound for the set of values of the sequence. Indeed, R (resp. $-R$) is an upper bound (resp. a lower bound) for the set $\{x_l, x_{l+1} \ldots, x_{n_1-1}\}$ just because R is $\geq$ than the absolute values of all these elements of the sequence, by the very definition of R above. Furthermore, R (resp. $-R$) is an upper bound (resp. a lower bound) for the other elements of the sequence, because these elements are lying in the interval $I = [y-1, y+1]$, R (resp. $-R$) is an upper bound (resp. a lower bound) for I, again, by definition of R. □

Example 4.28. The sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := \sqrt{n^3}$ cannot be convergent as it is not bounded.

Remark 4.29. The viceversa of the above proposition is not true: that is, if a sequence $(x_n)_{n\geq l}$ is bounded, then it is not necessary convergent. An example of that is given by the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := (-1)^n$, see Example 4.25.
In Section 4.7 we shall see that a monotone bounded sequence $(x_n)_{n\geq l}$ is always convergent. Of course, the sequence (x_n) defined by $x_n := (-1)^n$ is not monotone.

4.4.1 Limits and algebra

In this section we show that (finite) limits of sequences respect the standard operations.

Proposition 4.30. *Let (x_n) and (y_n) be two convergent sequences and let $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$ be their limits. Then:*

(1) the sequence $(x_n + y_n)$ is also convergent, and $\lim_{n\to\infty}(x_n + y_n) = x + y$,

(2) the sequence $(x_n \cdot y_n)$ is also convergent, and $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$,

(3) if $y \neq 0$, then the sequence $\left(\frac{x_n}{y_n}\right)$ is also convergent, and $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$, and

(4) if there is an $n_0 \in \mathbb{N}$, such that $x_n \leq y_n$ for each integer $n \geq n_0$, then $x \leq y$.

Remark 4.31. Let us note that, since $y \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $y_n \neq 0$ for $n \geq n_0$. Hence dividing the quotient $\frac{x_n}{y_n}$ makes sense for $n \geq n_0$, provided that x_n is defined for such choice of index.

Proof. We prove only (1). We refer to 2.3.3 and 2.3.6 in the book for the proofs of the others. Fix $0 < \varepsilon \in \mathbb{R}$. Let us try to explain how the proof should intuitively go. We need to show that for big enough an index $n \in \mathbb{N}$, $|(x_n + y_n) - (x + y)|$ is smaller than ε. However, as (x_n), (y_n) are both convergent with limit x, y, respectively, we know that $|x_n - x|$ and $|y_n - y|$ are small for big n; moreover,

$$|(x_n + y_n) - (x + y)| = \underbrace{|(x_n - x) + (y_n - y)| \le |x_n - x| + |y_n - y|}_{\text{triangle inequality}}. \tag{4.31.b}$$

So, to make $|(x_n + y_n) - (x + y)|$ smaller than ε, it suffices to make the sum $|x_n - x| + |y_n - y|$ smaller than ε. That we can attain for example if we make both $|x_n - x|$ and $|y_n - y|$ smaller than $\frac{\varepsilon}{2}$. The choice of $\frac{\varepsilon}{2}$ is rather arbitrary: the proof would work with any two positive numbers that add up to ε, for example with $\frac{\varepsilon}{3}$ and $\frac{2\varepsilon}{3}$, but for simplicity, we shall stick with $\frac{\varepsilon}{2}$. After this initial discussion, we proceed to the formal proof.
We work with $\varepsilon > 0$ fixed above. Thus, there exist integers $n'_{\frac{\varepsilon}{2}}$ and $n''_{\frac{\varepsilon}{2}}$, such that

$$\forall n \ge n'_{\frac{\varepsilon}{2}}, \quad |x - x_n| \le \frac{\varepsilon}{2}, \quad \text{and}$$
$$\forall n \ge n''_{\frac{\varepsilon}{2}}, \quad |y - y_n| \le \frac{\varepsilon}{2}.$$

Let us define $n_\varepsilon := \max\left\{n'_{\frac{\varepsilon}{2}}, n''_{\frac{\varepsilon}{2}}\right\}$. Then, (4.31.b) implies that for every $n \ge n_\varepsilon$,

$$|(x_n + y_n) - (x + y)| \le |x_n - x| + |y_n - y| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $(x_n + y_n)$ satisfies Definition 4.21 for convergence with respect to the finite limit $x + y$. □

Property (4) in Proposition 4.30 implies the following immediate corollary.

Corollary 4.32. *Let $(x_n)_{n\ge l}$ be a converging sequence and let $x := \lim_{n\to\infty} x_n$ be its limit. If there exists $n_0 \in \mathbb{N}$ such that $x_n \ge 0$, $\forall n \ge n_0$, then $x \ge 0$*

Example 4.33. In Corollary 4.32, it may well happen that $x = 0$ even if $x_n > 0$, $\forall n \in \mathbb{N}$ as shown by the sequence $x_n = \frac{1}{n}$.

Example 4.34. With the above machinery we can already compute the limits of series that are defined as fractions of polynomials; a fracion whose numerator and denominator are both polynomials is called a rational function.

(1) $x_n := \frac{n^2+2n+3}{4n^2+5n+6}$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^2 + 2n + 3}{4n^2 + 5n + 6} = \underbrace{\lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{4 + \frac{5}{n} + \frac{6}{n^2}}}_{\substack{\text{dividing both the numerator} \\ \text{and the denominator by } n}} =$$

$$\underbrace{\frac{\lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{3}{n^2}\right)}{\lim_{n\to\infty} \left(4 + \frac{5}{n} + \frac{6}{n^2}\right)}}_{\substack{\text{using Proposition 4.30.3 as} \\ \text{both the numerator and de-} \\ \text{nominator have finite limit}}} = \underbrace{\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{3}{n^2}}{\lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{5}{n} + \lim_{n\to\infty} \frac{6}{n^2}}}_{\text{addition rule for finite limits}} =$$

$$\underbrace{\frac{1 + \lim_{n\to\infty} \frac{2}{n} + 3 \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right)^2}{4 + \lim_{n\to\infty} \frac{5}{n} + 6 \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right)^2}}_{\substack{\text{product rule for finite limits \&} \\ \text{limits of constant sequences}}} = \frac{1 + 0 + 0}{4 + 0 + 0} = \frac{1}{4}$$

Here are a few comments on the manipulation we just performed:

(i) dividing the numerator and the denominator by n is an operation that one cannot perform for $n = 0$. So, after the second equality sign the expression that we wrote does not make sense for $n = 0$. But this is not an issues, for when we study a sequence for the purpose of understanding its convergence, we are only interested in the values of the index for big enough values of the index n. So you are free to substitute the 0-th term with any real number, e.g., 0, after the second equality sign. The same issue will show up many other times in this sectio, for example, when we are computing limits of sequences of the form $\frac{2}{n}$, or $\frac{3}{n^2}$. Hence, from now onwards, whenever we work with sequences to discuss their convergence, we will not worry too much about what may happen to a finite number of values of the sequence, whenever we perform some algebraic manipulations, or we show that certain estimates holds, etc.

(ii) for any number $c \in \mathbb{R}$, $\lim_{n\to\infty} \frac{c}{n} = 0$: infact, given a fixed $\varepsilon \in \mathbb{R}^*_+$, we may choose $n_\varepsilon := \left[\frac{c}{\varepsilon}\right] + 1$, and for this choice we have for each integer $n \geq n_\varepsilon$:

$$\left|\frac{c}{n}\right| < \frac{c}{\frac{c}{\varepsilon}} = \varepsilon.$$

(iii) in the step where we use that limits behave well with respect to fractions, we should check first that the limit of the denominator is not 0. However, following our argument, we see that this limit is 4, so we are fine.

(2) $x_n = \frac{n+2}{3n^2+4n+5}$. Here we will not give the above explanations again (as they are the same):

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n+2}{3n^2+4n+5} = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{2}{n^2}}{3 + \frac{4}{n} + \frac{5}{n^2}} = \frac{0+0}{3+0+0} = 0$$

(3) $x_n = \frac{n^2+2n+3}{4n+5}$. For $n \geq 1$, we have $0 \leq \frac{3}{n}$ and $1 \geq \frac{5}{n}$. Hence, for $n \geq 1$:

$$x_n = \frac{n^2+2n+3}{4n+5} = \frac{n+2+\frac{3}{n}}{4+\frac{5}{n}} \geq \frac{n+2}{5}$$

This shows that (x_n) is not bounded and hence cannot be convergent by Proposition 4.27.

Using the method of the above exercise one can show the following result on limits of sequences defined by means of rational functions.

Proposition 4.35. *If (x_n) and (y_n) are sequences given by polynomials*

$$x_n := P(n), \quad P(X) = a_0 + a_1 X + \cdots + a_p X^p, \text{ with } a_p \neq 0, \text{ and}$$
$$y_n := Q(n), \quad Q(X) = b_0 + b_1 X + \cdots + b_q X^q, \text{ with } b_q \neq 0,$$

then

(1) if $p \leq q$, then $\left(\frac{x_n}{y_n}\right)$ is convergent, and

(i) if $p = q$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a_p}{b_q}$,

(ii) if $p < q$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$,

(2) if $p > q$, then $\left(\frac{x_n}{y_n}\right)$ is not bounded and thus it does not converge.

Proof. See page 22 of the book for a precise proof. The book contains states an unnecessary assumption: it is requested that $y_n \neq 0$ for all $n \in \mathbb{N}$, but in fact it is enough if $y_n \neq 0$ for some $n \in \mathbb{N}$, as, in that case, y_n is a given by evaluating a non-zero polynomial at natural numbers, and a non-zero polynomial has at most as many roots as its degree. □

4.5 Squeeze theorem

Theorem 4.36 (Squeeze Theorem). *Let (x_n), (y_n), and (z_n) be three sequences. Assume that:*

(1) the sequences (x_n), (z_n) are convergent, and $\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} z_n$; and

(2) there exists $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$,

$$x_n \leq y_n \leq z_n.$$

Then the sequence (y_n) is convergent, and

$$\lim_{n\to\infty} y_n = a.$$

Proof. For each $\varepsilon > 0$, there are natural numbers n'_ε and n''_ε, such that

$$\begin{aligned} &\forall n \geq n'_\varepsilon, \quad a - \varepsilon < x_n, \quad \text{and,} \\ &\forall n \geq n''_\varepsilon, \quad a + \varepsilon > z_n. \end{aligned}$$

Set $n_\varepsilon := \max\{n'_\varepsilon, n''_\varepsilon, n_0\}$. Then, for each integer $n \geq n_\varepsilon$

$$a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon,$$

which in particular implies that $|y_n - a| < \varepsilon$.

$\square$

Example 4.37. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n := \frac{1}{n} + \frac{1}{\sqrt{n}}$. We show that $\lim\limits_{n\to\infty}\left(\frac{1}{n} + \frac{1}{\sqrt{n}}\right) = 0$.
In fact, we may squeeze x_n as follows

$$0 \leq \frac{1}{n} + \frac{1}{\sqrt{n}} \leq \frac{2}{\sqrt{n}}, \quad \forall n \geq 1$$

Indeed:

(1) $0 \leq \frac{1}{n} + \frac{1}{\sqrt{n}}$ holds for every integer $n \geq 1$.

(2) For every integer $n \geq 1$ we also have:

$$\frac{1}{n} + \frac{1}{\sqrt{n}} \leq \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}}}_{n \leq n^2 \Leftrightarrow \sqrt{n} \leq n} = \frac{2}{\sqrt{n}};$$

On the other hand,

(i) the limit of the constant sequence of value 0 is 0;

(ii) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ by the computation of Example 4.24.

Hence, we can apply the Squeeze Theorem 4.36 to conclude that $(x_n)_{n\in\mathbb{N}}$ converges and its limit is 0.

Example 4.38. In general, we can show that a geometric sequence $(x_n)_{n\in\mathbb{N}}$, $x_n := aq^n$ is convergent if and only if $a = 0$ or $-1 < q \leq 1$.
Indeed:

(1) When $a = 0$ or $q = 0, 1$, then the sequence is constant. If $q = -1$, then $x_n = (-1)^n a$ and the sequence does not converge.

(2) If $|q| > 1$, we have already shown in Example 4.14 that the sequence is not bounded. Hence, Proposition 4.27 implies that the sequence is also non-convergent.

(3) If $|q| < 1$, we show that the sequence converges and that $\lim_{n\to\infty} aq^n = 0$.
To prove that, we should understand when $|aq|^n < \varepsilon$ for a given $0 < \varepsilon \in \mathbb{N}$. But,

$$|aq|^n < \varepsilon \quad \Longleftrightarrow \quad \frac{|a|}{\varepsilon} < \left(\frac{1}{|q|}\right)^n \tag{4.38.a}$$

As $|q| < 1$, then $\left|\frac{1}{q}\right| > 1$ and we can apply Bernoulli's inequality, Proposition 4.16, showing that

$$\left(\frac{1}{|q|}\right)^n \geq 1 + n\left(\frac{1}{|q|} - 1\right) > n\left(\frac{1}{|q|} - 1\right). \tag{4.38.b}$$

Putting (4.38.a), (4.38.b) together, then the inequality $|aq|^n < \varepsilon$ holds as long as

$$\frac{|a|}{\varepsilon} < n\left(\frac{1}{|q|} - 1\right). \tag{4.38.c}$$

Since the inequality in (4.38.c) is satisfied for all integer $n \geq n_\epsilon$, where

$$n_\varepsilon = \left[\frac{\frac{|a|}{\varepsilon}}{\left(\frac{1}{|q|} - 1\right)}\right] + 1,$$

we can conclude that for all $n \geq n_\varepsilon$, $|aq^n| < \varepsilon$.

Example 4.39. Let $(y_n)_{n\in\mathbb{N}}$ be the sequence defined by $y_n := \frac{2^n}{n!}$.
We claim that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. Indeed, we have for all integers $n \geq 3$:

$$\underbrace{0}_{x_n} \leq \underbrace{\frac{2^n}{n!}}_{y_n} \leq \frac{2^n}{2 \cdot 3^{n-2}} = \frac{3^2 \cdot 2^n}{2 \cdot 3^2 \cdot 3^{n-2}} = \underbrace{\frac{9}{2} \cdot \left(\frac{2}{3}\right)^n}_{z_n}$$

Furthermore $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{9}{2} \cdot \left(\frac{2}{3}\right)^n = \frac{9}{2} \cdot \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = \frac{9}{2} \cdot 0 = 0$ by Example 4.38. So, the Squeeze Theorem 4.36 concludes our claim.

Example 4.40. Let $(y_n)_{n\in\mathbb{N}}$ be the sequence defined by $y_n := \sqrt[n]{n}$. we show that $\lim_{n\to\infty} x_n = 1$.
To prove the above claim, we show that we can squeeze the sequence (y_n) as follows:

$$\underbrace{1}_{x_n} \leq y_n \leq z_n := 1 + \frac{1}{\sqrt{n}}, \quad \forall n \gg 1.$$

As the limit of both sides is 1, and x_n is not smaller than 1, it is enough to prove the second inequality, for high enough values of n, that is, we shall prove that there exists $n_0 \in \mathbb{N}$ such that the above inequality holds $\forall n \geq n_0$. To do that, consider the following equivalence of inequalities:

$$\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}} \quad \Longleftrightarrow \quad n \leq \left(1 + \frac{1}{\sqrt{n}}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{(\sqrt{n})^i}$$

Note that the sum on the right hand side for $i = 4$ is

$$\frac{n(n-1)(n-2)(n-3)}{24} \frac{1}{(\sqrt{n})^4} = \frac{n(n-1)(n-2)(n-3)}{24n^2}.$$

So, we know the desired inequality $\left(\text{i.e., that } \sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}}\right)$ as soon as $n \geq 4$ and $n \leq \frac{n(n-1)(n-2)(n-3)}{24n^2}$. The latter is equivalent to

$$\frac{24n^2}{(n-1)(n-2)(n-3)} \leq 1.$$

However, we have just learned that

$$\lim_{n\to\infty} \frac{24n^2}{(n-1)(n-2)(n-3)} = 0,$$

so there is an integer n_1, such that for each $n \geq n_1$,

$$\left|\frac{24n^2}{(n-1)(n-2)(n-3)}\right| \leq 1.$$

[There is a different proof in the book, on page 24: check that out, too!].

Corollary 4.41. *Let (x_n) be a convergent sequence, and let (y_n) be a bounded sequence. If $\lim_{n\to\infty} x_n = 0$, then the sequence $(x_n y_n)$ is convergent and $\lim_{n\to\infty} x_n y_n = 0$.*

Remark 4.42. Let us recall, see also the exercises, that for a sequence (x_n), $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$. In fact, if we assume that $\lim_{n\to\infty} x_n = 0$, then we can use a similar argument to show that $\lim_{n\to\infty} |x_n| = 0$.

Proof of Corollary 4.41. Let us note that showing that $\lim_{n\to\infty} x_n y_n = 0$ is equivalent to showing that $\lim_{n\to\infty} |x_n y_n| = 0$. This follows immediately from Remark 4.42.
As y_n is bounded, there is an integer $M > 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Hence, we may squeeze $|x_n y_n|$:

$$0 \leq |x_n y_n| \leq M \cdot |x_n|,$$

Since $\lim_{n\to\infty} M \cdot |x_n| = M \cdot \lim_{n\to\infty} |x_n| = 0$, then also $\lim_{n\to\infty} |x_n y_n| = 0$. □

Example 4.43. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n := \frac{1}{n^2}\sin(n)$. We show that $\lim_{n\to\infty} \frac{1}{n^2}\sin(n) = 0$.
Let us note that we do not know whether $\sin(n)$ does or does not converge in itself – it is possible to prove that indeed it does not converge. So, we may not apply the previous multiplication rule of limits. However, we may apply the previous corollary, as $\lim_{n\to\infty} \frac{1}{n^2} = 0$, and $\sin(n)$ is bounded (by -1 and 1).

Example 4.44. Let us define the sequence $(x_n)_{n\in\mathbb{N}}$ recursively as

$$\begin{cases} x_{n+1} = \frac{\sin(x_n)}{2}, \\ x_0 = 1. \end{cases}$$

Then,

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\frac{|\sin(x_n)|}{2}}{|x_n|} \leq \frac{1}{2},$$

where the last inequality follows from the fact that $\frac{|\sin(x)|}{|x|} \leq 1$ for all $x \in \mathbb{R}$. To show this, just notice that $|x|$ measures the length of the circle segment of angle (measured in radiants) x, where we count multiple revolutions too, and $|\sin(x)|$ gives the absolute value of the y-coordinate of the endpoint of the circle segment, as shown in the figure below.
In particular,

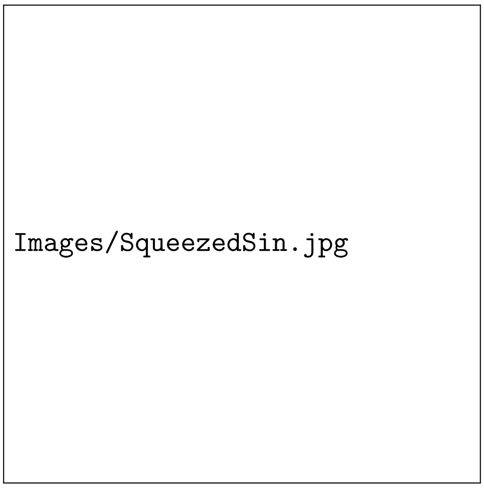

Figure 8: $|\sin(\theta)| \leq |\theta|$

$$|x_{n+1}| = \frac{|x_{n+1}|}{|x_n|}|x_n| \leq \frac{1}{2}|x_n|. \tag{4.44.d}$$

Iterating the observation in (4.44.d), we obtain

$$|x_{n+1}| \leq \frac{1}{2}|x_n| \leq \frac{1}{2^2}|x_{n-1}| \leq \frac{1}{2^3}|x_{n-2}| \leq \cdots \leq \frac{1}{2^n}|x_1| \leq \frac{1}{2^{n+1}}.$$

So, we may use the Squeeze Theorem 4.36 to show that $\lim_{n\to\infty} x_n = 0$, squeezing since

$$0 \leq x_n \leq \frac{1}{2^n}, \quad \forall n \in \mathbb{N}.$$

4.5.1 Limits of recursive sequences

Example 4.45. The Fibonacci sequence $(x_n)_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} x_{n+1} = x_n + x_{n-1} \\ x_0 = x_1 = 1. \end{cases}$$

If we define the sequence $(y_n)_{n\in\mathbb{N}}$ by $y_n := \frac{x_{n+1}}{x_n}$, then the sequence (y_n) admits a recursive definition as follows

$$\begin{cases} y_{n+1} = 1 + \frac{1}{y_n} \\ y_0 = 1. \end{cases} \tag{4.45.e}$$

We call $(y_n)_{n\in\mathbb{N}}$ the sequence of Fibonacci quotients.

Proposition 4.46. *If (y_n) is the sequence of Fibonacci quotients, then $\forall n \in \mathbb{N}$, $1 \leq y_n \leq 2$.*

Proof. We prove the above statements by induction on $n \in \mathbb{N}$.

- *Starting step*: the $n = 0$ case; by definition we have $2 \geq y_0 = 1$.

- *Inductive step*: we can assume that we know that statement for n and then we prove it for $n+1$ below:

$$y_{n+1} = 1 + \frac{1}{y_n} \geq 1 + \frac{1}{2} \geq 1,$$

and

$$y_{n+1} = 1 + \frac{1}{y_n} \leq 1 + \frac{1}{1} = 2,$$

$\square$

Example 4.47. Let us continue with Example 4.45. As we know that the sequence of Fibonacci quotients is bounded, we can ask whether it converges or not.
If a limit exists, can we use the recursive relation in (4.45.e) to find what that limit is? Let us try! Let us assume now that (y_n) is convergent, and $\lim_{n\to\infty} y_n = y$. Then, as $y_n \geq 1$, it follows that $y \geq 1$. Furthermore, by (4.45.e),

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \underbrace{\left(1 + \frac{1}{y_n}\right)}_{\text{recursive relation in (4.45.e)}} \underbrace{= 1 + \frac{1}{\lim_{n\to\infty} y_n} = 1 + \frac{1}{y}}_{\text{algebraic rules of limit}}.$$

This yields that the limit y satisfies the equation $y = 1 + \frac{1}{y}$ which we can rewrite as $y^2 - y - 1 = 0$ (since we know that $y \neq 0$) and whose solutions are

$$y = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

As we have seen that $1 \leq y \leq 2$, then this forces the equality $y = \frac{1+\sqrt{5}}{2}$. Thus, if the limit of (y_n) exists, then $y = \frac{1+\sqrt{5}}{2}$ However, we have not proven yet that (y_n) converges. As we have figured out that if (y_n) converges the only possible limit is $\frac{1+\sqrt{5}}{2}$, we may show that that the sequence $z_n := \left| y_n - \frac{1+\sqrt{5}}{2} \right|$ converges to 0. But then,

$$z_{n+1} = \left| y_{n+1} - \frac{1+\sqrt{5}}{2} \right| \underbrace{= \left| 1 + \frac{1}{y_n} - 1 - \frac{1}{\frac{1+\sqrt{5}}{2}} \right|}_{\substack{\text{we apply the definition of the sequence to } y_{n+1}\text{, and} \\ \text{then as we found } \frac{1+\sqrt{5}}{2} \text{ as the solution of } y = 1 + \frac{1}{y} \\ \text{we may replace } \frac{1+\sqrt{5}}{2} \text{ by } 1 + \frac{1}{\frac{1+\sqrt{5}}{2}}}} = \left| \frac{1}{y_n} - \frac{1}{\frac{1+\sqrt{5}}{2}} \right| = \frac{\left| y_n - \frac{1+\sqrt{5}}{2} \right|}{y_n \frac{1+\sqrt{5}}{2}} \leq \frac{|z_n|}{\frac{1+\sqrt{5}}{2}}$$

Iterating this reasoning, we get that

$$z_{n+1} \leq \frac{|z_n|}{\frac{1+\sqrt{5}}{2}} \leq \frac{|z_{n-1}|}{(\frac{1+\sqrt{5}}{2})^2} \leq \frac{|z_{n-2}|}{(\frac{1+\sqrt{5}}{2})^3} \leq \cdots \leq \frac{|z_{n-k}|}{(\frac{1+\sqrt{5}}{2})^{k+1}}.$$

and thus,

$$0 \leq z_n \leq \frac{|z_0|}{\left(\frac{1+\sqrt{5}}{2}\right)^n},$$

where

$$\lim_{n\to\infty} \frac{|z_0|}{\left(\frac{1+\sqrt{5}}{2}\right)^n} = 0.$$

So, the Squeeze Theorem (Theorem 4.36) shows that $\lim_{n\to\infty} z_n = 0$. This in turn implies, by the definition of z_n that $\lim_{n\to\infty} y_n = \frac{1+\sqrt{5}}{2}$. Summarizing, we showed that for the Fibonacci sequence (x_n)

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}$$

The number $\frac{1+\sqrt{5}}{2}$ is also known as the Golden ratio.

The general approach to finding the limit of a recursive sequence (x_n),

$$\begin{cases} x_i = f(x_{i-1}, x_{i-2}, \dots, x_{i-n}) \\ x_0 = c_0 \\ x_1 = c_1 \\ x_2 = c_2 \\ \vdots \\ x_{n-1} = c_{n-1} \end{cases}$$

is similar to the one we just explained in Example 4.47.
It can be summarized in the following 3-step recipe:

(1) assuming that there exists a finite limit for (x_n), $\lim_{n\to\infty} x_n = x$, then find the solutions of the equation

$$x = f(x, x, x, \dots, x). \tag{4.47.f}$$

In setting up such equation, one has to be careful as to whether the equation itself and its solutions are well-defined – e.g., one has to be careful when x appear in the denominator of a fraction: for which values of x is $f(x, x, x, \dots, x)$ makes sense? Can we make sure that those values of x for which $f(x, x, x, \dots, x)$ is not well-defined are values which cannot be attained by $\lim_{n\to\infty} x_n$?
If the above equation does not admit any solutions, then the sequence (x_n) cannot admit limit;

(2) we try to exclude all but one of the possibilities for among the values of x obtained in the previous point by using some argument coming from the explicit definition of (x_n);

(3) if we found a unique solution $\bar{x}$ of(4.47.f), we can try to make a direct verification hat $\lim_{n\to\infty} x_n = \bar{x}$ by showing that the $\bar{x}$ satisfies the definition of limit for (x_n).

Example 4.48. This method of finding the limit does not always work.
For example, consider the recursive sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} x_{n+1} = \frac{1}{2}(x_n + x_{n-1}) \\ x_0 = C. \end{cases}$$

Then applying Step 1 in the above recipe gives the equation $x = \frac{1}{2}(x + x)$. Of course, this equation is satisfied for any value of $x \in \mathbb{R}$. Hence, we cannot use it to restrict th epossible values of the limit of $(x_n)_{n\in\mathbb{N}}$.

Example 4.49. Here is an example of a recursive sequence where our recipe does not work. Let $a, b \in (0, +\infty)$ and let $(x_n)_{n\in\mathbb{N}}$ be the recursive sequence defined by the recurrence relation

$$\begin{cases} x_{n+1} = ax_n^2 \\ x_0 = b. \end{cases}$$

Claim 1. For all $n \in \mathbb{N}$, $x_n = a^{2^n-1}b^{2^n}$.

Proof. We prove the claim by induction on $n \in \mathbb{N}$.

- *Starting step*: For $n = 0$, $x_0 = a^{2^0-1}b^{2^0} = b$.
- *Inductive step*: Assuming that $x_k = a^{2^k-1}b^{2^k}$ for all $0 \le k < n$, then

$$\begin{aligned} x_n =& ax_{n-1}^2 \\ =& a \cdot \left(a^{(2^{n-1}-1)} \cdot b^{2^{n-1}}\right)^2 \\ =& a \cdot a^{(2 \cdot 2^{n-1}-2)} \cdot b^{2 \cdot 2^{n-1}} \\ =& a^{(2^n-1)}b^{2^n}. \end{aligned}$$

□

Now, applying the first step of our recipe, we assume that $\lim_{n\to\infty} x_n = x$ and we solve the equation

$$x = ax^2.$$

Solutions are $x = 0$ and $x = \frac{1}{a}$ – the latter is well defined since $a \neq 0$.
We can actually compute $\lim_{n\to\infty} x_n$ directly: we have to distinguish 3 different cases:

(1) if $ab = 1$ then

$$\lim_{n\to+\infty} x_n = \lim_{n\to\infty} \frac{(ab)^{2^n}}{a} = \frac{1}{a}.$$

Hence, in this case the limit of $(x_n)_{n\in\mathbb{N}}$ corresponds to one of the solutions that we found above;

(2) if $ab < 1$ then

$$\lim_{n\to+\infty} x_n = \lim_{n\to\infty} \frac{(ab)^{2^n}}{a} = 0.$$

Also in this case the limit of $(x_n)_{n\in\mathbb{N}}$ corresponds to one of the solutions that we found above;

(3) if $ab > 1$ then $x_n = \frac{(ab)^{2^n}}{a}$ which is not bounded; even better,

$$\lim_{n\to+\infty} x_n = \frac{(ab)^{2^n}}{a} = +\infty.$$

In this case, our algorithm could not possibly work since $(x_n)_{n\in\mathbb{N}}$ being unbounded cannot possibly converge to a finite limit.

4.5.2 Unbounded sets and infinite limits

Definition 4.50. Let (x_n) be a sequence.

(1) We say that (x_n) approaches $+\infty$ if for all $C \in \mathbb{R}$ there is an index $n_C \in \mathbb{N}$ such that for all integers $n \geq n_C$, $x_n \geq C$.

(2) We say that (x_n) approaches $-\infty$ if for all $C \in \mathbb{R}$ there is an index $n_C \in \mathbb{N}$ such that for all integers $n \geq n_C$, $x_n \leq C$.

Notation 4.51. If a sequence (x_n) approaches $+\infty$ (resp. $-\infty$), we write

$$\lim_{n\to\infty} x_n = +\infty \quad (\text{resp. } \lim_{n\to\infty} x_n = -\infty).$$

If a sequence (x_n) satisfies $\lim\limits_{n\to\infty} x_n = \pm\infty$, then it cannot possibly converge to a finite limit, as Definition 4.50 implies that (x_n) is unbounded.

Example 4.52. Let $(x_n)_{n\in\mathbb{N}}$ be a geometric sequence, $x_n := aq^n$.

(1) The Bernoulli inequality, see Proposition 4.16, implies that for every geometric progression $(x_n)_{n\in\mathbb{N}}$, $x_n := aq^n$, with $a > 0$ and $q > 1$ $\lim x_n = +\infty$. An example is the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := 3 \cdot 2^n$. In fact, $x_n \geq 3(1 + n(2-1)) = 3 + 3n$, and by the Archimidean property, cf. Proposition 2.30, given $C \in \mathbb{R}$, then

$$\begin{aligned} \text{if } C \leq 0, \text{ then } \forall n \in \mathbb{N}, &\quad 3n + 3 > 0 \geq C, \\ \text{if } C > 0, \text{ then } \forall n \geq \left[\frac{C}{3}\right], &\quad 3n + 3 > 3\frac{C}{3} = C. \end{aligned}$$

(2) Similarly, $\lim x_n = -\infty$ for every geometric progression $x_n = aq^n$ with $a < 0$ and $q > 1$. An example is the sequence defined by $x_n := -3 \cdot 2^n$.

(3) On the other hand, if $a \neq 0$ and $q < 0$, then (x_n) is unbounded but it neither approaches $+\infty$ not it approeaches $-\infty$. For example $x_n = (-2)^n$ is non-convergent but it also does not admit limit equal to $+\infty$ or $-\infty$.

The infinite limits satisfy some algebraic rules, and do not satisfy others. Check out page 29 and 30 of the book for full list.

Proposition 4.53. *Let $(x_n), (y_n)$ be two sequences.*

(1) Assume that $\lim\limits_{n\to\infty} x_n = +\infty$ and that (y_n) is bounded from below. Then,

(i) $\lim\limits_{n\to\infty} x_n + y_n = +\infty$;

(ii) if there exists $A \in \mathbb{R}_+^*$ and $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $y_n \geq A$, then $\lim\limits_{n\to\infty} x_n \cdot y_n = +\infty$;

(iii) if (y_n) is bounded, then $\lim\limits_{n\to\infty} \frac{y_n}{x_n} = 0$.

(2) Assume that $\lim\limits_{n\to\infty} x_n = -\infty$ and that (y_n) is bounded from above. Then,

(i) $\lim\limits_{n\to\infty} x_n + y_n = -\infty$;

(ii) if there exists $A \in \mathbb{R}_+^*$ and $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $y_n \geq A$, then $\lim\limits_{n\to\infty} x_n \cdot y_n = -\infty$;

(iii) if (y_n) is bounded, then $\lim\limits_{n\to\infty} \frac{y_n}{x_n} = 0$.

Example 4.54. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_n := 2^n + \sin(n)$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (2^n + \sin(n)) = +\infty,$$

because $\lim_{n\to\infty} 2^n = +\infty$ and $\sin(n) \geq -1$.

Remark 4.55. Part (3) for both of the above propositions claims that if $\lim_{n\to\infty} |x_n| = +\infty$ and (y_n) is bounded then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$. It is important to remark that one cannot drop the assumptions on the boundedness of (y_n). That is, if we do not assume that (y_n) is bounded, then we cannot conclude anything about $\lim_{n\to\infty} \frac{y_n}{x_n}$, as shown by the next examples. In fact, taking

(1) $x_n := n$, $y_n := n$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} 1 = 1$;

(2) $x_n := n$, $y_n := n^2$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} n = +\infty$;

(3) $x_n := n$, $y_n := \sqrt{n}$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$;

(4) $x_n := (-1)^n n$, $y_n := n$, then $\frac{y_n}{x_n} = (-1)^n$, thus, $(\frac{y_n}{x_n})$ does not converge.

Example 4.56. Here we show examples of sequences (x_n) and (y_n), for which $\lim_{n\to\infty} x_n = +\infty$, $\lim_{n\to\infty} y_n = -\infty$ and for which the sequence $(x_n + y_n)$ displays all possible behaviors in terms of its convergence (or lack thereof). In fact, taking

(1) $x_n := n$, $y_n := -n$ $\lim_{n\to\infty} (x_n + y_n) = 0$;

(2) $x_n := 2n$, $y_n := -n$ $\lim_{n\to\infty} (x_n + y_n) = +\infty$;

(3) $x_n := n$, $y_n := -2n$ $\lim_{n\to\infty} (x_n + y_n) = -\infty$;

(4) $x_n := 2n$, $y_n := (-1)^n n$, then

$$x_n + y_n = \begin{cases} n & \text{for } n \text{ odd,} \\ 3n & \text{for } n \text{ even.} \end{cases}$$

Hence, $x_n + y_n$ is unbounded, thus, non-converging, and its limit cannot be $\pm\infty$.

It is a homework to cook up similar examples for multiplication and division. For example, a famous case where "anything can happen" for multiplication is that of sequence (x_n), (y_n) such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = 0$.

Similarly to the argument for finite limits, we can prove squeeze theorems for infinite limits:

Theorem 4.57 (Squeeze Theorem for sequences approaching infinities)**.** *Let (x_n) and (y_n) be two sequences.*

(1) Assume that there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \quad x_n \leq y_n.$$

(i) If $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} y_n = +\infty$.

(ii) If $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.

(2) Assume that $\lim_{n\to\infty} \frac{x_n}{y_n} = q$.

(i') If $\lim_{n\to\infty} x_n = +\infty$ *and* $q \in \mathbb{R}_+^*$*, then* $\lim_{n\to\infty} y_n = +\infty$.

(ii') If $\lim_{n\to\infty} y_n = -\infty$ *and* $q \in \mathbb{R}_+^* \cup \{+\infty\}$*, then* $\lim_{n\to\infty} x_n = -\infty$.

Proof. (1) Let us prove (i). The other case is proven analogously.
Fix $C \in \mathbb{R}$. As $\lim_{n\to\infty} x_n = +\infty$, there exists $n_C \in \mathbb{N}$ such that $\forall n \geq n_C$, $x_n \geq C$. Taking $n'_C := \max n_0, n_C$, then $\forall n \geq n'_C$, $y_n \geq x_n \geq C$. Hence, $\lim_{n\to\infty} y_n = +\infty$.

(2) Let us prove (i') when $q \in \mathbb{R}_+^*$. All the other cases are proven analogously.
Hence, we assume that $\lim_{n\to\infty} \frac{x_n}{y_n} = q > 0$ and $\lim_{n\to\infty} x_n = +\infty$. In particular, the latter implies that there exists n_0 such that $\forall n \geq n_0$, $x_n > 0$.
Let us take $\epsilon = \frac{q}{2}$. Hence, there exists $n_{\frac{q}{2}} \in \mathbb{N}$ such that $\forall n \geq n_{\frac{q}{2}}$

$$-\frac{q}{2} \leq q - \frac{x_n}{y_n} \leq \frac{q}{2}.$$

Hence, $\forall n \geq n_{\frac{q}{2}}$,

$$\frac{q}{2} \leq \frac{x_n}{y_n} \leq 3\frac{q}{2}.$$

Thus, $\forall n \geq \max n_0, n_{\frac{q}{2}}$,

$$\begin{cases} y_n \geq \frac{q}{2x_n} > 0, & \text{since } n \geq n_0 \\ x_n \leq \frac{3q}{2} y_n, , & \text{since } y_n > 0 \text{ and } n \geq n_{\frac{q}{2}}. \end{cases}$$

Hence, by part (1) of the theorem, $\lim_{n\to\infty} \frac{3q}{2} \cdot y_n = \frac{3q}{2} \cdot \lim_{n\to\infty} y_n = \frac{3q}{2} y_n = +\infty$.

□

Example 4.58. (1) Let $(x_n)_{n\in\mathbb{N}}$ be the sequence define by $x_n := \frac{n!}{2^n}$ We compute $\lim_{n\to\infty} \frac{n!}{2^n}$.
We have $\frac{n!}{2^n} \geq \frac{2\cdot 3\cdot 3\cdots 3\cdot 3}{2^n} = \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}$, and $\lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^{n-2} = +\infty$ according to Example 4.52.
Hence, Theorem 4.57 part (1.i) yields $\lim_{n\to\infty} \frac{n!}{2^n} = +\infty$.

(2) Similarly, but using part (1.ii) of Theorem 4.57, then $\lim_{n\to\infty} -\frac{n!}{2^n} = -\infty$.

4.6 More convergence criteria

We can apply the Squeeze Theorem 4.36 to obtain more convergence criteria.

Corollary 4.59 (Quotient criterion)**.** *Let* (x_n) *be a sequence. Assume that*

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = q \in \mathbb{R}_+ \cup \{+\infty\}.$$

(1) If $q < 1$*, then both* (x_n) *and* $(|x_n|)$ *converge and the limit is* 0 *for both sequences.*

(2) If $q > 1$ *or* $q = +\infty$*, then* (x_n) *and* $(|x_n|)$ *both are non-converging sequences. Moreover,* $\lim_{n\to\infty} |x_n| = +\infty$.

Remark 4.60. As in the statement of the Corollary we are assuming that the sequence $y_n := \frac{|x_{n+1}|}{|x_n|}$ converges, then since $y_n \geq 0$, $\forall n \gg 1$, then the limit q of y_n is automatically a non-negative real number, cf. Corollary 4.32. Thus $q \geq 0$.

Proof. We show here only the $0 \le q < 1$ case; the other case is similar, and is left as a homework.
Fix $\varepsilon := \frac{1-q}{2}$. In particular $\varepsilon > 0$ and $q + \varepsilon < 1$. There is an index $n_\varepsilon \in \mathbb{N}$, such that

$$\forall n \ge n_\varepsilon, \quad \left| \frac{|x_{n+1}|}{|x_n|} - q \right| < \epsilon, \quad \text{or equivalently,}$$
$$\forall n \ge n_\varepsilon, \quad q - \varepsilon < \frac{|x_{n+1}|}{|x_n|} < q + \varepsilon,$$

thus, $|x_{n+1}| < (q+\varepsilon)|x_n|$. Denoting $\overline{q} := q + \varepsilon$, then

$$\overline{q} < 1 \quad \text{and} \quad \forall i \in \mathbb{N},\ |x_{n_\varepsilon + i}| \le |x_{n_\varepsilon}|\, \overline{q}^i.$$

Hence, as

$$\forall n \ge n_\varepsilon, \quad 0 \le |x_n| \le |x_{n_\varepsilon}| \overline{q}^{n-n_\varepsilon},$$

we may apply the Squeeze Theorem 4.36 to $|x_n|$ since

$$\lim_{n\to\infty} |x_{n_\varepsilon}|\overline{q}^{n-n_\varepsilon} = \underbrace{\frac{|x_{n_\varepsilon}|}{\overline{q}^{n_\varepsilon}} \lim_{n\to\infty} \overline{q}^n = \frac{|x_{n_\varepsilon}|}{\overline{q}^{n_\varepsilon}}}_{|\overline{q}|<1 \Rightarrow \lim_{n\to\infty} |\overline{q}|^n = 0} \cdot 0 = 0.$$

□

Example 4.61. We present some examples showing that if in Corollary 4.59 $q = 1$, then we cannot conclude anything about the behavior of the sequence (x_n).

(1) If $x_n := n$, then (x_n) is non-convergent and $\lim_{n\to\infty} x_n = +\infty$, while

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

(2) If $x_n := (-1)^n n$, then (x_n) is not bounded and its limit does not exist in $\overline{\mathbb{R}}$, while

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

(3) If $x_n := \frac{n+1}{n}$, then (x_n) is convergent to 1, while

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\frac{n+2}{n+1}}{\frac{n+1}{n}} = \lim_{n\to\infty} \frac{(n+2)n}{(n+1)^2} = 1.$$

(4) If $x_n := (-1)^n$, then (x_n) is bounded but it does not admit a finite limit, while

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} 1 = 1.$$

Hence, all possible behaviors of a sequence, in terms of its convergence or lack thereof, can appear when $q = 1$ in Corollary 4.59.

Another consequence

Corollary 4.62 (Root criterion)**.** *Let (x_n) be a sequence. Assume that*

$$\lim_{n\to\infty} \sqrt[n]{|x_n|} = q \in \mathbb{R}_+ \cup \{+\infty\}.$$

(1) If $q < 1$, then both (x_n) and $(|x_n|)$ converge and their limit is 0.

(2) If $q > 1$ or $q = +\infty$, then (x_n) and $(|x_n|)$ both are non-converging sequences. Moreover, $\lim_{n\to\infty} |x_n| = +\infty$.

Proof. We prove part (2). The proof of the case is analogous.
We start assuming that $q \in (1, +\infty)$.
If $q = +\infty$, instead, then there exists $n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$, $\sqrt[n]{|x_n|} \geq 2$ or, equivalently, $|x_n| \geq 2^n$. Since $\lim_{n\to\infty} 2^n = +\infty$, by Theorem 4.57, then $\lim_{n\to\infty} |x_n| = +\infty$. □

4.7 Monotone sequences

Let us recall that we say that a sequence (x_n) is monotone if it is increasing or decreasing, cf. Definition 4.7.

Theorem 4.63. *Let $(x_n)_{n\geq l}$ be a monotone sequence.*

(1) If $(x_n)_{n\geq l}$ is bounded and increasing (resp. decreasing), then $(x_n)_{n\geq l}$ is convergent and

$$\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N},\ n \geq l\} \qquad (\textit{resp.}\ \lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N},\ n \geq l\}).$$

(2) If $(x_n)_{n\geq l}$ is unbounded and increasing (resp. decreasing) then $\lim_{n\to\infty} x_n = +\infty$ (resp. $\lim_{n\to\infty} x_n = -\infty$).

Proof. We prove only the increasing case. We leave as a homework to change the words in it to obtain a proof for the decreasing case.
Set $S := \sup\{x_n \mid n \in \mathbb{N},\ n \geq l\}$ and let $0 < \varepsilon \in \mathbb{R}$ be arbitrary. By definition, S is the smallest upper bound, so $S - \varepsilon$ is not an upper bound. Hence, there exists $n_\varepsilon \in \mathbb{N}$ such that $S - \varepsilon < x_{n_\varepsilon}$. In particular, for any integer $n \geq n_\varepsilon$:

$$S - \varepsilon \underbrace{< x_{n_\varepsilon}}_{\text{definition of } n_\varepsilon} \underbrace{\leq x_n}_{(x_n) \text{ is monotone}} \underbrace{\leq S}_{S \text{ is the supremum}} < S + \varepsilon.$$

□

Example 4.64 (Nepero's number e)**.** Let us consider the sequence $(x_n)_{n\geq 1}$ defined by

$$x := \left(1 + \frac{1}{n}\right)^n,\ n \in \mathbb{N}^*.$$

Claim. The sequence $(x_n)_{n\geq 1}$ is strictly increasing.

Proof. We need to show that $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$, $\forall n \in \mathbb{N}^*$. Indeed,

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n^i} = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \frac{1}{n^i} = \sum_{i=0}^{n} \frac{1}{i!} \frac{n(n-1)\dots(n-(i-1))}{n^i} \\
&= \underbrace{1}_{=\binom{n}{0}\frac{1}{n^0}} + \underbrace{1}_{=\binom{n}{1}\frac{1}{n}=\frac{n}{n}} + \sum_{i=2}^{n} \frac{1}{i!} \underbrace{\frac{n}{n}}_{=1} \overbrace{\frac{(n-1)}{n} \frac{(n-2)}{n} \frac{(n-(i-1))}{n}}^{i-1 \text{ terms}} \qquad (4.64.\text{a}) \\
&= 1 + 1 + \sum_{i=2}^{n} \frac{1}{i!} \underbrace{\left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{i-1}{n}\right)}_{i-1 \text{ terms}}
\end{aligned}$$

Similarly,

$$\left(1+\frac{1}{n+1}\right)^{n+1} = \sum_{i=0}^{n+1} \frac{1}{i!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{i-1}{n+1}\right) \tag{4.64.b}$$
$$= 1+1+\left(\sum_{i=2}^{n}\frac{1}{i!}\underbrace{\left(1-\frac{1}{n+1}\right)}_{>\left(1-\frac{1}{n}\right)}\underbrace{\left(1-\frac{2}{n+1}\right)}_{>\left(1-\frac{2}{n}\right)}\dots\underbrace{\left(1-\frac{i-1}{n+1}\right)}_{>\left(1-\frac{i-1}{n}\right)}\right)+\left(\frac{1}{n+1}\right)^{n+1}$$
$$> 2+\sum_{i=2}^{n}\frac{1}{i!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{i-1}{n+1}\right)$$

□

Having proved our claim, then $(x_n)_{n\geq 1}$ is a monotone increasing sequence. Is it bounded? Yes, it is: indeed,

$$\left(1+\frac{1}{n}\right)^{n} =2+\sum_{i=2}^{n}\frac{1}{i!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{i-1}{n}\right)$$
$$\leq \sum_{i=0}^{n}\frac{1}{i!} \leq 1+\sum_{i=1}^{n}\frac{1}{2^{i-1}} = 1+\frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 3-\frac{1}{2^n} \leq 3,$$

where, for evaluating the sum, we used the formula that we proved in Proposition 1.6

$$(1+\dots+a^{n-1}) = \frac{1-a^n}{1-a},$$

for $a=\frac{1}{2}$. Hence, $(x_n)_{n\geq 1}$ is not only increasing, but also bounded above by 3. Thus, $\lim\limits_{n\to\infty} x_n$ exists, according to Theorem 4.63.

Definition 4.65. We define $e := \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Theorem 4.63 also gives another method for showing the existence of limits for recursive sequences:

Example 4.66. We consider the recursive sequence $(x_n)_{n\in\mathbb{N}}$ defined as

$$\begin{cases} x_{n+1} = \frac{1}{2}\left(x_n+\frac{1}{x_n}\right) \\ x_0 = 2. \end{cases}$$

First we claim that $x_n > 0$ for all integers $n\in\mathbb{N}$. This is certainly true for $n=0$, and if we assume it for $n-1$, then the recursive formula gives it to us also for n. Hence, by induction, $\forall n\in\mathbb{N}$, $x_n>0$. In particular, the division in the definition does make sense.
Next, we claim that $x_n\geq 1$ for all integers $n\geq 1$. Indeed, a similar induction shows that this claim: indeed, for $n=0$, we have $x_0=2\geq 1$. Furthermore,

$$x_{n+1} = \frac{1}{2}\left(x_n+\frac{1}{x_n}\right)\geq 1 \Leftrightarrow x_n+\frac{1}{x_n}\geq 2 \Leftrightarrow x_n^2+1\geq 2x_n \Leftrightarrow (x_n-1)^2\geq 0, \tag{4.66.c}$$

where we used that we already know that $x_n>0$, when we multiplied by x_n. So, by (4.66.c), the induction step works too. That is, assuming $x_n\geq 1$, we obtain that $x_{n+1}\geq 1$ holds as well.
Next, we claim that the sequence is decreasing indeed,

$$x_n - x_{n+1} = x_n-\frac{1}{2}\left(x_n+\frac{1}{x_n}\right) = \frac{1}{2}\left(x_n-\frac{1}{x_n}\right)\geq 0,$$

where we obtained the last inequality using that $x_n \geq 1 \geq \frac{1}{x_n}$.

So, (x_n) is decreasing (hence bounded from above) and also bounded from below by 1. In particular, x_n is convergent, and $\lim_{n\to\infty} x_n \geq 1$. Hence to find the actual limit we may just apply limit to the recursive equation to obtain that if y is the limit, then

$$y = \frac{1}{2}\left(y + \frac{1}{y}\right) \Leftrightarrow \frac{y}{2} = \frac{1}{2y} \Leftrightarrow y^2 = 1$$

As we also know that $y \geq 1$, $y = 1$ has to hold. So, $\lim_{n\to\infty} x_n = 1$.

4.8 Liminf, limsup

Let $(x_n)_{n\geq l}$ be a bounded sequence. We define two new sequences

$$(y_n)_{n\geq l} := \sup\{x_k | n \leq k \in \mathbb{N}\}, \quad (z_n)_{n\geq l} := \inf\{x_k | n \leq k \in \mathbb{N}\}.$$

The sequence $(y_n)_{n\geq l}$ (resp. $(z_n)_{n\geq l}$) is a decreasing (increasing) sequence, as sup (inf) is taken over smaller and smaller sets or real numbers as n increases. Moreover, if $C \in \mathbb{R}$ is an upper bound (resp. lower bound) for (x_n), then it is also an upper bound (lower bound) for (y_n) (for (z_n)). Hence, Theorem 4.63 implies that the sequence (y_n) (resp. (z_n)) is convergent since it is monotone and bounded. This observation justifies the following definition.

Definition 4.67. With the notation just introduced,

(1) we call the limit of the sequence (y_n) the *limsup* of the sequence (x_n) and we denote it by

$$\lim_{n\to\infty} \sup x_n;$$

(2) we call the limit of the sequence (z_n) the *liminf* of the sequence (x_n) and we denote it by

$$\lim_{n\to\infty} \inf x_n.$$

Example 4.68. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined as $x_n = (-1)^n$.
Then,

$$\lim_{n\to\infty} \sup\{x_k | n \leq k \in \mathbb{N}\} = \lim_{n\to\infty} \sup\{-1, 1\} = \lim_{n\to\infty} 1 = 1,$$

and

$$\lim_{n\to\infty} \inf\{x_k | n \leq k \in \mathbb{N}\} = \lim_{n\to\infty} \sup\{-1, 1\} = \lim_{n\to\infty} -1 = -1.$$

Hence,

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} y_n = 1,$$

and

$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} z_n = -1.$$

In general, it is not always easy to compute the liminf and limsup of a sequence. Nonetheless, when a sequence is converging, this task becomes significantly simpler.

Proposition 4.69. *Let $(x_n)_{n\geq l}$ be a converging sequence with $\lim_{n\to\infty} x_n = l \in \mathbb{R}$. Then,*

$$\lim_{n\to\infty} \inf x_n = l = \lim_{n\to\infty} \sup x_n.$$

4.9 Subsequences

Definition 4.70. Let $(x_n)_{n\geq l}$ be a sequence. A subsequence $(y_k)_{k\in\mathbb{N}}$ of $(x_n)_{n\geq l}$ is a sequence sequence defined by $y_k := x_{n_k}$ where $n_k \in \mathbb{N}$ is defined by a function

$$\begin{aligned} f\colon \mathbb{N} &\to \{n \in \mathbb{N} \mid n \geq l\} \\ k &\mapsto f(k) =: n_k \end{aligned}$$

which is a strictly increasing function of k.

To say that f is strictly increasing simply means that $\forall k \in \mathbb{N}$, $f(k) < f(k+1)$.

Thus, a subsequence of $(x_n)_{n\geq l}$ is a new sequence $(y_k)_{k\in\mathbb{N}}$ constructed taking the values of $(x_n)_{n\geq l}$ along a subset of the indices of $(x_n)_{n\geq l}$, where we remember the order in which those values appear.

Example 4.71. (1) for the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := (-1)^n$, then both the constant 1 sequence and the constant -1 sequences are subsequences.
In fact for

(i) for $f(k) := 2k$, then $y_k := x_{n_k} = x_{2k} = (-1)^{2k} = 1$; and
(ii) for $f(k) := 2k = 1$, then $y_k := x_{n_k} = x_{2k+1}(-1)^{2k+1} = -1$.

(2) for the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := n^2$, then $y_k := x_{n_k} = k^6$ is the subsequence obtained by setting $n_k := k^3$.

(3) for the sequence $(x_n)_{n\geq 1}$ defined by $x_n = \left(1 + \frac{2}{n}\right)^n$ and $n_k := 2k$, then

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \left(1 + \frac{2}{2k}\right)^{2k} = \lim_{k\to\infty} \left(\left(1 + \frac{1}{k}\right)^k\right)^2 = \left(\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k\right)^2 = e^2.$$

We can ask whether $(x_n)_{n\geq 1}$ converges and, if so, what its limit is? Is $\lim\limits_{n\to\infty} x_n = e^2$?

The next proposition illustrates the (simple) connection between the convergence of a sequence and that of a subsequence.

Proposition 4.72. *Let (x_n) be a sequence.*
If $\lim\limits_{n\to\infty} x_n = a \in \overline{\mathbb{R}}$, then for any subsequence (y_k), $y_k := x_{n_k}$, $\lim\limits_{k\to\infty} y_k = a$.

Let us recall that $a \in \overline{\mathbb{R}}$ means that either a is a real number or $a = \pm\infty$.

The proof of Proposition 4.72 is just about invoking the definition of limit, cf. Definition 4.21 and 4.50, thus we do not spell out the details here.

Example 4.73. Let $(x_n)_{n\geq 1}$ be the sequence defined as $x_n := (-1)^n \left(1 + \frac{1}{n}\right)^n$.

(1) If $n_k = 2k$, then the subsequence $(y_k)_{k\geq 1}$ defined by $y_k := x_{2k} = \left(1 + \frac{1}{2k}\right)^{2k}$ and $\lim\limits_{k\to\infty} y_k = e$;

(2) if $n_k = 2k+1$, then the subsequence $(y_k)_{k\geq 1}$ defined by $y_k := x_{2k+1} = -\left(1 + \frac{1}{2k+1}\right)^{2k+1}$ and $\lim\limits_{k\to\infty} y_k = -e$.

Hence, the sequence $(x_n)_{k\geq 1}$ cannot converge.

We just saw an example of a sequence which does not converge, but which admits converging subsequences – which converge to different limits. Given a sequence (x_n), does it always admit a converging subsequence? The answer, for a general sequence (x_n) is no. In fact, Proposition 4.72 shows that if $\lim\limits_{n\to\infty} x_n = \pm\infty$, then any subsequence will have the same limit, thus, (x_n) will not admit any converging subsequence.

Remark 4.74. It actually follows from the definition, that if a sequence (x_n) is unbounded then it admits a subsequence (y_k), $y_k := x_{n_k}$ such that either $\lim_{k\to\infty} y_k = +\infty$ or $\lim_{k\to\infty} y_k = -\infty$. [Try to prove this claim!]

Hence, in view of the claim, we can ask whether for a bounded sequence (x_n), there always exists a convergent subsequence (y_k), $y_k := x_{n_k}$. Indeed, we can always answer this question affirmatively, as shown by the following celebrated result.

Theorem 4.75 (Bolzano-Weierstrass). *Let (x_n) be a bounded sequence. Then (x_n) contains a convergent subsequence.*

Proof. We define n_k by induction $k \in \mathbb{N}$. We set $n_0 = 0$ - this is the starting step of the induction. So, let us assume n_{k-1} is defined. Let us then define $s_k := \sup\{x_n | n > n_{k-1}\}$. Then there is a integer $n_k > n_{k-1}$ such that

$$x_{n_k} > s_k - \frac{1}{k}.$$

We claim that (x_{n_k}) is convergent. Indeed, this follows from the squeeze principle, as we have

$$s_k - \frac{1}{k} < x_{n_k} < s_k,$$

if we can prove that (s_k) converges. As $s_k := \sup x_n \mid n > n_{k-1}$, then $s_{k+1} \leq s_k$, as the subset of $\mathbb{R}$ of which we are taking the supremum gets smaller with k. Hence, (s_k) is decreasing. Moreover, (s_k) is bounded, since $\inf\{x_n \mid n \in \mathbb{N}\} \leq s_k \leq \sup\{x_n \mid n \in \mathbb{N}\}$. Hence, $\lim_{k\to\infty} s_k = l \in \mathbb{R}$, and

$$l = \lim_{k\to\infty} s_k = \lim_{k\to\infty} s_k - \lim_{k\to\infty} \frac{1}{k} = \lim_{k\to\infty} (s_k - \frac{1}{k}),$$

so that also $\lim_{k\to\infty} x_{n_k} = l$. $\square$

Example 4.76. Sometimes, given a sequence (x_n), it is possible to write down explicitly some convergent subsequences.
For example, defining $x_n := \sin\left(\frac{n\pi}{4}\right)\left(1 + \frac{1}{n}\right)^n$, then setting

(1) $n_k := 8k + 1$, $\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_{n_k} = \frac{1}{\sqrt{2}}e$;

(2) $n_k := 8k + 2$, then $\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_{n_k} = e$,

(3) $n_k = 8k + 5$, then $\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_{n_k} = -\frac{1}{\sqrt{2}}e$.

Example 4.77. Other times, given a sequence (x_n), it is not quite possible to write down explicitly converging subsequences. One example where this is not immediate is given for example by the sequence $x_n := \sin n$ – you can read here a discussion of how to obtain a converging subsequence, and how "difficult" that should be.
In general, the Bolzano-Weierstrass Theorem 4.75 implies that some convergent subsequence exists but it does not a priori indicate how to explicitly obtain one. [Try to write down a converging subsequence of the sequence $x_n := \sin(n)\left(1 + \frac{1}{n}\right)^n$.]

Example 4.78. Let $a > 0$ be an integer. Then defining the sequence $(x_n)_{n\geq 1}$ by $x_n =: \left(1 + \frac{a}{n}\right)^n$, we can consider the subsequence $(y_k)_{k\geq 1}$ defined by $y_k := x_{ak}$, to obtain:

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} x_{ak} = \lim_{k\to\infty} \left(1 + \frac{a}{ak}\right)^{ak} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^{ak} = \lim_{k\to\infty} \left(\left(1 + \frac{1}{k}\right)^k\right)^a = e^a.$$

It is not hard to show that x_n is increasing and bounded for $a > 0$ – the proof is similar to the case where $a = 1$, using binomial expansion. In particular, x_n is convergent, as it is bounded – again the proof of this is similar to the case $a = 1$. However, if (x_n) is convergent we may compute the limit $\lim_{n\to\infty} x_n$ by computing the limit of any of subsequence of (x_n). Thus, $\lim_{n\to\infty} x_n = e^a$.

4.10 Cauchy convergence

Definition 4.79. A sequence (x_n) is a *Cauchy sequence* if for every $\varepsilon \in \mathbb{R}^*_+$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every integer $n, m \geq n_\varepsilon$, $|x_n - x_m| \leq \varepsilon$.

Let us start with a few examples of Cauchy sequences.

Example 4.80. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_n := 1 - \frac{1}{n}$, then for all $n, m \geq \left[\frac{2}{\varepsilon}\right]$,

$$|x_n - x_m| = \left|1 - \frac{1}{n} - 1 + \frac{1}{m}\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \underbrace{\frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{n,m\geq\left[\frac{2}{\varepsilon}\right]\Rightarrow\frac{1}{n},\frac{1}{m}<\frac{\varepsilon}{2}} = \varepsilon.$$

Hence, $(x_n)_{n\geq 1}$ is a Cauchy sequence. It is easy to compute that the sequence converge and it has limit 1.

Cauchy sequences naturally appear when we try to approximate the decimal representation of a real number, by means of rational numbers.

Example 4.81. Let $x \in \mathbb{R}$ be a real number. Let us think of x by means of a decimal representation. We can define a sequence $(x_n)_{n\in\mathbb{N}}$, in the following way:

- $x_0 = [x]$;
- for $n \geq 1$, x_n is defined as the truncation of the decimal representation of x at the n-th decimal digit.

With this definition, we can verify that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. In fact, for any $n, m \in \mathbb{N}$, $n < m$, then

$$|x_m - x_n| < 10^{-n}.$$

Thus, for a given $\epsilon > 0$, it suffices to take $n_\epsilon \in \mathbb{N}$ such that $10^{-n_\epsilon} < \epsilon$ – this is always possible since $\lim_{n\to\infty} 10^{-n} = 0$ – and thus

$$\forall n, m \geq n_\epsilon, \quad |x_n - x_m| < 10^{-n_\epsilon} < \epsilon.$$

The important fact about Cauchy sequences is that they are always convergent.

Theorem 4.82. *Let (x_n) be a sequence. Then, the following two properties are equivalent:*

(1) (x_n) is convergent;

(2) (x_n) is a Cauchy sequence.

In view of this theorem, we will indicate that a sequence (x_n) is a Cauchy sequence (or, simply, Cauchy) by saying that it is *Cauchy convergent.* Of course, by the above statement, all converging sequences are Cauchy convergent, and viceversa.

Proof. (1) $\implies$ (2). First we assume that (x_n) is convergent, and then we show that it is Cauchy convergent. Let $x := \lim_{n\to\infty} x_n$ and $0 < \varepsilon \in \mathbb{R}$ arbitrary. Then there is an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all integers $n \geq n_{\frac{\varepsilon}{2}}$, we have $|x_n - x| \leq \frac{\varepsilon}{2}$. Then, for any integers $n, m \geq n_{\frac{\varepsilon}{2}}$ we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(2) $\implies$ (1). Let us assume that (x_n) is Cauchy convergent. We divide this part of the proof into three steps:

(1) We first claim that then (x_n) is bounded. Indeed, there is an $n_1 \in \mathbb{N}$ such that for all integers $n \geq n_1$, $|x_n - x_m| \leq 1$. Then, an upper bound for $|x_n|$ is

$$\max\{|x_0|, \dots, |x_{n_1-1}|, |x_{n_1}| + 1\}.$$

(2) As (x_n) is bounded, then by Bolzano-Weierstrass, it contains a convergent subsequence x_{n_k} converging to $x \in \mathbb{R}$.

(3) We show that $\lim_{n\to\infty} x_n = x$.
Fix then a $0 < \varepsilon \in \mathbb{R}$. As (x_n) is Cauchy, there is an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all integers $n, m \geq n_{\frac{\varepsilon}{2}}$,

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

Now, there is a k such that $n_k \geq n_{\frac{\varepsilon}{2}}$ and $|x_{n_k} - x| \leq \frac{\varepsilon}{2}$. For this value of k and any integer $n \geq n_{\frac{\varepsilon}{2}}$ we have:

$$|x_n - x| \leq |(x_n - x_{n_k}) + (x_{n_k} - x)| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

5 Series

Let us start this section with the following motivating example.

Example 5.1 (Zeno's paradox)**.** Achilles races a tortoise. Achilles runs at 10 m/s, while the tortoise moves at 0.1 m/s. Achilles gives the tortoise a head start of 100m.

Step 1 Achilles runs to the tortoise's starting point, in 10s, while, at the same time, the tortoise has walked 1m forward.

Step 2 Achilles advances to where the tortoise was at the end of Step 1, in 0.1s, while the tortoise goes yet 0.001m further.

Step 3 Achilles advances to where the tortoise was at the end of Step 2, in 0.001s, while the tortoise goes yet 0.00001m further.

Step n Achilles advances to where the tortoise was at the end of Step $n-1$, in $\frac{10}{100^{n-1}}s$, while the tortoise goes yet $\frac{1}{100^{n-1}}m$ further.

The philosopher Zeno doubted that Achilles could ever overtake the tortoise, since however many steps Achilles would ever complete, the tortoise would remain ahead of him.
It should be intuitively clear, though, that the more steps Achilles and the tortoise take, the closer they get. So, if they could run for infinitely many steps of the above observations of the run, Achilles would reach the tortoise.
So, the question is whether by taking infinitely many steps of the above observations the time that has passed since the start of the run is going to infinity or it is bounded.
After the n-th step, the amount of time s_n that has passed since the start of the race is $(10 + 0.1 + 0.001 + 0.00001 + \cdots + \frac{10}{100^{n-1}})s$. We can rewrite this as

$$s_n = \sum_{i=0}^{n-1} \frac{10}{100^i}.$$

Hence, to understand whether Achilles ever reaches the tortoise, we need to understand the convergence of the sequence (s_n).

To understand how to solve the problem above, we now introduce the concept of series.

Definition 5.2. Let $(x_n)_{n\geq l}$ be a sequence. The series associated to $(x_n)_{n\geq l}$ is the sequence $(s_n)_{n\geq l}$ defined by the formula

$$s_n := \sum_{i=l}^{n} x_i.$$

Given a sequence (x_n) and the associated series (s_n) defined above, we will refer to the sequence (s_n) as the sequence of the truncated sums of (x_n). We will also use the symbol $\sum_{i=0}^{\infty} x_i$ to refer to the sequence (s_n). Depending on the context, we will also use the symbol $\sum_{i=0}^{\infty} x_i$ to denote the limit of the series, that is, $\sum_{i=0}^{\infty} x_i := \lim_{n\to\infty} s_n$, provided that such limit exists.

Example 5.3. The following are a few examples of sequences (x_n) and of their sequences of truncated sums (s_n).

(1) (Geometric series) Taking $x_k := \frac{1}{2^k}$, then $s_n = \sum_{k=0}^{n} \frac{1}{2^k} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right)$; in general, for $q \in \mathbb{R}$, we can define $x_n = q^n$ then $s_n = \sum_{k=0}^{n} q^k$.

(2) (Harmonic series) Taking $x_k := \frac{1}{k}$, then $s_n = \sum_{k=1}^{n} \frac{1}{k}$;

(3) Taking $x_k := (-1)^k \frac{1}{k}$, then $s_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$;

(4) Taking $x_k := \frac{1}{k^2}$, then $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$;

(5) Taking $x_k := \frac{1}{k^s}$, for a fixed $s \in \mathbb{Q}_+^*$, then $s_n = \sum_{k=1}^{n} \frac{1}{k^s}$, see. **??**;

(6) (Another definition of e) Taking $x_k := \frac{1}{k!}$, then $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. We shall show in **??**, that $\sum_{k=0}^{\infty} \frac{1}{k!} = e$.

In the case of the first example one has an explicit expression for s_n without involving sums. However, in the other cases, we are not able to provide such formulas. So, one just has to take it as it is, so as a sequence obtained by adding the first n elements of the given other sequence.

We can define a notion of convergence for series, using the notion of convergence already introduced for sequences.

Definition 5.4. Let $(x_n)_{n \geq l}$ be a sequence.

(1) The series $(s_n)_{n \geq l}$, $s_n := \sum_{k=l}^{n} x_k$ associated to (x_n) is convergent if $(s_n)_{n \geq l}$ converges to a finite limit.

(2) The series $(s_n)_{n \geq l}$, $s_n := \sum_{k=l}^{n} x_k$ associated to (x_n) approaches $+\infty$ (resp. $-\infty$) if $\lim_{n \to \infty} s_n = +\infty$ (resp. $\lim_{n \to \infty} s_n = -\infty$).

Notation 5.5. Given a sequence $(x_n)_{n \geq l}$, such that the series $(s_n)_{n \geq l}$ associated to $(x_n)_{n \geq l}$ is convergent with $\lim_{n \to \infty} s_n = y$, we will write

$$\sum_{k=0}^{\infty} x_k = y.$$

to denote .

In the course of this section we will discover several techniques to determine when a series converges (or not).

A first natural condition from convergence stems from the following simple observation: when a sequence (x_n) has values in the positive real numbers $\mathbb{R}_+$, then the series (s_n) is increasing, hence it converges if and only if it is bounded.

Example 5.6. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_n := \frac{1}{2^n}$. Then

$$s_n := \sum_{k=0}^{n} \frac{1}{2^k} = 2\left(1 - \frac{1}{2^{n+1}}\right).$$

This identity implies that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2\left(1 - \frac{1}{2^{n+1}}\right) = 2 =: \sum_{k=0}^{\infty} \frac{1}{2^k}.$$

Similarly, taking $x_n := q^n$, for $q \in \mathbb{R}$, then

$$s_n := \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q},$$

cf. (1.6.f). If $|q| < 1$, then we showed already that $\lim\limits_{n\to\infty} q^n = 0$. Thus,

$$\lim_{n\to\infty} s_n = \frac{1}{1-q} =: \sum_{k=0}^{\infty} q^k$$

The above observation can be naturally extended to yield the following proposition.

Proposition 5.7. *Let $(x_n)_{n\geq l}$ be a sequence. Assume that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $x_n \geq 0$. Then,*

$$\sum_{k=l}^{\infty} x_k = \begin{cases} y \in \mathbb{R} & \text{if and only if } (s_n)_{n\geq l},\ s_n := \sum_{k=l}^{n} x_k \text{ is a bounded sequence} \\ +\infty & \text{if and only if } (s_n)_{n\geq l} \text{ is not bounded.} \end{cases}$$

Proof. As $\forall n \geq n_0$, $x_n \geq 0$, then $(s_n)_{n\geq n_0}$ is increasing starting from n_0. Thus, we can conclude by Theorem 4.63. □

Using Cauchy's convergence criterion for sequences, see Theorem 4.82, we have the following basic convergence criterion for series.

Proposition 5.8. *Let $(x_n)_{n\geq l}$ be a sequence. Then, the following conditions are equivalent:*

(1) $\sum\limits_{k=l}^{\infty} x_k$ *is convergent;*

(2) $(s_n)_{n\geq l}$ *is a Cauchy sequence;*

(3) for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that for all integers $m, n \geq n_\varepsilon$, with $m > n$,

$$\left|\sum_{k=n+1}^{m} x_k\right| < \varepsilon.$$

Example 5.9. Let $(x_n)_{n\geq 1}$ be sequence defined by $x_n := \frac{1}{n}$. We show that $\sum\limits_{k=1}^{\infty} \frac{1}{k} = +\infty$.

Since $\forall k \geq 1$, $\frac{1}{k} > 0$, then we know that either $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ either converges to a finite limit $y \in \mathbb{R}$ or $\sum\limits_{k=1}^{\infty} \frac{1}{k} = +\infty$. Thus, let us assume, by contradiction, that $\sum\limits_{k=1}^{\infty} \frac{1}{k} = y \in \mathbb{R}$. Hence, by **??**, for

$\varepsilon = \frac{1}{4}$, Cauchy's condition for the convergence of series is satisfied. That is, there exists some index $n_{\frac{1}{4}} \in \mathbb{N}$ such that for all $n, m \geq n_{\frac{1}{4}}$, with $m > n$, then

$$\left| \sum_{i=n+1}^{m} \frac{1}{i} \right| < \frac{1}{4}.$$

In particular, the above inequality must hold for $n := n_{\frac{1}{4}}$ and $m = 2n$, in which case,

$$\frac{1}{4} > \left| \sum_{k=n+1}^{2n} \frac{1}{k} \right| = \underbrace{\sum_{k=n+1}^{2n} \frac{1}{k} \geq \sum_{k=n+1}^{2n} \frac{1}{2n}}_{k \leq 2n \Rightarrow \frac{1}{k} \geq \frac{1}{2n}} = \frac{1}{2}$$

which provides the sought contradiction.

An immediate consequence of **??** is the following necessary condition for convergence of a series.

Proposition 5.10. *Let $(x_n)_{n \geq l}$ be a sequence. If $\sum_{n=l}^{\infty} x_n$ is convergent, then $\lim_{n \to \infty} x_n = 0$.*

Proof. Indeed, by **??**, for every $0 < \varepsilon \in \mathbb{R}$, there is an $n_\varepsilon \in \mathbb{N}$ such that for all integers $m, n \geq n_\varepsilon$ with $m > n$,

$$\left| \sum_{k=n+1}^{m} x_k \right| \leq \varepsilon.$$

In particular, if we choose $n := m - 1$, then we obtain that $\forall m \geq n_\epsilon + 1$,

$$\varepsilon \geq \left| \sum_{k=m-1}^{m} x_k \right| = |x_m|.$$

This implies that $\lim_{n \to \infty} x_n = 0$. □

Example 5.11. *The series $\sum_{n=0}^{\infty} \cos(n)$ is not convergent.* By **??**, it suffices to show that $x_n := \cos(n)$ does not converge to 0. Let us assume by contradiction that instead it does. Then, so do all its subsequences. However, consider the subsequence given by $n_k := \lfloor 2k\pi \rfloor$. Thus,

$$x_{n_k} = \cos(\lfloor 2k\pi \rfloor) \geq \cos(2k\pi - 1) = \cos(-1) > 0,$$

where the inequality follows from the fact that $\cos(x)$ is an increasing function in the interval $2k\pi - \frac{\pi}{2} \leq x \leq 2k\pi$, and moreover, as $\frac{\pi}{2} > 1$, $2k\pi - 1$ is in this interval.
As $x_{n_k} \geq \cos(-1) > 0$, $\forall n_k$, then (x_{n_k}) cannot converge to 0; but this is in contradiction with the the assumption that x_n converge to 0.

One can use **??** to give a version of the Squeeze Theorem for series.

Theorem 5.12 (Squeeze theorem for series)**.** *Let $(x_n)_{n \geq l}$, $(y_n)_{n \geq l}$ be sequences. Assume there exists $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$, $0 \leq x_n \leq y_n$.*

(1) If $\sum_{k=l}^{\infty} y_k$ is convergent, then $\sum_{k=l}^{\infty} x_k$ is also convergent.

Images/graphcos.png

Figure 9: To check that $\cos(x)$ is increasing, by using periodicity, it suffices to check that the same holds over the inverval $[-\frac{\pi}{2}, 0]$.

(2) If $\sum_{k=l}^{\infty} x_k = +\infty$*, then also* $\sum_{k=l}^{\infty} y_k = +\infty$*.*

Proof. For every $n, m \geq n_0$ with $m > n$

$$0 \leq \left| \sum_{k=n+1}^{m} x_k \right| = \sum_{k=n+1}^{m} x_k \leq \sum_{k=n+1}^{m} y_k = \left| \sum_{k=n+1}^{m} y_k \right|.$$

So, if the property in **??**.3 is verified for y_k then it must also holds for x_k. On the other hand, if the property in **??**.3 is not satisfied for the sequence of truncated sums of $(x_n)_{n\geq l}$, then it must also fail for the sequence of truncated sums of $(y_n)_{n\geq l}$. $\square$

Definition 5.13. If $0 < s$ is a rational number, say $s = \frac{a}{b}$ then we define $n^s := \sqrt[b]{n^a}$ for all $n \in \mathbb{N}$.

Example 5.14. $2^{\frac{2}{3}} = \sqrt[3]{4}$ and this is the only positive real solution to the equation $X^3 - 4 = 0$.

Remark 5.15. The above definition does not depend on the representation of s as $\frac{a}{b}$. That is, if we replace $\frac{a}{b}$ by $\frac{ca}{cb}$ (where $c \in \mathbb{N}$), then:

$$\sqrt[cb]{n^{ca}} = \sqrt[b]{\sqrt[c]{n^{ca}}} = \sqrt[b]{n^a}.$$

Moreover, any $x, y \in \mathbb{R}_+$, and $s, t \in \mathbb{Q}$, then:

(1) $x^0 = 1$;

(2) if $x > y$, $s > 0$, then $x^s > y^s$;

(3) if $x > y$, $s < 0$, then $x^s < y^s$;

(4) if $x > 1$ and $s > t$, then $x^s > x^t$;

(5) if $x < 1$ and $s > t$, then $x^s < x^t$.

Example 5.16. *If $0 < s = \frac{a}{b} < 1$ is a rational number, then $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is divergent.*
In fact, with the assumption $0 < s \leq 1$, we can use the Squeeze **??**: indeed, for each $n \geq 1$,

$$\frac{1}{n^s} = \underbrace{\frac{1}{(\sqrt[b]{n})^a} \geq \frac{1}{(\sqrt[b]{n})^b}}_{b>a \text{ and } \sqrt[b]{n}\geq 1 \Rightarrow (\sqrt[b]{n})^a < (\sqrt[b]{n})^b} = \frac{1}{n},$$

and since $\sum\limits_{k=1}^{\infty} \frac{1}{k} = +\infty$, then the Squeeze Theorem for series implies that for all $0 < s < 1,\ s \in \mathbb{Q}$ also $\sum\limits_{k=1}^{\infty} \frac{1}{k^s} = +\infty$.

Example 5.17. *If $s > 1$ be a rational number, then $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is convergent.*
Indeed, when $s > 1$,

$$\begin{aligned} s_n := \sum_{k=1}^{n} \frac{1}{k^s} \leq \sum_{k=1}^{2n+1} \frac{1}{k^s} &= 1 + \sum_{k=1}^{n} \frac{1}{(2k)^s} + \sum_{k=1}^{n} \underbrace{\frac{1}{(2k+1)^s}}_{2k+1>2k} \\ &\leq 1 + \underbrace{\sum_{k=1}^{n} \frac{1}{(2k)^s}}_{=\frac{1}{2^s}s_n} + \sum_{k=1}^{n} \frac{1}{(2k)^s} = 1 + \frac{2}{2^s}s_n = 1 + \frac{1}{2^{s-1}}s_n \end{aligned}$$

By taking the two ends of this chain of inequalities,

$$s_n \leq 1 + 2^{1-s}s_n \quad \text{or, equivalently, } s_n \leq \frac{1}{1-2^{1-s}}.$$

Hence, s_n is bounded from above. As it is also increasing, since we are summing positive terms, then (s_n) is convergent by Theorem 4.63.

Remark 5.18. We will show later on in the course that

(1) $\forall s \in (0, 1],\ \sum\limits_{k=1}^{\infty} \frac{1}{k^s} = +\infty$; and,

(2) $\forall s \in (1, +\infty),\ \sum\limits_{k=1}^{\infty} \frac{1}{k^s}$ converges.

Example 5.19. *The series $\sum\limits_{i=1}^{\infty} \sin\left(\frac{1}{i}\right)$ does not converge.*
We have seen in the exercises that

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

Using Definition 4.21, taking, for example, $\epsilon := \frac{1}{2}$, then there exists $n_{\frac{1}{2}} \in \mathbb{N}$ such that $\forall n \geq n_{\frac{1}{2}}$, $\left|\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} - 1\right| < \frac{1}{2}$, or equivalently,

$$-\frac{1}{2} < \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} - 1 < \frac{1}{2}$$

In particular, $\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} > \frac{1}{2}$ from which it follows that $\frac{1}{2}\frac{1}{n} < \sin\left(\frac{1}{n}\right)$. Hence, the Squeeze Theorem and **??** imply that $\sum\limits_{i=1}^{\infty} \sin\left(\frac{1}{i}\right)$ does not converge.

The strategy adopted in **??** can be generalized to prove the following convergence criterion.

Proposition 5.20. *Let (x_n), (y_n) be sequences. Assume that $x_n, y_n > 0$, $\forall n \geq n_0$. Let (z_n) be the sequence defined as $z_n := \frac{x_n}{y_n}$.*

(1) If (z_n) is bounded and $\sum_{k=0}^{\infty} y_k$ converges then $\sum_{k=0}^{\infty} x_k$ converges.

(2) If $\lim_{n\to\infty} \frac{x_n}{y_n} = \rho \neq 0$ and $\sum_{k=0}^{\infty} y_k$ diverges then $\sum_{k=0}^{\infty} x_k$ diverges.

Remark 5.21. One cannot drop the assumption that $\rho \neq 0$ in part (2) of **??**. Indeed, taking $x_n := 2^{-n}$, $y_n := \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{2^n} = 0,$$

but we know that $\sum_{k=0}^{\infty} x_k$ converges, while $\sum_{k=0}^{\infty} y_k$ does not.

Example 5.22. *The series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is convergent.*

First let us look at the odd terms of s_n, that is, the subsequence $s'_m := s_{2m+1}$ of s_n. Then,

$$\begin{aligned} s'_m = s_{2m+1} :=& -1 + \sum_{k=2}^{2m+1} (-1)^k \frac{1}{k} = -1 + \sum_{k=1}^{m} \left((-1)^{2k} \frac{1}{2k} + (-1)^{2k+1} \frac{1}{2k+1} \right) \\ =& -1 + \sum_{k=1}^{m} \left(\frac{1}{2k} - \frac{1}{2k+1} \right) = -1 + \sum_{k=1}^{m} \frac{1}{2k(2k+1)} \\ \leq& -1 + \sum_{k=1}^{m} \frac{1}{2k \cdot 2k} = -1 + \frac{1}{4} \sum_{k=1}^{m} \frac{1}{k^2} \leq \underbrace{-1 + \frac{1}{1 - 2^{1-2}}}_{\text{by }\textbf{??}} = 1 \end{aligned}$$

Hence, s_{2m+1} is an increasing bounded sequence. Theorem 4.63 implies that (s'_m) is convergent according. Let us define $S := \lim_{m\to\infty} s_{2m+1}$.
What happens when we consider also the even terms of (s_n)? By the definition, we have

$$s_{2m} = s_{2m-1} + \frac{1}{2m}.$$

In particular, we may write the sequence s_n as the sum $s_n = x_n + y_n$ of two other sequence defined by

$$x_n := \begin{cases} s_n & \text{if } n \text{ is odd }, \\ s_{n-1} & \text{if } n \text{ is even }, \end{cases} \quad \text{and} \quad y_n := \begin{cases} 0 & \text{if } n \text{ is odd }, \\ \frac{1}{n} & \text{if } n \text{ is even }. \end{cases}$$

Claim. $\lim_{n\to\infty} x_n = S$.

Proof of the claim. x_n is increasing and its values are the same as those of s_{2m+1}. Hence, any upper bound for s_{2m+1} is automatically an upper bound for x_n too. In particular, x_n is not only increasing but also bounded. Therefore, x_n is convergent by Theorem 4.63 and then it must converge to the same limit as its subsequence s_{2m+1}, see Proposition 4.72. □

By the definition of y_n we also have $\lim_{n\to\infty} y_n = 0$ [prove this!]. By the algebraic rules for finite limits, cf. Proposition 4.30, s_n is convergent and

$$\underbrace{\lim_{n\to\infty} s_n = \overbrace{\lim_{n\to\infty} (x_n + y_n)}_{s_n = x_n + y_n}}^{\text{Proposition 4.30}} = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = S + 0 = S.$$

The previous example can be generalized to the following criterion for the convergence of series.

Proposition 5.23 (Leibniz criterion)**.** *Let (x_n) be a decreasing sequence. Assume that $\lim_{n\to\infty} x_k = 0$. Then, $\sum_{k=0}^{\infty}(-1)^k x_k$ is convergent.*

Proof. We refer to the book for the proof, but many of the main ideas are already presented in **??**. ☐

Example 5.24. Let $s \in \mathbb{Q}_+^*$. Then **??** implies that $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k^s}$ is convergent, since $\forall k \in \mathbb{N}^*$, $\frac{1}{k^s} > \frac{1}{(k+1)^s}$ and $\lim_{k\to\infty} \frac{1}{k^s} = 0$.

Example 5.25. *The series* $\sum_{i=1}^{\infty}(-1)^i \sin\left(\frac{1}{i}\right)$ *converges.* By Leibnitz criterion, it suffices to check that the sequence $(x_n)_{n\geq 1}$, $x_n := \sin\left(\frac{1}{n}\right)$ is decreasing, but this follows from the definition of the function $\sin(x)$, cf. as we already know that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$, for example, since $\sin\left(\frac{1}{n}\right) \leq \frac{1}{n}$.

Definition 5.26. Let (x_n) be a sequence. We say that $\sum_{k=0}^{\infty} x_n$ is *absolutely convergent* if the series $\sum_{k=0}^{\infty} |x_n|$ converges.

Example 5.27. In **??** we showed that $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$ is convergent. On the other hand, **??** implies that $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$ is not absolute convergent. Hence, **??** is a non-trivial criterion to establish the convergence of series whose terms are not all positive or all negative.

We now show that it is not possible for the viceversa of this to happen: that is, an abolutely convergent series is convergent.

Proposition 5.28. *If $\sum_{n=0}^{\infty} x_n$ is absolute convergent, then it is convergent.*

Proof. As we are assuming that $\sum_{k=0}^{\infty} |x_k|$ converges, then **??** implies that for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\forall m > n \geq n'_\epsilon$

$$\sum_{k=n+1}^{m} |x_k| < \varepsilon. \tag{5.28.a}$$

Let us fix $\epsilon > 0$. To prove that $\sum_{k=0}^{\infty} x_k$ converges as well, it suffice to find an index $n_\epsilon \in \mathbb{N}$ such that $\forall m > n \geq n_\epsilon$,

$$\left| \sum_{k=n+1}^{m} x_k \right| < \varepsilon. \tag{5.28.b}$$

By the triangle equality, Proposition 2.57,

$$\left|\sum_{k=n+1}^{m} x_k\right| \leq \sum_{k=n+1}^{m} |x_k|.$$

Hence, taking $n_\epsilon := n'_\epsilon$, then $\forall m > n \geq n_\epsilon$,

$$\left|\sum_{k=n+1}^{m} x_k\right| \leq \sum_{k=n+1}^{m} |x_k| < \epsilon,$$

that is, ?? implies ??. □

Example 5.29. *We show that* $\sum_{k=0}^{\infty} \frac{1}{k!} = e$. This is one of the rare occasions when we will be actually able to compute the value of the limit of an infinite sum.

First, the sequence (s_n), $s_n := \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$ is a strictly increasing sequence, since $\frac{1}{k!} > 0$, $\forall k \in \mathbb{N}$. Furthermore, (s_n) is bounded, because

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{k!} &= 1 + \sum_{k=1}^{n} \frac{1}{k!} = 1 + \sum_{k=1}^{n} \frac{1}{1 \cdot 2 \cdot 3 \cdots k} = 1 + \sum_{k=1}^{n} \underbrace{\frac{1}{1}}_{=1} \cdot \overbrace{\frac{1}{2} \cdot \underbrace{\frac{1}{3}}_{<\frac{1}{2}} \cdots \underbrace{\frac{1}{k}}_{<\frac{1}{2}}}^{i-1 \text{ terms}} \\ &\leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n} \frac{1}{2^k} \leq 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 3.\end{aligned}$$

Thus, $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$ is convergent.

We showed earlier, cf. Example 4.64, that

$$\left(1 + \frac{1}{n}\right)^n = 2 + \sum_{i=2}^{n} \frac{1}{i!}\left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \cdots \cdot \left(1 - \frac{i-1}{n}\right)$$

But, for any $n \in \mathbb{N}^*$, and for any $j \in \{1, 2, 3, \ldots, i-1\}$ then $1 - \frac{i-1}{n} \leq 1 - \frac{j}{n} < 1$, so that

$$2 + \sum_{i=2}^{n} \frac{1}{i!}\left(1 - \frac{i-1}{n}\right)^{i-1} \leq \left(1 + \frac{1}{n}\right)^n \leq 2 + \sum_{i=2}^{n} \frac{1}{i!} = \sum_{i=0}^{n} \frac{1}{i!}$$

Using ??, then $\left(1 - \frac{i-1}{n}\right)^{i-1} \geq \left(1 - (i-1)\frac{i-1}{n}\right)$ so that we can rewrite the above chain of inequality as

$$2 + \sum_{i=2}^{n} \frac{1}{i!}\left(1 - \frac{(i-1)^2}{n}\right) \leq \left(1 + \frac{1}{n}\right)^n \leq 2 + \sum_{i=2}^{n} \frac{1}{i!} = \sum_{i=0}^{n} \frac{1}{i!}$$

or, equivalently,

$$\sum_{i=0}^{n} \frac{1}{i!} - \frac{1}{n}\sum_{i=2}^{n} \frac{1}{i!}\left(1 - \frac{(i-1)^2}{n}\right) \leq \left(1 + \frac{1}{n}\right)^n \leq \sum_{i=0}^{n} \frac{1}{i!}$$

Substituting $s_n := \sum_{i=0}^{n} \frac{1}{i!}$, $z_n := \sum_{i=2}^{n} \frac{(i-1)^2}{i!}$, then

$$s_n + \frac{z_n}{n} \leq \left(1 + \frac{1}{n}\right)^n \leq s_n.$$

If we can prove that (z_n) is bounded, then $\lim_{n\to\infty} \frac{z_n}{n} = 0$ and the Squeeze Theorem 4.36 implies that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. Why should we think that (z_n) is bounded? Well $z_n = \sum_{i=2}^n \frac{(i-1)^2}{i!}$ which means that (z_n) is the series associated to the sequence defined by $x_i = \frac{(i-1)^2}{i!}$. As $x_i \geq 0$, for any $i \in \mathbb{N}$, then boundedness of (z_n) is equivalent to the convergence of $\displaystyle\sum_{i=2}^{n} \frac{(i-1)^2}{i!}$. This will be proven in **??**.

In the course of the above example. we have used this other version of Proposition 4.16.

Proposition 5.30 (Bernoulli inequality (negative case))**.** *Fix $n \in \mathbb{N}$. Given $-1 < x \leq 0$, then $(1+x)^n \geq 1 + nx$.*

Proof. We prove the statement by induction on $n \in \mathbb{N}$.

- STARTING STEP: for $n = 0$, then $(1+x)^0 = 1 = 1 + 0 \cdot x$.
- INDUCTIVE STEP: let us assume that we know the statement of the proposition holds for any $j \leq n$; we need to prove that the statement holds also for $n+1$. But then,

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = (1 + (n+1)x + \underbrace{nx^2}_{\geq 0}) \geq 1 + (n+1)x.$$

□

There are two further criteria for the convergence of series that will be very useful in the course.

Proposition 5.31 (Cauchy's criterion)**.** *Let (x_n) be a sequence.*

(1) If $\left(\sqrt[n]{|x_n|}\right)$ is bounded and $\lim_{n\to\infty} \sqrt[n]{|x_n|} = \rho < 1$, then $\displaystyle\sum_{n=0}^{\infty} x_n$ is absolutely convergent.

(2) If $\left(\sqrt[n]{|x_n|}\right)$ is bounded and $\lim_{n\to\infty} \sqrt[n]{|x_n|} = \rho > 1$ or if $\left(\sqrt[n]{|x_n|}\right)$ is not bounded, then $\displaystyle\sum_{n=0}^{\infty} x_n$ is divergent.

Remark 5.32. Let (x_n) be a sequence such that $\lim_{n\to\infty} \sqrt[n]{x_n} = 1$. Then, it is not possible to predict the behavior of $\displaystyle\sum_{n=0}^{\infty} x_n$, as the following examples illustrate.

(1) $x_n := (-1)^n \implies \displaystyle\sum_{n=0}^{\infty} x_n$ diverges but (s_n) is bounded;

(2) $x_n := n \implies \displaystyle\sum_{n=0}^{\infty} x_n$ diverges and (s_n) is unbounded;

(3) $x_n := \frac{1}{n} \implies \displaystyle\sum_{n=0}^{\infty} x_n$ diverges, (s_n) is unbounded and $\lim_{n\to\infty} x_n = 0$;

(4) $x_n := \frac{1}{n^2} \implies \displaystyle\sum_{n=0}^{\infty} x_n$ converges absolutely, cf. **??**;

(5) $x_n := \frac{(-1)^n}{n} \implies \sum_{n=0}^{\infty} x_n$ converges, but it does not converges absolutely, cf. **??**-**??**.

Proof of Cauchy's criterion. (1) For the convergence statements, we try to apply the Squeeze theorem for series. It suffices to show that there exists $n_0 \in \mathbb{N}$ and $0 < q < 1$ such that for any $n \geq n_0$, $0 \leq |x_n| \leq q^n$. The only question is how to choose q.
As $0 \leq \rho < 1$, let $\epsilon := \frac{1-\rho}{2} > 0$. Then, there exists $n_\epsilon \in \mathbb{N}$, such that $\forall n \geq n_\epsilon$, $|\sqrt[n]{|x_n|} - \rho| < \epsilon$. In particular, this implies that $0 \leq \sqrt[n]{|x_n|} < \rho + \epsilon < 1$, or equivalently, that $0 \leq |x_n| < (\rho + \epsilon)^n$, as desired.

(2) For the divergence statements one just shows that $|x_n|$ does not converge to 0. There are infinitely many elements with $\sqrt[n]{|x_n|} \geq 1$; this is equivalent to $|x_n| \geq 1$.

$\square$

Example 5.33. Take $x_n := \frac{q^n}{n!}$, $q \in \mathbb{R}$. Then,

$$\sqrt[n]{|x_n|} = \sqrt[n]{\left|\frac{q^n}{n!}\right|} = \frac{|q|}{\sqrt[n]{n!}}.$$

Claim $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Proof. This is an exercise in Week 7's exercise sheet. $\square$

Then $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$, thus, $\lim_{n\to\infty} \sqrt[n]{x_n} = 0$, and Cauchy's criterion implies that $\sum_{k=0}^{\infty} \frac{q^k}{k!}$ converges absolutely for any $q \in \mathbb{R}$.

Proposition 5.34 (D'Alembert's criterion). *Let (x_n) be a sequence such that $\left(\frac{|x_{n+1}|}{|x_n|}\right)$ is convergent and let the limit be ρ.*

(1) If $\lim_{n\to\infty} \left(\frac{|x_{n+1}|}{|x_n|}\right) = \rho < 1$, then $\sum_{k=0}^{\infty} x_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty} \left(\frac{|x_{n+1}|}{|x_n|}\right) = \rho > 1$, then $\sum_{k=0}^{\infty} x_n$ is divergent.

The ideas behind the proofs. (1) For the convergence statements, we try to apply the Squeeze theorem for series by showing for any $n \geq$ of some index n_0, $0 \leq |x_n| \leq q^n$ for some $0 < q < 1$. The only question is what q should one choose. Take $\frac{x_{n+1}}{x_n} \leq \rho + \varepsilon$ after finitely many steps, say after $n \geq n_\varepsilon$. So, we have $|x_n| \leq q^{n-n_\varepsilon} |x_{n_\varepsilon}|$ if we set $q = \rho + \varepsilon$ here too.

(2) There exists $n_1 \in \mathbb{N}$, such that $\forall n \geq n_1$, $\frac{|x_{n+1}|}{|x_n|} > 1$. Hence, $\forall n \geq n_1$, $|x_{n+1}| > |x_n|$, i.e., starting from n_1 the sequence $(|x_n|)$ is strictly increasing. As $x_n \neq 0$ for all $n \gg 1$, then $\lim_{n\to\infty} x_n$ if it exists cannot be 0.

$\square$

Remark 5.35. A remark completely analogous to **??** holds also for D'Alembert's criterion. By this, we mean that if $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = 1$, then anything can happen for $\sum_{n=0}^{\infty} x_n$. That is, it is possible to find sequences (x_n) such that:

(1) (x_n) is unbounded;

(2) (x_n) is bounded and $\sum\limits_{n=0}^{\infty} x_n$ diverges;

(3) (x_n) is bounded and $\sum\limits_{n=0}^{\infty} x_n$ converges absolutely;

(4) (x_n) is bounded and $\sum\limits_{n=0}^{\infty} x_n$ converges but not absolutely.

[This is an exercise in Week 7's exercise sheet].

Example 5.36. The series $\displaystyle\sum_{k=0}^{\infty} \frac{k}{2^k}$ is convergent. This can be proved using both criteria in **??** or **??**. In fact,

$$\lim_{k\to\infty} \sqrt[k]{|x_k|} = \lim_{k\to\infty} \sqrt[k]{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k}}{2} = \frac{\lim\limits_{k\to\infty} \sqrt[k]{k}}{2} = \frac{1}{2},$$
$$\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \lim_{k\to\infty} \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2}.$$

Example 5.37. Let (x_n) $x_n = \frac{n^n}{n!}$. Does the series $\sum_{i=1}^{\infty} x_i$ converge? For any $n \in \mathbb{N}^*$, $n^n \geq n!$. Hence, $\forall n \in \mathbb{N}^*$, $x_n \geq 1$. Thus, the series cannot converge, by **??**.
Let us take instead $y_n := \frac{n!}{n^n}$. Does the series $\sum_{i=1}^{\infty} y_i$ converge instead? In this case,

$$\frac{|y_{n+1}|}{|y_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(n)!} = \frac{(n+1)(n!)}{(n+1)(n+1)^n} \cdot \frac{n^n}{(n)!} = \frac{n^n}{(n+1)^n} = \left(1 + \frac{1}{n}\right)^{-n}.$$

Hence, $\lim\limits_{n\to\infty} \frac{|y_{n+1}|}{|y_n|} = e^{-1} < 1$, so that the series $\displaystyle\sum_{i=1}^{\infty} y_i$ converges (absolutely).

Example 5.38. The series $\displaystyle\sum_{i=1}^{\infty} \frac{(i-1)^2}{i!}$ converges.
It suffices to apply D'Alembert's criterion to $x_n := \frac{(n-1)^2}{n!}$. In fact,

$$\frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{n!}{(n+1)!}.$$

Hence,

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \lim_{n\to\infty} \frac{1}{n+1} = 1 \cdot 0 = 0,$$

so that the series $\displaystyle\sum_{i=1}^{\infty} x_i$ converges (absolutely).

6 Real functions of one variable

In this section, we are going to consider functions $f\colon E \to \mathbb{R}$ where E is a subset of $\mathbb{R}$ and study their properties. We first start by recalling general basic properties of functions.

6.1 Limits of functions and continuity

In this section we will define and discuss the notion of limit of a function at a given point. The notion of limit aims to give a mathematically precise measure of what the local behavior of a function is around a given point.

Example 6.1. The starting point of our investigation is the function $f(x) := \frac{\sin(x)}{x}$ near $x = 0$. At first sight, $f(0)$ would seem not to be defined, as x appears in the denominator of $\frac{\sin(x)}{x}$. On the other hand, looking at the graph of the function in **??**, it would appear that the closer

Images/sinx_sux_gr.png

Figure 10: $f(x) = \frac{\sin(x)}{x}$.

x is to 0, the closer $\frac{\sin(x)}{x}$ is to 1.
We can be even more precise if, for example, we consider the sequence (y_n), $y_n := \frac{1}{n}$, $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} y_n = 0$ and we can actually show that also the limit $\lim\limits_{n\to\infty} f(y_n)$ exists. Indeed, you proved in the exercise sheets that

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

So, even though $\frac{\sin(x)}{x}$ is not defined at $x = 0$, if we set

$$f(x) = \frac{\sin(x)}{x}, \text{ for } x \neq 0, \quad f(0) = 1,$$

then it would appear that $f(x)$ becomes a nice "continuous" function around at $x = 0$, meaning that we could draw the graph of f with just one continuous stroke of the pen.

The goal of this section is for us to turn the ideas contained in the previous example into some precise mathematical concepts and definitions and derive further consequences starting from those. In particular, we will define precisely why, in the previous example, $f(0) = 1$ makes $f(x)$ "continuous".

First, we need to make of how to unsure that a function f is defined around a point $x_0 \in \mathbb{R}$, so that it makes sense for us to talk about "the behavior of f around x_0".

Definition 6.2. A function $f\colon E \to \mathbb{R}$ is defined on *a punctured neighborhood* of $x_0 \in \mathbb{R}$ if for some positive real number $\delta \in \mathbb{R}^*_+$, E contains a set of the form $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.

Remark 6.3. Equivalently, we can restate the above definition in the following way:
A function $f\colon E \to \mathbb{R}$ is defined on *a punctured neighborhood* of $x_0 \in \mathbb{R}$ if for some positive real number $\delta \in \mathbb{R}^*_+$, an interval of the form $(x_0 - \delta, x_0 + \delta)$ is contained in $E \cup \{x_0\}$.

Example 6.4. The function $f(x) := \frac{\sin(x)}{x}$ is defined on any pointed neighborhood of 0. Indeed, f is defined on $E := \mathbb{R} \setminus \{0\}$ so that

$$(-\delta, +\delta) \setminus \{0\} \subseteq E, \quad \forall \delta \in \mathbb{R}^*_+.$$

We are then ready to give the formal definition of limit.

Definition 6.5. Let $f\colon E \to \mathbb{R}$ and $l \in \mathbb{R}$. Assume that E contains a punctured neighborhood of $x_0 \in \mathbb{R}$. Then, $\lim\limits_{x \to x_0} f(x) = l$ if one of the following two equivalent conditions holds:

(1) For every $0 < \varepsilon \in \mathbb{R}$ there exists $\delta_\varepsilon \in \mathbb{R}^*_+$ such that

$$\forall x \in E, 0 < |x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon.$$

(2) For every sequence $(x_n) \subseteq E \setminus \{x_0\}$ for which $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = l$.

Remark 6.6. Roughly speaking the two definitions mean the following:

(1) whenever f is defined at x and x is close to x_0, then $f(x)$ is close to l. More precisely: for every $\varepsilon > 0$ there is a $\delta > 0$ such that if x is closer to x_0 than δ (and f is defined at x), then $f(x)$ is closer to l than ε.

(2) whenever a sequence (y_n) is contained in $E \setminus \{x_0\}$ and it converges to x_0, then the sequence $(f(y_n))$, given by the values of the function f along the x_n, converges to l.

Remark 6.7. We explain why the two conditions in **??** are equivalent.

?? $\Rightarrow$ **??**: Let us fix a sequence (y_n) which is contained in $E \setminus \{x_0\}$ and for which $\lim\limits_{n\to\infty} x_n = x_0$. We have to show that $\lim\limits_{n\to\infty} f(x_n) = l$. Let us fix $\varepsilon > 0$. Then, this yields a $\delta > 0$ as in definition (i). For this δ, there is an n_δ such that $|x_0 - x_n| < \delta$ for $n \geq n_\delta$, and hence for all such n, $|l - f(x_n)| < \varepsilon$.

NOT **??** $\Rightarrow$ NOT **??**: The negation of (i) is that there is an $\varepsilon > 0$ such that for each $\delta > 0$ we can find $y_\delta \in (x_0 - \delta, x_0 + \delta[\setminus\{x_0\}$ such that $|f(y_\delta) - l| \geq \varepsilon$. Defining $y_n := y_{\frac{1}{n}}$, then the sequence (y_n) converges to x_0, but all values $f(y_n)$ have distance at least ε from l, so the sequence $(f(x_n))$ cannot converge to l.

We will work more often with the use definition **??** more as it is simpler. Luckily, it is almost always enough for proving that a limit does not exist. We usually use definition **??** only when **??** does not work.

Example 6.8. We show that $\lim\limits_{x\to 2} x^2 = 4$ using point **??** of **??**. In order to do this, we proceed as follows: let us fix $\varepsilon > 0$; at this point, we need to find $\delta > 0$ such that

$$\text{if } 0 < |x - 2| < \delta \text{ then, } |x^2 - 4| < \varepsilon.$$

Let us note that

$$x^2 - 4 = (x - 2)(x + 2).$$

Furthermore, if $0 < |x-2| < 1$, then $3 < x+2 < 3$; thus, if $0 < |x-2| < 1$, then

$$|x^2-4| = |x-2||x+2| < 5|x-2|. \tag{6.8.a}$$

Therefore, taking $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$, we conclude that, if $0 < |x-2| < \delta$ then

$$|x-2| < 1 \quad \text{and } |x-2| < \frac{\varepsilon}{5}.$$

Hence, it follows that

$$\underbrace{|x^2-4| \quad < \quad 5|x-2|}_{\text{Using ??, since } |x-2|<1} < \quad 5\frac{\varepsilon}{5} = \varepsilon.$$

Example 6.9. We can repeat the same computation as in the previous example, also using **??** of **??**. In general, proving the existence (and finiteness of the limit using sequence) can be rather tricky: you can try for example to use that definition to compute the limit of $\frac{\sin(x)}{x}$ at $x = 0$. Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = 2$. Then, by the algebraic properties of the limit, that is, by Proposition 4.30, we know that $\lim_{n\to\infty} x_n^2 = \left(\lim_{n\to\infty} x_n\right)^2 = 2^2 = 4$.

Example 6.10. We can generalize the arguments from the previous examples to show that for any $x_0 \in \mathbb{R}$, $\lim_{x\to x_0} x^n = x_0^n$ for any $n \in \mathbb{N}$.

Using the notion of limit, we can also define the notion of continuity of a function f at a point in $D(f)$.

Definition 6.11. Let $f\colon E \to \mathbb{R}$ be a function, $E \subset \mathbb{R}$. The function f is *continuous* at a point $x_0 \in E$, if $\lim_{x\to x_0} f(x)$ exists, it is finite and $\lim_{x\to x_0} f(x) = f(x_0)$.

Remark 6.12. Implicit in **??** is the fact that the limit of $f(x)$ at x_0 exists. In particular, the subset $E \subseteq \mathbb{R}$ on which f is assumed to be defined must contain not only x_0 but also a punctured neighborhood of x_0.

Remark 6.13. The main difference between **??** and **??** is the fact that, while in **??** we do not require the function f to be defined at the point x_0 at which we are trying to compute the limit and when taking the limit we only look at the value of f on points of $D(f) \setminus \{x_0\}$, in the case of **??**, instead, we very much want to allow the value $f(x_0)$ to play a role. More precisely, we have the following characterization of continuity at a point, via conditions analogous to those in **??**.

Proposition 6.14. *Let $f\colon E \to \mathbb{R}$ be a function and let $x_0 \in E$. Assume that there exists a open interval of the form $(x_0-\delta, x_0+\delta[$, $\delta > 0$ contained in E. Then, f is continuous at x_0 if and only if one of the following two equivalent definitions hold:*

(1) For every $\varepsilon \in \mathbb{R}_+^*$ there is a $\delta_\varepsilon \in \mathbb{R}_+^*$ such that

$$\forall x \in E \text{ such that } |x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon.$$

(2) For every sequence $(x_n) \subseteq E$ for which $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

In view of the above proposition, **??** yields the following immediate corollary.

Corollary 6.15. *Fix $n \in \mathbb{N}$. Then $f(x) = x^n$ is continuous at every $x_0 \in \mathbb{R}$.*

Using the next definition, we can rephrase the previous corollary by saying that, for a fixed $n \in \mathbb{N}$, the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Definition 6.16. Let $f: E \to \mathbb{R}$. Assume that $\forall x_0 \in E$, E contains an open ball centered at x_0. Then we say that f *is continuous* if it is continuous at every $x_0 \in E$.

Example 6.17. (1) Let us define the function $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} 0 & x \neq 0, \\ 1 & x = 0. \end{cases}$$

Then, f is not continuous at $x = 0$.
In fact, $\lim\limits_{x \to 0} f(x) = 0$, as in the definition we assumed $0 < |x - x_0| \leq \delta$, so the function value 1 for $x_0 = 0$ does not cause any problem.

(2) Let us define the function $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, the set of points of $\mathbb{R}$ at which f is continuous is empty.
For example, let us consider the point $0 \in \mathbb{R}$ and let us define the sequences $(y'_n), (y''_n) \subset \mathbb{R} \setminus \{0\}$, to be

$$y'_n = \frac{1}{n}, \quad y''_n = \frac{1}{\sqrt{n^2 + 1}}, \qquad n \geq 1.$$

As $\forall n \in \mathbb{N}^*$, $\frac{1}{n} \in \mathbb{Q}$, $\frac{1}{\sqrt{n^2+1}} \in \mathbb{R} \setminus \mathbb{Q}$, then $\forall n \in \mathbb{N}^*$, $f(y'_n) = 0$, while $f(y''_n) = 1$.
Hence, $\lim_{n\to\infty} y'_n = 0 = \lim_{n\to\infty} y''_n$, while $\lim_{n\to\infty} f(y'_n) = 0$, $\lim_{n\to\infty} f(y''_n) = 1$, hence the limit $\lim_{x\to 0} f(x)$ does not exist, and moreover, f is not continuous at 0.
One can repeat the same reasoning at any point $x_0 \in \mathbb{R}$, by taking y'_n to be a sequence of rational numbers converging to x_0 (for example, $y'_n = x_0 + \frac{1}{n}$, if x_0 is rational, or y'_n to be the truncation of the decimal form of x_0 at the n-th decimal digit, if x_0 is irrational) and y''_n to be a sequence of irrational numbers converging to x_0 (for example, $y''_n = x_0 + \frac{1}{\sqrt{n^2+1}}$, if x_0 is rational, or $y''_n = x_0 + \frac{1}{n}$ if x_0 is irrational). Then, $\forall n \in \mathbb{N}^*$, $f(y'_n) = 0$, while $f(y''_n) = 1$, and $\lim_{n\to\infty} y'_n = x_0 = \lim_{n\to\infty} y''_n$, while $\lim_{n\to\infty} f(y'_n) = 0$, $\lim_{n\to\infty} f(y''_n) = 1$, hence the limit $\lim_{x\to x_0} f(x)$ does not exist, and moreover, f is not continuous at x_0.
This implies that $\lim_{x\to x_0} f(x)$ does not exist at any point $x_0 \in \mathbb{R}$, in particular, f is not continuous at any point of $\mathbb{R}$.

Example 6.18. *We claim that* $\lim\limits_{x\to 0} \cos(x) = 1$.
Indeed, let (x_n) be a sequence converging to 0. Then,

$$0 \leq |\cos(x_n) - 1| = \left|2\sin^2\left(\frac{x_n}{2}\right)\right| \leq 2\frac{x_n^2}{4} = \frac{x_n^2}{2},$$

using the inequality $|\sin(x)| \leq |x|$. So, squeeze theorem tells us that $\lim\limits_{n\to\infty} |\cos(x_n) - 1| = 0$.

Example 6.19. *The limit* $\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ *does not exist.*
Indeed, consider the sequences $x_n := \frac{1}{\pi\left(2n+\frac{1}{2}\right)}$ and $x'_n := \frac{1}{\pi\left(2n+\frac{3}{2}\right)}$. Then, first $\lim\limits_{n\to\infty} x_n = 0$ and $\lim\limits_{n\to\infty} x'_n = 0$. However,

$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{\frac{1}{\pi\left(2n+\frac{1}{2}\right)}}\right) = \sin\left(\pi\left(2n + \frac{1}{2}\right)\right) = 1,$$

but

$$\lim_{n\to\infty} \sin\left(\frac{1}{x'_n}\right) = \sin\left(\frac{1}{\frac{1}{\pi\left(2n+\frac{3}{2}\right)}}\right) = \sin\left(\pi\left(2n+\frac{3}{2}\right)\right) = -1.$$

So, point **??** of **??** is not satisfied, and hence the limit does not exist

6.1.1 Limits and algebra

?? allows us to translate all the statements about limits of sequences to limits of functions. Indeed, let us say we are have functions f, g defined around a point $x_0 \in \mathbb{R}$ – but we are not necessarily assuming that f, g are defined at x_0 – and we want to prove that if l and k are the limits of $f(x)$ and $g(x)$ (at x_0), then $l + k$ is the limit of $(f + g)(x)$. Let us take a sequence (y_n) converging to x_0. We know that $\lim_{n\to\infty} f(y_n) = l$ and $\lim_{n\to\infty} g(y_n) = k$. But then, Proposition 4.30 implies that $\lim_{n\to\infty} f(x_n) + f'(x_n) = l + k$, that is, $\lim_{n\to\infty} (f + f')(x_n) = l + k$, which is exactly generalizing the statement about limit of sequences and addition, to the case of limit of functions.

Unsurprisingly, at this point, we can do the same with all other properties that we proved for limits of sequences. We collect all the statements one can show along the same arguments:

Proposition 6.20. *Let f and g be two functions such that a punctured neighborhood of x_0 is in the domain of both f and g. Assume that the limits of f and g at x_0 exist and they are l and k, respectively. Then,*

(1) the limit of $f + g$ exists at x_0 and $\lim_{x\to x_0} (f + g)(x) = l + k$

(2) the limit of $f \cdot g$ exists at x_0 and $\lim_{x\to x_0} (f \cdot g)(x) = l \cdot k$

(3) if $k \neq 0$, then the limit of $\frac{f}{g}$ exists at x_0 and $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{l}{k}$

(4) if $f(x) \leq g(x)$ for any x in a punctured neighborhood of x_0, then $l \leq k$.

(5) **Squeeze Theorem***: if there is a third function $h(x)$ such that there is also a punctured neighborhood of x_0 in the domain of h, and:*

(i) on some punctured neighborhood of x_0 we have $f(x) \leq h(x) \leq g(x)$, and

(ii) $l = k$,

then $\lim_{x\to x_0} h(x) = l$.

Example 6.21. The main example for using point **??** of **??** is that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Indeed, we have already seen, cf. Figure 8, that

$$0 \leq \sin(x) \leq x \leq \tan(x), \quad \text{for } x \in [0, \frac{\pi}{2}],$$
$$\tan(x) \leq x \leq \sin(x) \leq 0, \quad \text{for } x \in [-\frac{\pi}{2}, 0];$$

which implies that

$$|\cos(x)| \leq \left|\frac{\sin(x)}{x}\right| \leq 1, \quad \text{for } x \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{6.21.b}$$

Since $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos(x) \geq 0$, whereas $\sin(x)$ and x are odd function, so that also $\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$, $\frac{\sin(x)}{x} \geq 0$, the chain of inequalities in **??** holds also once we remove the absolute values. So, by the Squeeze Theorem for limits we can conclude that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1,$$

since $\lim_{x \to 0} \cos(x) = 1$, see **??**.

Example 6.22. *For any $k \in \mathbb{N}^*$, $\lim_{x \to 0} x^k \sin\left(\frac{1}{x}\right) = 0$.* Indeed, as $|\sin\left(\frac{1}{x}\right)| = 1$, $\forall x \in \mathbb{R}^*$, then

$$-x^k \leq x^k \sin\left(\frac{1}{x}\right) \leq x^k, \quad \forall x \in \mathbb{R}^*.$$

and the conclusion follows from the Squeeze Theorem.

The above proposition has all the nice consequences about continuity.

Proposition 6.23. *If $f, g \colon E \to \mathbb{R}$ are continuous functions at $x_0 \in E$. Then the following function are also continuous at x_0:*

(1) $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{R}$,

(2) $f \cdot g$, and

(3) $\frac{f}{g}$, if $g|_E$ is nowhere zero (meaning that for all $x \in E : g(x) \neq 0$), then.

Example 6.24. We collect here some example of continuous functions, on their respective domains, that is, each of these functions is continuous at any point where they are defined:

- $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r$, that is, $p(x)$ is a polynomial in one variable x.
- $f(x) := \frac{1}{x}$ is continuous on $\mathbb{R} \setminus 0$.
- $\frac{x}{x^2 - 3x + 1}$ is continuous on $\mathbb{R} \setminus \left\{\frac{3 \pm \sqrt{5}}{2}\right\}$,
- In general, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ is continuous on $\{x \in \mathbb{R} | q(x) \neq 0\}$ (which is the whole real line minus finitely many points).

6.1.2 Limit and composition

Let us recall the following definition of composition of functions.

Definition 6.25. If $f \colon E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ are functions such that $\mathrm{R}(f) \subseteq G$ then we may define the *composition* $g \circ f$ (order matters!!) of f with g by

$$(g \circ f)(x) = g(f(x)).$$

Example 6.26. Let us take $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$, and $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = y^3 + y + 1$. Then we have

$$(g \circ f)(x) = (x^2 + 1)^3 + (x^2 + 1) + 1 = x^6 + 3x^4 + 4x^2 + 3. \tag{6.26.c}$$

Let us look at an example about whether composition of continuous functions is continuous or not.

Example 6.27. Consider the functions defined in **??**. *We want to show that $g \circ f$ is continuous at $x = 0$.* As $(g \circ f)(0) = 3$ by **??**, we then need to show that $\lim_{x \to 0} (g \circ f)(x) = 3$. and this can be immediately deduced from the last part of the equation in **??**. So, indeed $g \circ f$ is continuous at $x = 0$.

In general the situation is just as nice as in **??**.

Proposition 6.28. *Let $f \colon E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ be two functions. Assume that:*

(1) $f(E) \subseteq G$,

(2) f is continuous at x_0,

(3) g is continuous at $y_0 := f(x_0)$.

Then $g \circ f$ is continuous at x_0.

Proof. We verify condition **??** of **??**. Let $(z_n) \subseteq E$ be a sequence such that

$$\lim_{n \to \infty} z_n = x_0. \tag{6.28.d}$$

According to **??** and our assumption **??**, then

$$\lim_{n \to \infty} f(z_n) = y_0. \tag{6.28.e}$$

Hence

$$\underbrace{\lim_{n \to \infty} (g \circ f)(z_n) = \overbrace{\lim_{n \to \infty} g(f(z_n))}^{\text{?? and condition ??}}}_{??} \overbrace{= g(y_0)}.$$

□

Remark 6.29. Let us examine a bit further **??**. In the proof of **??** we showed that if

$$\lim_{x \to x_0} f(x) = y_0 \qquad \text{and} \qquad \lim_{y \to y_0} g(y) = l, \tag{6.29.f}$$

then $\lim_{x \to x_0} g \circ f(x) = l$ holds under the assumption that f and g are continuous. We may be tempted to think that an analogous statement to **??** should also for the limit of a composition of functions, just assuming the condition in **??**. However, as we will see in **??**, this is not true. The reason is that in **??**, contrary to **??**, there is nothing said about the behavior at x_0 and y_0. So, we have to assume that $f(x)$ avoids y_0 in a punctured neighborhood of x_0.

The precise statement about composition of functions, in regards to limits, is as follows.

Proposition 6.30. *Let $f \colon E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ be functions and let $x_0 \in E$ be a point such that*

(1) $f(E) \subseteq G$,

(2) $\lim_{x \to x_0} f(x_0) = y_0$,

(3) $\lim_{y \to y_0} g(y_0) = l$

(4) there is a punctured neighborhood $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \subseteq E$ such that for every x in this neighborhood, $f(x) \neq y_0$.

Then, $\lim_{x \to x_0} (g \circ f)(x) = l$

Proof. We use part **??** of **??**. Thus, let us fix a sequence $(z_n) \subseteq E \setminus \{x_0\}$ such that

$$\lim_{n \to \infty} z_n = x_0. \tag{6.30.g}$$

In particular, by throwing away finitely many elements of the sequence, we may assume that

$$(z_n) \subseteq (x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \subseteq E. \tag{6.30.h}$$

By the assumption **??** in the statement of the proposition, and by **??**, it follows that

$$\lim_{n \to \infty} f(z_n) = y_0. \tag{6.30.i}$$

Lastly, by our assumption **??** and **??** we have

$$(f(z_n)) \subseteq G \setminus \{y_0\}. \tag{6.30.j}$$

Hence, by our assumption **??** and by **??**, we have

$$\lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} g(f(x_n)) = l.$$

□

The following example shows that condition **??** of **??** is necessary. That is, if we drop condition **??**, the statement of **??** would not hold.

Example 6.31. Consider:

$$g(x) = \begin{cases} 0, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0, \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

Then, $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} g(x) = 0$. However, $\lim_{x \to 0} (g \circ f)(x) \neq 0$, because the following two sequences induce function value sequences with different limits:

$$x_n := \frac{1}{\pi n} \qquad \text{and} \qquad y_n := \frac{1}{\pi n + \frac{\pi}{2}},$$

as

$$\lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} 1 = 1 \qquad \text{and} \qquad \lim_{n \to \infty} (g \circ f)(y_n) = \lim_{n \to \infty} 0 = 0.$$

Also, let us note that condition **??** of **??** below is not satisfied in this example, as $f(x) = 0$ for $x = \frac{1}{\pi n}$, so there is no punctured neighborhood of 0 on which the function of f avoids the value 0.

Example 6.32. A positive example for applying **??** is during the argument of showing that $\lim_{x \to 0} \frac{\sin(x^2)}{x^2} = 1$. Indeed, if we set $g(x) := \frac{\sin(x)}{x}$, and $f(x) = x^2$, then condition **??** of **??** is also satisfied, as $f(x) \neq 0$ for $x \neq 0$.

6.1.3 Infinite limits

Definition 6.33. A *neighborhood* of $+\infty$ (resp. $-\infty$) is an unbounded interval of the form $(a, +\infty)$ (resp. $(-\infty, a)$).

We extend the definition of limit to comprise the case where we allow ourselves to work with the extended real line $\overline{\mathbb{R}}$.

Definition 6.34. Let $x_0, l \in \overline{\mathbb{R}}$, and let $f : E \to \mathbb{R}$ be a function, $E \subset \mathbb{R}$. Assume that E contains a punctured neighborhood of x_0. We say that the limit of $f(x)$ at x_0 is l, if for any sequence $(y_n) \subseteq E \setminus \{x_0\}$[14], whenever $\lim_{n\to\infty} y_n = x_0$, then $\lim_{n\to\infty} f(y_n) = l$.

Example 6.35. *We show that* $\lim_{x\to 0} \frac{1}{x^2} = +\infty$. Indeed, if $(x_n) \subset \mathbb{R}^*$ is a sequence satisfying $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} \frac{1}{x_n^2} = +\infty$ by algebraic properties of limits of sequences.
On the other hand, the limit $\lim_{x\to 0} \frac{1}{x}$ does not exist. In fact, considering the sequence $x_n := \frac{1}{n}$, then $\lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} n = +\infty$, while for $y_n = \frac{-1}{n}$, then $\lim_{n\to\infty} \frac{1}{y_n} = \lim_{n\to\infty} -n = -\infty$.

Proposition 6.36. *Let $x_0 \in \overline{\mathbb{R}}$, and let $f, g : E \to \mathbb{R}$ be functions.*

(1) ADDITION RULE. *Assume that the following conditions are satisfied:*

- $\lim_{x\to x_0} f(x) = +\infty$ *(resp. $-\infty$), and*
- $g(x)$ *is bounded from below (resp. from above)*

then $\lim_{x\to x_0} (f+g)(x) = +\infty$ *(resp. $-\infty$).*

(2) PRODUCT RULE. *Assume that the following conditions are satisfied:*

- $\lim_{x\to x_0} |f(x)| = +\infty$,
- *there exists $\delta > 0$ such that $|g(x)| \geq \delta$ for all $x \in E$, and*
- $f(x)g(x) > 0$ *(resp. < 0) for all $x \in E$,*

then $\lim_{x\to x_0} f(x)g(x) = +\infty$ *(resp. $-\infty$).*

(3) FIRST DIVISION RULE. *If*

- $f(x)$ *is bounded,*
- $g(x)$ *is nowhere zero, and*
- $\lim_{x\to x_0} |g(x)| = +\infty$.

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$

(4) SECOND DIVISION RULE. *If*

- $\lim_{x\to x_0} g(x) = 0$,
- *there is a $\delta > 0$ such that $|f(x)| \geq \delta$ for all $x \in E$, and*
- $f(x)/g(x) > 0$ *(resp. < 0) for all $x \in E$,*

then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = +\infty$ *(resp. $-\infty$).*

[14] When $x_0 = \pm\infty$, then the condition that x_0 does not belong to E is automatically satisfied, since $E \subseteq \mathbb{R}$.

(5) SQUEEZE. *If $f(x) \le g(x)$, and*

- *if* $\lim_{x\to x_0} f(x) = +\infty$, *then* $\lim_{x\to x_0} g(x) = +\infty$
- *if* $\lim_{x\to x_0} g(x) = -\infty$, *then* $\lim_{x\to x_0} f(x) = -\infty$

Example 6.37. Here are a few examples.

- $\lim_{x\to 0} \underbrace{\frac{1}{x^2}}_{\to\infty} + \underbrace{\cos(x)}_{\text{bounded}} = +\infty$
- $\lim_{x\to+\infty} \underbrace{\cos(x)}_{\text{bounded}} \cdot \underbrace{(-x^2+x^3)}_{\to+\infty} = -\infty$, since $-x^2+x^3=0$ only for $x=0,1$.
- $\lim_{x\to+\infty} \frac{\overbrace{\arctan(x)}^{\text{bounded}}}{\underbrace{(-x)}_{\to-\infty}} = 0.$

Remark 6.38. (1) The assumptions stated in part **??** of **??** are important, as otherwise we can have all different kinds of limits. We give examples of this using the functions $f(x) = x^3$, $g(x) = x^2$ and $h(x) = x^3 + 1$. We have

- $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 = +\infty$, $\lim_{x\to+\infty} -f(x) = \lim_{x\to+\infty} -x^3 = -\infty$
- $\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} (x^3+1) = +\infty$, $\lim_{x\to+\infty} -h(x) = \lim_{x\to+\infty} -(x^3+1) = -\infty$, and
- $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} x^2 = +\infty$, $\lim_{x\to+\infty} -g(x) = \lim_{x\to+\infty} -x^2 = -\infty$.

On the other hand:

- $\lim_{x\to+\infty} f(x) - g(x) = \lim_{x\to+\infty} x^3 - x^2 = \lim_{x\to+\infty} x^2(x-1) = +\infty$,
- $\lim_{x\to+\infty} g(x) - f(x) = -\infty$, and
- $\lim_{x\to+\infty} f(x) - h(x) = -1$.

In particular, never use addition law for limits of the type $(+\infty) + (-\infty)$.

(2) The above assumptions for point **??** of **??** are also important. We give examples of this using the functions $f(x) = x$, $g(x) = \frac{\cos(x)}{x}$ and $h(x) = (-1)^{[x]}$. We have

- $\lim_{x\to+\infty} |f(x)| = +\infty$,
- $|g(x)|$ is not bounded from below, and
- $|h(x)|$ is bounded from below, but $f(x)h(x) \not> 0$.

Then:

- $\lim_{x\to+\infty} f(x)g(x) = \lim_{x\to+\infty} \cos(x)$ does not exist, and
- $\lim_{x\to+\infty} f(x)h(x) = \lim_{x\to+\infty} x(-1)^{[x]}$ does not exist, on the other hand
- the product law applies to $(f(x)h(x))(h(x))$ and yields $\lim_{x\to+\infty} (f(x)h(x))(h(x)) = +\infty$

Never try to use product rule to limits of the type $0 \cdot \infty$.

(3) The assumptions of the first division rule are also important. One can can show that in the $\frac{\pm\infty}{\pm\infty}$ case anything can happen for example using $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$,$(-1)^{\left[\frac{1}{x}\right]}\frac{1}{x}$ with limit at 0:

- $\lim\limits_{x\to 0}\frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim\limits_{x\to 0} x = 0$,
- $\lim\limits_{x\to 0}\frac{(-1)^{\left[\frac{1}{x}\right]}\frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to 0}(-1)^{\left[\frac{1}{x}\right]}$ does not exist and bounded,
- $\lim\limits_{x\to 0}\frac{\frac{1}{x^2}}{\frac{1}{x}} = \lim\limits_{x\to 0}\frac{1}{x}$ does not exist and unbounded, and
- $\lim\limits_{x\to 0}\frac{\frac{1}{x^3}}{\frac{1}{x}} = \lim\limits_{x\to 0}\frac{1}{x^2} = +\infty$.

Similar examples show that the assumptions are important for the second division rule. Never try to use division rules to limits of the form $\frac{\pm\infty}{\pm\infty}$ and $\frac{0}{0}$.

6.1.4 One sided limits

The main question is how to make sense of limits such as at 0 of $\sqrt{x}$, as here the domain does not contain a punctured neighborhood of 0. The solution for this is the introduction of the notions of left and right limits.

Definition 6.39. A function $f : E \to \mathbb{R}$ is defined to the left (resp. to the right) of $x_0 \in \mathbb{R}$, if E contains an interval of the form $(x_0 - \delta, x_0[$ (resp. $(x_0, x_0 + \delta[)$.

Definition 6.40. Let $f\colon E \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$. Assume that f is defined to the left (resp. right) of x_0. Let $l \in \overline{\mathbb{R}}$.

(1) We say that the limit of f for x that goes to x_0 from the *left* is l if for all sequences $(x_n) \subseteq \{x \in E | x < x_0\}$, whenever $\lim\limits_{n\to\infty} x_n = x_0$ then $\lim\limits_{n\to\infty} f(x_n) = l$. When this condition is satisfied, we write $\lim\limits_{x\to x_0^-} f(x) = l$.

(2) We say that the limit of f for x that goes to x_0 from the *right* is l if for all sequences $(x_n) \subseteq \{x \in E | x > x_0\}$, whenever $\lim\limits_{n\to\infty} x_n = x_0$ then $\lim\limits_{n\to\infty} f(x_n) = l$. When this condition is satisfied, we write $\lim\limits_{x\to x_0^+} f(x) = l$.

Example 6.41. Consider the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ defined as $f(x) := \sqrt{x}$. *We claim that* $\lim\limits_{x\to 0^+}\sqrt{x} = 0$.
Indeed, fix a sequence $(x_n) \subseteq \mathbb{R}_+^*$ such that $\lim\limits_{n\to\infty} x_n = 0$. We have to show that then $\lim\limits_{n\to\infty}\sqrt{x_n} = 0$ too. So, we need to show that for each $\varepsilon > 0$, there is an n_0 such that for every integer $n \geq n_0$, $\sqrt{x_n} \leq \varepsilon$. However, we know that $\lim\limits_{n\to\infty} x_n = 0$. So, we know that there is an n_0 such that $|x_n| < \varepsilon^2$ for all $n \geq n_0$. But then, for any such n we also have $\sqrt{x_n} < \varepsilon$.

Proposition 6.42. *Let $f\colon E \to \mathbb{R}$ be a function such that there is a punctured neighborhood of x_0 contained in E, and both*

$$l_1 := \lim_{x\to x_0^-} f(x) \qquad \text{and} \qquad l_2 := \lim_{x\to x_0^+} f(x)$$

exists. Then

$$\lim_{x\to x_0} f(x) = l \quad \Longleftrightarrow \quad l_1 = l_2 = l.$$

Example 6.43. Consider the function $f(x) = \{x\} =$. Both left and right limits exist at all points, and furthermore:

$$\lim_{x\to x_0^-} \{x\} = \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z} \\ 1 & \text{if } x \in \mathbb{Z} \end{cases} \qquad \text{and} \qquad \lim_{x\to x_0^+} \{x\} = \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

Hence, according to **??**,

$$\lim_{x\to x_0} \{x\} \text{ exists } \Leftrightarrow x \notin \mathbb{Z}.$$

Example 6.44. **??** together with **??** show that the function $f(x) = \sqrt{|x|}$ is continuous in 0. It is not hard to show that f is actually continuous everywhere in $\mathbb{R}$.

6.1.5 Monotone functions

For a monotone function f, the left (resp. the right) limits always exists at any point x_0 at which the function is defined at the left (resp. the right) of x_0.

Proposition 6.45. *Let $f\colon E \to \mathbb{R}$ be a monotone function. Then, at each point $x_0 \in E$:*

(1) if f is defined on the left of x_0, $\lim_{x\to x_0^-} f(x)$ exists,

(2) if f is defined on the right of x_0, $\lim_{x\to x_0^+} f(x)$ exists, and

(3) if f is defined in a neighborhood of $\pm\infty$, then $\lim_{x\to\pm\infty} f(x)$ exists.

Proof. We treat only the increasing case, as the decreasing one follows from that by regarding $-f$ instead of f. Also, we treat only the first case as the others are similar. Set:

$$l := \sup\{f(x)|x \in E, x < x_0\}. \tag{6.45.k}$$

Let

$$(x_n) \subseteq \{x \in E|x < x_0\} \quad \text{such that} \quad \lim_{n\to\infty} x_n = x_0. \tag{6.45.l}$$

We have to show that $\lim_{n\to\infty} f(x_n) = l$. Fix a $\varepsilon > 0$. Then, by the definition of l, there is an $x' \in \{x \in E|x < x_0\}$, such that

$$f(x') > l - \varepsilon. \tag{6.45.m}$$

According to **??**, there is an $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$ we have

$$x' < x_n < x_0. \tag{6.45.n}$$

However, then for all integers $n \geq n_0$ we have:

$$l \geq \underbrace{f(x_n)}_{\textbf{??}\text{ and }\textbf{??}} \geq \underbrace{f(x')}_{f\text{ is increasing and }\textbf{??}} \geq \underbrace{l - \varepsilon.}_{\textbf{??}}$$

This shows that $\lim_{n\to\infty} f(x_n) = l$ indeed.

□

Example 6.46. (1) Let

$$f(x) := \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Images/sgn_gr.png

Figure 11: $f(x) = \operatorname{sgn}(x)$.

Then

$$\lim_{x \to x_0^-} \operatorname{sgn}(x) = -1 \qquad \text{and} \qquad \lim_{x \to x_0^+} \operatorname{sgn}(x) = 1$$

Note that these limits exist and neither of them agree with $f(0) = 0$.

(2) Let $f(x) := \lfloor x \rfloor$. Then:

$$\lim_{x \to x_0^-} f(x) = \begin{cases} \lfloor x \rfloor & x \notin \mathbb{Z} \\ x - 1 & x \in \mathbb{Z} \end{cases} \qquad \text{and} \qquad \lim_{x \to x_0^+} f(x) = \begin{cases} \lfloor x \rfloor & x \notin \mathbb{Z} \\ x & x \in \mathbb{Z}. \end{cases}$$

So, the left and right limits exist, despite having different values whenever $x \in \mathbb{Z}$.

Images/floor_gr.png

Figure 12: $f(x) = \lfloor x \rfloor$.

6.1.6 More on continuity

First, we note that there are more algebraic rules of continuity (we already discussed addition, multiplication and division in **??**):

Proposition 6.47. *If $f, g\colon E \to \mathbb{R}$ are functions that are continuous at $x_0 \in E$, then so are:*

(1) $|f|$,

(2) $\max\{f, g\}$*, where*

$$\max\{f, g\}(x) := \max\{f(x), g(x)\}$$

(3) $\min\{f, g\}$ *(defined similarly),*

(4) $f^+ := \max\{f, 0\}$,

(5) $f^- := \min\{f, 0\}$.

Example 6.48. We can use for example the continuity of the absolute value for squeezing. For example, let

$$g(x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ x & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

We claim that $g(x)$ is continuous at $x_0 = 1$. The main idea is that we can try to apply the Squeeze Theorem for the limit of functions using the following chain of inequalities:

$$-|x - 1| \leq g(x) - 1 \leq |x - 1|.$$

According to point **??** of **??**, the function $|x - 1|$ is continuous everywhere over $\mathbb{R}$; thus,

$$\lim_{x \to 1} -|x - 1| = \lim_{x \to 1} |x - 1| = 0,$$

so that by **????**, it follows that $\lim\limits_{x \to 1} f(x) - 1 = 0$. Hence, $\lim\limits_{x \to 1} f(x) = 1 = f(1)$ and f is continuous at $x_0 = 1$.

6.1.7 Uniform continuity and Lipschitzianity

We introduce a stronger version of continuity.

Definition 6.49. A function $f\colon E \to \mathbb{R}$ is said to be *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for all $x, y \in E$ then

$$\text{if } |x - y| < \delta_\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Remark 6.50. The notion of uniform continuity defined above is much stronger than that of continuity, cf. **??**. More precisely, using the characterization of continuity for a function $f\colon E \to \mathbb{R}$ given in **??**, for any $x_0 \in E$ and any $\varepsilon > 0$ there exists $\delta > 0$, which depends on ε and x_0, such that for any $x \in E$

$$\text{if } |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

In **??**, the for a fixed $\varepsilon > 0$, the existence of $\delta > 0$ is no longer dependent on the choice of a base point $x_0 \in E$; instead, at this point such choice can be made independently (or rather, uniformly) from the points of E: it just depends on the choice of ε.

In view of the observation of the previous remark, we immediately have the following proposing showing that uniform continuity is a stornger property than continuity.

Proposition 6.51. *If $f\colon E \to \mathbb{R}$ is uniformly continuous then it is continuous.*

Example 6.52. *The function $f(x) := x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.* On the other hand, we have already seen that it is continuous as it is a polynomial.
Indeed, for any $x, y \in \mathbb{R}$,

$$|x^2 - y^2| = |x + y| \cdot |x - y|.$$

So, for any $\varepsilon > 0$ and $\delta > 0$, we may chose $x, y \in \mathbb{R}$ such that $|x+y| > \frac{2\varepsilon}{\delta}$, and $|x - y| = \frac{\delta}{2}$ – to do that, it suffices to choose two real numbers x, y that are very large but very close to each other. Thus, it follows that

$$|x - y| < \delta \quad \text{and} \quad |x^2 - y^2| > \frac{2\varepsilon}{\delta}\frac{\delta}{2} = \varepsilon.$$

We will see in the next section that if we consider a continuous function f over a closed bounded interval $[a, b]$ – rather than on an unbounded domain as in this case, where x^2 is considered over $\mathbb{R}$ – then f is absolutely continuous.

Example 6.53. *We show that* $\cos(x) : \mathbb{R} \to \mathbb{R}$ *is uniformly continuous and hence continuous.* Indeed,

$$|\cos(x) - \cos(y)| = 2\left|\sin\left(\frac{x+y}{2}\right)\right|\left|\sin\left(\frac{x-y}{2}\right)\right| \le 2\left|\sin\left(\frac{x-y}{2}\right)\right| \le 2|x - y|.$$

So, if we set $\delta = \frac{\varepsilon}{2}$, then we have

$$|x - y| \le \delta \Rightarrow |\cos(x) - \cos(y)| \le 2|x - y| \le 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

This result, together with **??**, implies that also functions such as $\cos(x^2)$, $\cos^2(x)$, etc. are continuous.

We introduce now a property that makes it particularly easy to show that a function is uniformly continuous.

Definition 6.54. A function $f\colon E \to \mathbb{R}$ is said to be *Lipschitz* if there exists a positive real number C such that for every $x, y \in E$, $|f(x) - f(y)| \le C|x - y|$.

When the conditions of **??** are satisfied we say that C is a Lipschitz constant for the Lipschitz function f.

Proposition 6.55. *Let $f\colon E \to \mathbb{R}$ be a function which is Lipschitz with Lipschitz constant C, where E is an open interval. Then f is uniformly continuous on E; hence f is also continuous on E.*

Proof. For a fixed positive real number ε in **??**, it suffices to take $\delta := \frac{\varepsilon}{C}$. □

Example 6.56. Let $f\colon [0, 1] \to \mathbb{R}$ be the function $f(x) = x^2$. Then for any $x, y \in [0, 1]$,

$$|x^2 - y^2| = |x - y||x + y| \le C|x - y|, \tag{6.56.o}$$

where $C := \sup\{|x + y| \mid x, y \in [0, 1]$. By definition, $C \le 2$ – it not hard to show that actually $C = 2$ – hence we can rewrite **??** as

$$|x^2 - y^2| = |x - y||x + y| \le 2|x - y|.$$

Hence, f is Lipschitz and thus uniformly continuous. Let us notice that

We will see in **??** that there exist functions that are uniformly continuous but not Lipschitz.

6.1.8 Left and right continuity

Lastly, we introduce left and right continuity, and we use this to define continuity on a closed interval.

Definition 6.57. Let $f\colon E \to \mathbb{R}$ be a function, and $x_0 \in E$.

(1) f *is left continuous at* x_0, if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$.

(2) f *is right continuous at* x_0, if $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$.

In **??** we defined what it means to be continuous on an open interval. For functions the domains of which are closed intervals the definition has to use left and right limits as well at the two endpoints:

Definition 6.58. A function $f\colon [a, b] \to \mathbb{R}$ is *continuous* if:

(1) f is continuous at any point contained in (a, b);

(2) f is left continuous at b; and,

(3) f is right continuous at a.

Example 6.59. The function $f\colon [-1, 1] \to \mathbb{R}$ defined as $f(x) := \sqrt{1 - x^2}$ is continuous. Indeed this is true by the following (where we use that $g(y) = \sqrt{y}$ is continuous on $\mathbb{R}^*_+$, which will be a consequence of our general theorem about the continuity of the inverse. Indeed, by applying the statement of **??** to $f(x) = x^2$ one obtains that $f^{-1} = g$ is continuous on $\mathbb{R}^*_+$):

(1) if $-1 < c < 1$, then $\sqrt{1 - x^2}$ at c is continuous because $\sqrt{1 - x^2}$ is the composition of $\sqrt{y}$ and $1 - x^2$, and the latter is continuous at c and the former is continuous at $1 - c^2$ (as $1 - c^2 > 0$).

(2) $\sqrt{1 - x^2}$ is left continuous at 1, because for all (x_n) converging to 1 from the left we have $\lim\limits_{n \to \infty} \sqrt{1 - x_n^2} = 0$, as $\lim\limits_{n \to \infty} 1 - x_n^2 = 0$, and $\lim\limits_{x \to 0^+} \sqrt{y} = 0$ according to **??**.

(3) $\sqrt{1 - x^2}$ is right continuous at -1 by almost verbatim the same argument as the previous point, one only needs to take $\lim\limits_{n \to \infty} x_n = -1$ instead of 1.

6.1.9 Consequences of Bolzano-Weierstrass

In this subsection we shall show how continuous functions defined over bounded closed intervals behave nicely. The proofs of all the results illustrated in this subsection heavily relies on Bolzano-Weierstrass Theorem 4.75. As we will be assuming, throughout this section, that the domain $D(f)$ of a function f is closed bounded interval, given a sequence $(x_n) \subseteq D(f)$, by Theorem 4.75 we will always be able to assume that we can pass to a converging subsequence $(x_{n_k}) \subseteq (x_n)$ whose limit belongs to $D(f)$, since we are assuming $D(f)$ is closed.

We start by showing that a continuous function defined over a closed bounded interval is always uniformly continuous.

Theorem 6.60. *Let $a, b \in \mathbb{R}$. If $f\colon [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

Proof. Assume that f is not uniformly continuous. Then there is a $\varepsilon > 0$ such that for every $\frac{1}{n}$ there are x_n and $y_n \in [a, b]$ such that $|x_n - y_n| \leq \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon$. By Bolzano-Weierstrass (Theorem 4.75) we may assume that $\lim\limits_{n \to \infty} x_n = x_0 \in [a, b]$. However, then the condition $|x_n - y_n| \leq \frac{1}{n}$ yields that we have also $\lim\limits_{n \to \infty} y_n = x_0$. Using again $|x_n - y_n| \leq \frac{1}{n}$ together with the continuity of f we obtain that $|f(x_0) - f(x_0)| \geq \varepsilon$. This is a contradiction. $\square$

Remark 6.61. For **??** to hold true, it is very important that the domain of f is a closed bounded interval $[a, b]$ for $a, b \in \mathbb{R}$. We have already seen that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not uniformly continuous on any interval of the form $(a, +\infty[$, $a \in \mathbb{R} \cup \{-\infty\}$.

Example 6.62. The following example shows that there exists uniformly continuous functions that are not Lipschitz.
Let us consider $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \sqrt{x}$. Let us fix a real number $a > 0$, and define $g : [0, a] \to \mathbb{R}$ by $g = f|_{[0,a]}$, $g(x) = \sqrt{x}$. **??** shows that g is uniformly continuous, as we are taking the domain of definition of g to be a closed bounded interval. To show that g is not Lipschitz, it suffices to show that for any $C \in \mathbb{R}_+^*0$ there exists $s, t \in [0, a], s \neq t$ such that $|g(s) - g(t)| > C|s - t|$, or, equivalently, $\frac{|g(s)-g(t)|}{|s-t|} > C$. Let us fix $C > 0$. Then, there exists $t \in]0, 1[$ such that $\frac{1}{\sqrt{t}} > C$, since $\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$. Taking $s = 0$, then $\frac{|g(s)-g(t)|}{|s-t|} = \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}} > C$, which is what we wanted to prove. It is not hard to show, that actually even $f(x) = \sqrt{x}$ is uniformly continuous but not Lipschitz. The proof is left as an exercise.

Theorem 6.63. *If $f : [a, b] \to \mathbb{R}$ is continuous for some $a, b \in \mathbb{R}$, then there are $c, d \in [a, b]$ such that*

$$M := \sup_{x \in [a,b]} f(x) = \max_{x \in [a,b]} f(x) = f(c),$$
$$m := \inf_{x \in [a,b]} f(x) = \min_{x \in [a,b]} f(x) = f(d).$$

Remark 6.64. The above theorem can be restated by saying that for a given function $f : [a, b] \to \mathbb{R}$, if f is continuous then the range $\mathrm{R}(f)$ of f is a closed and bounded interval, $\mathrm{R}(f) = [c, d]$.

Proof. We only prove the existence of $\max_{x \in [a,b]} f(x)$, the case of $\min_{x \in [a,b]} f(x)$ follows similarly.
First we prove that f is bounded from above, so that $\sup_{x \in [a,b]} f(x)$ must exist. Assume, by contradiction, that f is not bounded from above. That means that for each integer $n > 0$ there is $x_n \in [a, b]$ such that $f(x_n) \geq n$. As $(x_n) \subseteq [a, b]$ is a bounded sequence, by Bolzano-Weierstrass Theorem 4.75, there exists a convergent subsequence $(x_{n_k}) \subseteq (x_n)$. Set $c := \lim_{k \to \infty} x_{n_k}$. Then $c \in [a, b]$, and the following chain of equalities yields a contradiction:

$$\mathbb{R} \ni f(c) = \underbrace{\lim_{k \to \infty} f(x_{n_k})}_{f\,:\,[a,b] \to \mathbb{R} \text{ is continuous}} = \underbrace{+\infty}_{f(x_{n_k}) \geq n_k \geq k}.$$

This concludes the statement that f is bounded from above.
Having proved that f is bounded from above, $\sup_{x \in [a,b]} f(x)$ makes sense. Thus, we must prove that $\sup_{x \in [a,b]} f(x) = \max_{x \in [a,b]} f(x)$. By definition of supremum, there exists a sequence $(y_n) \subseteq [a, b]$ such that $f(y_n) \geq M - \frac{1}{n}$. In particular, $\lim_{n \to \infty} f(y_n) = M$. By Bolzano-Weierstrass Theorem 4.75, there exists a convergent subsequence $(y_{n_k}) \subseteq (y_n)$. Set $c := \lim_{k \to \infty} y_{n_k}$. Then $c \in [a, b]$, and

$$f(c) = \underbrace{\lim_{k \to \infty} f(y_{n_k})}_{f\,:\,[a,b] \to \mathbb{R} \text{ is continuous}} = \underbrace{\lim_{n \to \infty} f(y_n)}_{\text{Proposition 4.72}} = M.$$

$\square$

Remark 6.65. The conclusion of the above theorem does not hold, if we do not assume that the domain of f is a closed bounded interval $[a, b]$, $a, b \in \mathbb{R}$. For example, take $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{1}{x^2+1}$. Then f does not attain its minimum as $\mathrm{R}(f) =]0, 1]$: in fact, $f(x) > 0$, $\forall x \in \mathbb{R}$ and f converges to 0 as x goes to $\pm\infty$.

Theorem 6.66 (Intermediate value theorem)**.** *Let $a, b \in \mathbb{R}$. If $f\colon [a,b] \to \mathbb{R}$ is continuous, then it takes each value between $M := \max_{x\in[a,b]} f(x)$ and $m := \min_{x\in[a,b]} f(x)$ at least once. More precisely, for each $c \in [m, M]$, there exists $d \in [a, b]$ such that $f(d) = c$.*

Idea. We give only the idea and we refer to the precise proof to page 81-82 of the book.

We know by the above theorem that there are $a', b' \in [a, b]$ such that $m = f(a')$ and $M = f(b')$. Hence, by replacing a with a' and b with b' (and some algebraic manipulation in the case when $b' < a'$), we may assume that $f(a) = m$, $f(b) = M$ and $m < c < M$. Then, the idea is to consider

$$S := \{x \in [a, b] | f(x) < c\}$$

Set $d := \sup S$. By the definition of sup, there is a sequence $(x_n) \subseteq S$ converging to d from the left and let y_n be any sequence converging to d from the right. Applying continuity to the first sequence shows that $f(d) \leq c$, and by applying it to the second one shows that $f(d) \geq c$. So, $f(d) = c$.

□

Example 6.67. In other words, **??** says that $\mathrm{R}(f) = [m, M]$. Hence, for example, the image of an interval $[a, b]$ via a continuous function f (whose domain contains $[a, b]$) cannot be $[c, d] \cup [e, d]$, $c < d < e < f$ – that is, it cannot be the union of two disjoint intervals.

Example 6.68. If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, such that $f(0) = 1$, $f(1) = 3$, $f(2) = -1$, then f attains the value 2 at least two times. Indeed, our assumptions say that the maximum of $f|_{[0,1]}$ is at least 3 and the minimum of $f|_{[0,1]}$ is at most 1. Hence, **??** applied to $f|_{[0,1]}$ yields that there is at least one $c \in [0, 1]$ such that $f(c) = 2$. Similarly, **??** applied to $f|_{[0,1]}$ yields that there is at least one $d \in [1, 2]$ such that $f(d) = 2$. Furthermore, $c \neq d$, because $c = d$ can only happen if $c = d = 1$. However, $f(c) = 3 \neq 2$. Hence, c and d are two distinct real numbers at which f takes the value 2.

We will apply the above theoretical result to find solutions of equations of the form $f(x) = x$. For example one can ask, if there is a solution of $\cos(x) = x$ for some $x \in \left[0, \frac{\pi}{2}\right]$. **??** lets us answer this question.

Corollary 6.69 (Banach fixed point theorem for closed intervals)**.** *Let $a, b \in \mathbb{R}$. If $f\colon [a, b] \to [a, b]$ is a continuous function, then there exists $x \in [a, b]$ such that $f(x) = x$.*

Given a set S and function $f\colon S \to S$, an element $s \in S$ such that $f(s) = s$ is called a *fixed point.*

Proof. Set $g(x) := f(x) - x$. Then $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. So, by the intermediate value theorem, there is a real number $c \in [a, b]$ such that $0 = g(c)$. This is equivalent to $f(c) = c$. □

Example 6.70. The function $\cos(x)|_{\left[0,\frac{\pi}{2}\right]}\colon \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ can be regarded as $\cos(x)|_{\left[0,\frac{\pi}{2}\right]}\colon \left[0, \frac{\pi}{2}\right] \to \left[0, \frac{\pi}{2}\right]$, since $\mathrm{R}(\cos(x)|_{\left[0,\frac{\pi}{2}\right]}) = [0, 1] \subset \left[0, \frac{\pi}{2}\right]$. Then the above theorem says that there is a fixed point x for which $\cos(x) = x$.

6.2 Monotonicity and invertibility of continuous functions

Let us recall the following definition.

Definition 6.71. Let $f\colon E \to \mathbb{R}$ be a function, $E \subset \mathbb{R}$.

(1) f is *strictly increasing* if $f(x) < f(y)$ for all $x < y$ in E.

(2) f is *strictly decreasing* if $f(y) > f(x)$ for all $x < y$ in E.

(3) $f\colon E \to \mathbb{R}$ is *strictly monotone*, if it is strictly increasing or strictly decreasing.

Corollary 6.72. *Let $a, b \in \overline{\mathbb{R}}$. If $f\colon (a,b) \to \mathbb{R}$ is strictly monotone and continuous, then the range $\mathrm{R}(f)$ is an open interval.*

Proof. Set

$$S := \sup\{\ f(x) \mid x \in (a,b)\ \},$$
$$I := \inf\{\ f(x) \mid x \in (a,b)\ \}.$$

First, we show that $S, I \notin \mathrm{R}(f)$. We only prove statement about S since the statement about I can be proven analogously. So, let us assume by contradiction that $S = f(c)$ for some $c \in (a,b)$. Choose $c < d \in (a,b)$ – here we are using that the interval is open!. Then, as f is strictly increasing $f(d) > f(c) = S$, which is a contradiction with the definition of S.
We now show that $\mathrm{R}(f) = (I, S)$. Let us fix $p \in (I, S)$. By the definition of S and I, there exist $c, d \in (a,b)$ such that $f(c) < p < f(d)$. Then the Intermediate Value **??** implies that $p \in \mathrm{R}(f)$, since $p \in [f(c), f(d)] \subset \mathrm{R}(f)$. □

Theorem 6.73. *Let $f\colon E \to F$ be a continuous function on an interval E. Then, f is strictly monotone if and only if it is injective.*

Proof. We do not prove this in class, read the proof from page 84-85 of the book.

□

Theorem 6.74. *If $f\colon E \to F$ is continuous, strictly monotone and surjective function between intervals E, F. Then f^{-1} is also continuous.*

Let us recall that in the hypotheses of **??**, the inverse function f^{-1} exists by **??**.

Proof. We only show the case when E is an open interval (a,b), for some $a, b \in \overline{\mathbb{R}}$. In this case, F is also an open interval according to **??**. Fix $0 < \varepsilon \in \mathbb{R}$ and $y_0 \in F$. Set $x_0 := f^{-1}(y_0)$. According to **??**, there exist $c, d \in \mathbb{R}$ such that

$$\mathrm{R}(f|_{(x_0-\varepsilon, x_0+\varepsilon)}) = (c,d) \tag{6.74.a}$$

In particular, there exists $\delta > 0$ such that for every $y \in F$

$$\text{if } |y - y_0| \leq \delta \Rightarrow y \in (c,d). \tag{6.74.b}$$

For example, it suffices to take $\delta := \frac{\min\{|c-y_0|, |d-y_0|\}}{2}$: that is a choice of δ for which the above condition is satisfied.

We show that with the above choice of δ the definition of the continuity of f^{-1} at y_0 is satisfied. That is, for every $y \in F$,

$$|y - y_0| \leq \delta \Rightarrow \underbrace{y \in (c,d)}_{\mathbf{??}} \Rightarrow \underbrace{|f^{-1}(y) - x_0| \leq \varepsilon.}_{\mathbf{??}}$$

□

Example 6.75. Neither of the functions $\sin(x)$, $\cos(x)$, $\tan(x)$ and $\mathrm{cotan}(x)$ are invertible if considered as functions $\mathbb{R} \to \mathbb{R}$, as they are not injective in view of their periodicity. However, if we restrict their domains adequately they become strictly montone, and then, according to **??**, their inverses are continuous too:

(1) $\arcsin(x)$ is the inverse of $\sin(x)|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$. For example, $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, and $\arcsin\left(-\frac{1}{2}\right) \neq \frac{7\pi}{6}$, despite having $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$ too.

(2) $\arccos(x)$ is the inverse of $\cos(x)|_{[0,\pi]}$.

(3) $\arctan(x)$ is the inverse of $\tan(x)|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$.

Images/arcsin_gr.png

Figure 13: $f(x) = \arcsin(x)$.

Images/arccos_gr.png

Figure 14: $f(x) = \arccos(x)$.

7 Differentiation

Let $f\colon E \to \mathbb{R}$ be a real valued one variable function. We would like to approximate it with a linear one. That is, we would like to write

$$f(x) = f(x_0) + a(x - x_0) + r(x), \tag{7.0.a}$$

where a is a real number, and the error function $r(x)$ is small in a neighborhood of x_0. The question is: how small would we like $r(x)$ to be so that we obtain a "good" approximation? What kind of function do then realize formula **??** with our chosen conditions on $r(x)$?

Well, if we want our approximation to at least compute the right value of f at x_0, since

$$\lim_{x \to x_0} x - x_0 = 0,$$

we need to impose that $\lim\limits_{x \to x_0} r(x) = 0$. Even better, we would like $r(x)$ to be smaller than a linear function, otherwise the linear approximation in **??** will not be very precise. But what does

Images/arctan_gr.png

Figure 15: $f(x) = \arctan(x)$.

it precisely mean that $r(x)$ should be smaller than a linear function? The precise mathematical wording is the following:

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0. \tag{7.0.b}$$

Even better, taking $r_1(x) := \frac{r(x)}{x - x_0}$, we can rewrite the above condition as

$$r(x) = (x - x_0) r_1(x), \quad \text{and} \quad \frac{r_1(x)}{x - x_0} = 0. \tag{7.0.c}$$

The graph of the function $g(x) := f(x_0) + a(x - x_0)$ is a line in the cartesian plane. Considering

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Figure 16: A differentiable function and the tangent line to the graph.

the graph of $f(x)$, then if we can show that that for f the error function $r(x)$ is smaller than linear, that is, if $r(x)$ satisfies the condition of **??**, then the line representing the graph of g will be tangent to the graph of f at the point $(x_0, f(x_0))$.

At this point the central question is: for what functions f do a and $r(x)$ exists satisfying **??**, **??**, respectively?

If both **??** and **??** hold, then

$$\frac{r(x)}{x-x_0}+a=\frac{f(x)-f(x_0)}{x-x_0},\quad x\neq x_0, \tag{7.0.d}$$

and, moreover, by taking the limit for $x \to x_0$ on both sides of this equation, using **??**, it follows that

$$a=\underbrace{\lim_{x\to x_0}\frac{r(x)}{x-x_0}+a}_{\text{by ??}}=\underbrace{\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}}_{\text{by ??}}. \tag{7.0.e}$$

So, the existence of the real number a together with the sub-linear[15] behavior of the error term described in **??** imply that the limit on the right of **??** exists and it is finite. This discussion motivates the following definition.

Definition 7.1. Let $f\colon E \to \mathbb{R}$ be a function and let $x_0 \in E$.

(1) The function f is differentiable at x_0, if the limit

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \tag{7.1.f}$$

exists and it is finite. We call the value of the limit in **??** the derivative of f at x_0 and we denote it by $f'(x_0)$.

(2) We say that $f\colon E \to \mathbb{R}$ is differentiable if it is differentiable at all points $x_0 \in E$.

(3) The function

$$f'\colon \{x\in E \mid f \text{ is differentiable at } x\} \to \mathbb{R},\quad x\mapsto f(x)'$$

is called the derivative function of f. the domain of f' is composed of all points of E where the above limit exists.

Remark 7.2. (1) The derivative $f'(x_0)$ of f at x_0 can be also defined to be the unique real number c satisfying

$$f(x)=f(x_0)+c\cdot(x-x_0)+r(x), \tag{7.2.g}$$

where the function $r(x)$ satisfies $\lim\limits_{x\to x_0}\frac{r(x)}{x-x_0}=0$. As above, we can write $r(x)=(x-x_0)r_1(x)$ and $\lim\limits_{x\to 0} r_1(x)=0$. In the reminder of this section, we will also use the notation $\varepsilon_1(x)$ to denote the function $r_1(x)$.

(2) The definition of the derivative $f'(x_0)$ in **??** can be summarized from a geometrical viewpoint by saying that the derivative is the limit (when it exists) for $x\to x_0$ of the slope of the unique line passing through $(x_0, f(x_0))$ and the point $(x, f(x))$ corresponding to x on the graph.

Example 7.3. Constant functions are differentiable everywhere. In fact, if $f\colon \mathbb{R}\to\mathbb{R}$, $f(x)=C$, $\forall x\in\mathbb{R}$, $C\in\mathbb{R}$, then for $x_0\in\mathbb{R}$

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{C-C}{x-x_0}=0.$$

[15] Sublinear stands for "less than linear", that is, the condition defined in **??**

Images/secants.jpg

Example 7.4. *We show that* $(x^2)' = 2x$.
For any $x_0 \in \mathbb{R}$, we need to compute the limit $\lim\limits_{x\to x_0} \frac{x^2-x_0^2}{x-x_0}$. Thus,

$$\lim_{x\to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x\to x_0} \frac{(x-x_0)(x+x_0)}{x-x_0} = \lim_{x\to x_0} x + x_0 = 2x_0.$$

Example 7.5. *Similarly, if* $a \in \mathbb{Z}_+$*, then* $(x^a)' = ax^{a-1}$.
Indeed,

$$\begin{aligned}
\lim_{x\to x_0} \frac{x^a - x_0^a}{x - x_0} &= \lim_{x\to x_0} \frac{(x-x_0)(x^{a-1} + x^{a-2}x_0 + x^{a-3}x_0^2 + \cdots + x^1 x_0^{a-2} + x_0^{a-1})}{x - x_0} \\
&= \lim_{x\to x_0} x^{a-1} + x^{a-2}x_0 + x^{a-3}x_0^2 + \cdots + x x_0^{a-2} + x_0^{a-1} \\
&= \underbrace{\lim_{x\to x_0} x^{a-1} + \lim_{x\to x_0} x^{a-2}x_0 + \lim_{x\to x_0} x^{a-3}x_0^2 + \cdots + \lim_{x\to x_0} x x_0^{a-2} + \lim_{x\to x_0} x_0^{a-1}}_{\text{by the addition rule for finite limits and the fact that } \forall c \in \mathbb{N},\ \lim_{x\to x_0} x^c = x_0^c} \\
&= a x_0^{a-1}.
\end{aligned}$$

Example 7.6. *We show that* $\sin(x)' = \cos(x)$.

$$\begin{aligned}
\lim_{x\to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x - x_0} \\
&= \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right) \cdot \underbrace{\lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{\lim_{t\to 0}\frac{\sin(t)}{t}=1} = \cos(x_0),
\end{aligned}$$

where we could break up the limit in the multiplication thanks to **??**.

Example 7.7. *Similarly,* $\cos(x)' = -\sin(x)$.

$$\begin{aligned}
\lim_{x\to x_0} \frac{\cos(x) - \cos(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{-2\sin\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x - x_0} \\
&= \lim_{x\to x_0} -\sin\left(\frac{x+x_0}{2}\right) \cdot \underbrace{\lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{\lim_{t\to 0}\frac{\sin(t)}{t}=1} = -\sin(x_0),
\end{aligned}$$

where we could break up the limit in the multiplication thanks to **??**.

Differentiability is a stronger condition than continuity, as the following proposition readily shows.

Proposition 7.8. *If $f\colon E \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0.*

Proof. This is a consequence of the following computation:

$$\begin{aligned}\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \underbrace{f(x_0) + (x - x_0)f(x_0)' + r(x)}_{\text{by }\mathbf{??}} =& \underbrace{\;f(x_0) + \lim_{x\to x_0} r(x)}_{\mathbf{??}\text{ and }\lim_{x\to x_0} x - x_0 = 0} \\ =& f(x_0) + \underbrace{\lim_{x\to x_0} \frac{r(x)}{x - x_0}}_{=0\text{by }\mathbf{??}} \cdot \lim_{x\to x_0}(x - x_0) = f(x_0).\end{aligned}$$

□

Example 7.9. The viceversa of **??** is not true. That is, if f is continuous at x_0, it does not necessarily have to be differentiable.
For example, let us consider the function $f(x) := |x|$. The function f is continuous on $\mathbb{R}$, in particular, it is continuous at $x_0 = 0$. On the other hand, f is not differentiable at 0, because that would imply that $\lim\limits_{x\to 0} \frac{|x|}{x}$ exists. However, since

Images/abs_val_gr.png

Figure 17: $f(x) = |x|$.

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \neq 1 = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} \frac{|x|}{x}.$$

?? implies that f is not differentiable at 0.

Example 7.10. (1) The function $f(x) \coloneqq \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ is not differentiable at 0, since it is not continuous at 0. On the other hand, outside 0, f is differentiable, since over $\mathbb{R}^*$, f is constant.

(2) The function $f(x) \coloneqq \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is not differentiable at any point of $\mathbb{R}$ since it is not continuous at any point of $\mathbb{R}$.

7.1 Computing derivatives

In this section we show how to compute derivatives. We first start by studying how derivatives behave with respect to the usual algebraic operations on $\mathbb{R}$, and then continue by studying how to compute derivatives with respect to composition and taking the inverse.

7.1.1 Addition

Proposition 7.11. *If $f, g\colon E \to \mathbb{R}$ are differentiable at x_0, then so is $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{R}$, and furthermore*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

Proof.

$$\begin{aligned}(\alpha f + \beta g)'(x_0) =& \lim_{x\to x_0} \frac{(\alpha f + \beta g)(x) - (\alpha f + \beta g)(x_0)}{x - x_0} \\ =& \alpha \underbrace{\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{=f'(x_0)} + \beta \underbrace{\lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0}}_{=g'(x_0)} = \alpha f'(x_0) + \beta g'(x_0)\end{aligned}$$

where we could split the limit of the sum into the sum of the limits by the assumption on the differentiability of f, g at x_0, using **??**. ☐

Example 7.12. $\left(5x^3 + 6x^2\right)' = \left(5x^3\right)' + \left(6x^2\right)' = 15x^2 + 12x$

7.1.2 Multiplication

Proposition 7.13. *If $f, g\colon E \to \mathbb{R}$ are differentiable at x_0, then so is $f \cdot g$, and furthermore*

$$(f \cdot g)'(x_0) = (fg' + f'g)$$

Proof.

$$\begin{aligned}(f \cdot g)'(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x)(g(x) - g(x_0)) + (f(x) - f(x_0))g(x_0)}{x - x_0} \\ &= \left(\underbrace{\lim_{x\to x_0} f(x)}_{=f(x_0)}\right) \cdot \left(\underbrace{\lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0}}_{=g'(x_0)}\right) + g(x_0) \left(\underbrace{\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{=f'(x_0)}\right) \\ &= f(x_0)g'(x_0) + g(x_0)f'(x_0)\end{aligned}$$

where the fact that $\lim_{x\to x_0} f(x) = f(x_0)$ follows from **??** and we could split the limits of sum and multiplications using the differentiability of f, g at x_0 and **??**. ☐

Example 7.14.

$$\left(x^2 \cos(x)\right)' = \left(x^2\right)' \cos(x) + x^2 \left(\cos(x)\right)' = 2x\cos(x) + x^2(-\sin(x)) = x(2\cos(x) - x\sin(x))$$

7.1.3 Division

Proposition 7.15. *If $f, g\colon E \to \mathbb{R}$ are differentiable at x_0, and $g(x_0) \neq 0$, then $\frac{f}{g}$ is also differentiable at x_0, and furthermore*

$$\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf' - fg'}{g^2}\right)(x_0)$$

In particular,

$$\left(\frac{1}{g}\right)'(x_0) = \left(\frac{-g'}{g^2}\right)(x_0)$$

Proof. We compute the now familiar limit

$$\begin{aligned}\lim_{x\to x_0} \frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)(x - x_0)} \\ &= \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)(x - x_0)}.\end{aligned}$$

Grouping together, in the denominator of the last member of the previous equation, those terms that depend on $g(x_0)$ and $f(x_0)$, respectively, we obtain,

$$\begin{aligned}&\lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)(x - x_0)} \\ =&\frac{g(x_0)}{g(x_0)\cdot \underbrace{\lim_{x\to x_0} g(x)}_{=g(x_0)}}\left(\underbrace{\lim_{x\to x_0}\frac{f(x) - f(x_0)}{x - x_0}}_{=f'(x_0)}\right) \\ &- \frac{f(x_0)}{g(x_0)\cdot \underbrace{\lim_{x\to x_0} g(x)}_{=g(x_0)}}\left(\underbrace{\lim_{x\to x_0}\frac{g(x) - g(x_0)}{x - x_0}}_{=g'(x_0)}\right) \\ =&\left(\frac{gf' - fg'}{g^2}\right)(x_0),\end{aligned}$$

where the fact that $\lim_{x\to x_0} g(x) = g(x_0)$ follows from **??** and we could split the limits of sum and multiplications using the differentiability of f, g at x_0 and **??**. ☐

Example 7.16. If $b > 0$ is an integer and $x \neq 0$, then:

$$\left(\frac{1}{x^b}\right)' = -\frac{\left(x^b\right)'}{x^{2b}} = -\frac{bx^{b-1}}{x^{2b}} = \frac{-b}{x^{b+1}}.$$

That is, by setting $a = -b$ we obtain $(x^a)' = ax^{a-1}$. In particular, this shows that

$$(x^a)' = ax^{a-1}$$

holds for all integer a, not just the non-negative ones.

Example 7.17. If $x \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$, or, equivalently, if $\cos(x) \neq 0$, then

$$\begin{aligned}\tan(x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' =&\frac{\cos(x)(\sin(x))' - \sin(x)(\cos(x))'}{\cos(x)^2} \\ =&\frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos(x)^2} = \frac{1}{\cos^2(x)}.\end{aligned}$$

7.1.4 Composition of functions and derivatives

Proposition 7.18. *Let $f\colon E \to \mathbb{R}$, $g\colon G \to \mathbb{R}$ be functions such that that $f(E) \subseteq G$. Assume that f is differentiable at $x_0 \in E$, and g is differentiable at $f(x_0)$ then $g \circ f\colon E \to \mathbb{R}$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x)) \cdot f'(x_0).$$

Idea of the proof.

$$\begin{aligned}\lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0)) \cdot f'(x_0)\end{aligned}$$

□

Example 7.19. Let $f(x) = x^2$ and $g(y) = \cos(y)$. Then $f'(x) = 2x$ and $g'(y) = -\sin(y)$. In particular,

$$\cos(x^2)' = (g \circ f)'(x) = (g' \circ f)(x) \cdot f'(x) = -\sin(x^2)2x.$$

Example 7.20. Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$, $f(x) = x^a$ and $g(y) = y^{\frac{1}{b}}$. Then according to **??** and **??**, $f(x)' = ax^{a-1}$ and $g(y)' = \frac{1}{b}y^{\frac{1}{b}-1}$. Hence,

$$\left(x^{\frac{a}{b}}\right)' = \left((x^a)^{\frac{1}{b}}\right)' = (g \circ f)'(x) = (g' \circ f)(x) \cdot f'(x) = \frac{1}{b}\,(x^a)^{\frac{1}{b}-1}\,ax^{a-1} = \frac{a}{b}x^{\frac{a}{b}-a+a-1} = \frac{a}{b}x^{\frac{a}{b}-1}$$

So, *the formula $(x^r)' = rx^{r-1}$ holds also when r is any rational number* (as it did for $r \in \mathbb{Z}$ in **??**).

7.1.5 Inversion of functions and derivatives

Proposition 7.21. *Let $f\colon E = (a, b) \to F$ be a bijective continuous function (so f is strictly monotone, and f^{-1} exists and is continuous by **??**), and let $x_0 \in E$ be such that $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $y_0 := f(x_0)$, and we have*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Proof. The idea behind the proof of the proposition is that if we set $y = f(x)$ and $y_0 = f(x_0$, we have

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)}} = \frac{1}{\frac{f(f^{-1}(y))-f(f^{-1}(y_0))}{f^{-1}(y)-f^{-1}(y_0)}}.$$

Check page 109 for the precise proof. □

Example 7.22. If $f(x) = x^b$ for some integer $b \geq 1$, then $f^{-1}(y) = \sqrt[b]{y} = y^{\frac{1}{b}}$. So, $f'(x) = bx^{b-1}$, and

$$\left(y^{\frac{1}{b}}\right)' = \left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{b\left(y^{\frac{1}{b}}\right)^{b-1}} = \frac{1}{b}y^{-\frac{b-1}{b}} = \frac{1}{b}y^{\frac{1}{b}-1}.$$

So, for $c = \frac{1}{b}$ (where $b \in \mathbb{Z}^*_+$), the formula for $(y^c)' = cy^{c-1}$. That is the formula is the same as in the case of c being an integer.

Example 7.23. Let $f(x) = \sin(x)|_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$. Then f is invertible and $f^{-1}(y) = \arcsin(y)$. Also, $f'(x) = \cos(x)|_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}$, thus, for any $y \in]-1, 1[$,

$$\arcsin'(y) = \frac{1}{\cos(\arcsin(y))} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(y))}} = \frac{1}{\sqrt{1 - y^2}}.$$

7.1.6 The exponential function

For our last example in this section, we will discuss in details the exponential and logarithmic functions. Let us remind the reader that we defined

$$e^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Definition 7.24. For $x \in \mathbb{R}$, we define

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Remark 7.25. Applying **??**, to $x = 0$ yields $e^0 = 1$. Furthermore, according to **??**, $e^1 = e$.

Proposition 7.26. *For any $x, y \in \mathbb{R}$, $e^{x+y} = e^x \cdot e^y$.*

Proof. This is an exercise in Week 10 exercise sheet. □

Corollary 7.27. *For any $x \in \mathbb{R}$, $e^{-x} = \frac{1}{e^x}$.*

Proof.

$$e^x \cdot e^{-x} = \underbrace{e^{x+(-x)}}_{??} = e^0 = \underbrace{1}_{??}.$$

Dividing by e^x yields the statement (e^x cannot be 0, since then $e^x \cdot e^{-x} = 1$ could not hold). □

Corollary 7.28. *For every $x \in \mathbb{R}$, $e^x > 0$.*

Proof. For $x \geq 0$, then all the terms in the infinite sum in **??** is at least zero, and the first term is 1. This implies the statement for $x \geq 0$.

So, we may assume from now that $x < 0$. We have $e^x = \frac{1}{e^{-x}}$ by **??**. However, as now $-x > 0$ holds, the previous paragraph tells us that $e^{-x} > 0$, and hence also $\frac{1}{e^{-x}} > 0$. □

Proposition 7.29. $(e^x)' = e^x$

Proof. We need to show that

$$\lim_{x\to x_0} \frac{e^x - e^{x_0}}{x - x_0} = e^{x_0}.$$

This is equivalent to showing that

$$0 = \lim_{x\to x_0} \frac{e^x - e^{x_0}}{x - x_0} - e^{x_0} = \left(\lim_{x\to x_0} \frac{e^{x-x_0} - 1}{x - x_0} - 1\right) e^{x_0}.$$

By setting $y = x - x_0$, what we need to show is that

$$\lim_{y\to 0} \frac{e^y - 1}{y} - 1 = 0. \tag{7.29.a}$$

However, for $0 < |y| \leq 1$:

$$0 \leq \left|\frac{e^y - 1}{y} - 1\right| = \left|\frac{\sum_{k=0}^{\infty} \frac{y^k}{k!} - 1}{y} - 1\right| = \left|\sum_{k=2}^{\infty} \frac{y^{k-1}}{k!}\right| \leq \sum_{k=2}^{\infty} \frac{|y|^{k-1}}{k!} \leq |y| \sum_{k=2}^{\infty} \frac{|y|^{k-2}}{(k-2)!}$$

$$= |y| \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \leq |y| e$$

Hence, by Squeeze Theorem **????**, it follows that **??** holds indeed. □

Proposition 7.30. *We have* $\lim_{x\to+\infty} e^x = +\infty$*, and* $\lim_{x\to-\infty} e^x = 0$.

Proof. According to **??**, for all $x > 0$, $e^x \geq 1 + x$. As $\lim_{x\to+\infty} 1 + x = +\infty$, squeeze (point **??** of **??**) shows that $\lim_{x\to+\infty} e^x = +\infty$. Then **??**, **??** and point **??** of **??** show that $\lim_{x\to-\infty} e^x = 0$. □

Proposition 7.31. *The function* $e^x : \mathbb{R} \to \mathbb{R}$ *is strictly increasing.*

Proof. Choose $y > x \in \mathbb{R}$. We have to show that $e^y > e^x$. This is shown by the following computation:

$$e^y - e^x = \underbrace{(e^{y-x} - 1)}_{>\ 1 \text{ by ??, using } y > x} \cdot \underbrace{e^x}_{>\ 0 \text{ by ??}} > 0$$

□

Corollary 7.32. *The range* $R(f)$ *of* $f := e^x : \mathbb{R} \to \mathbb{R}$ *is* $(0, +\infty) = \mathbb{R}_+^*$.

Proof. Follows immediately from **??** and **??**. □

Definition 7.33.

(1) We define the (natural) logarithm function $\log(x)\colon \mathbb{R}_+^* \to \mathbb{R}$ to be the inverse of the exponential function $f(x) = e^x$.

(2) For any $a \in \mathbb{R}_+^*$, the a-based exponential functions a^x is defined as

$$a^x := e^{x\cdot\log(a)}.$$

The logarithm in base a of x is the inverse function of the a-based exponential function a^x,

$$\log_a(x) := \frac{\log(x)}{\log(a)}.$$

(3) For any $a \in \mathbb{R}$, the a-th power functions is defined as

$$x^a := e^{a\cdot\log(x)}.$$

Remark 7.34. In the special cases where the functions of **??** have been already defined (so x^a when $a \in \mathbb{Q}$, and a^x when $a = e$), they agree with the previously defined functions. This will be an exercise on the exercise sheet.

Example 7.35. (1) If $f(x) = e^x$, then $f^{-1}(x) = \log(x)$ and $f'(x) = e^x$ (**??**). Hence:

$$(\log(x))' = \frac{1}{e^{\log(x)}} = \frac{1}{x}.$$

(2) Let $h\colon \mathbb{R}_+^* \to \mathbb{R}$, $h(x) := x^x$. Then, definining $f(x) = x\log(x)$, $g(y) := e^y$,

$$h(x) = (f \circ g)(x).$$

Thus,

$$h'(x) = (f \circ g)'(x) = f'(g(x))g'(x) = x^x(\log(x) + 1).$$

Definition 7.36. (1) The *hyperbolic trigonometric* functions are defined below, and they are called hyperbolic sine/cosine/tangent/cotangent:

$$\begin{aligned}
\sinh(x) &:= \frac{e^x - e^{-x}}{2} \\
\cosh(x) &:= \frac{e^x + e^{-x}}{2} \\
\tanh(x) &:= \frac{\sinh(x)}{\cosh(x)} \\
\coth(x) &:= \frac{\cosh(x)}{\sinh(x)}
\end{aligned}$$

The domains of all the above functions is $\mathbb{R}$, except for coth which is defined over $\mathbb{R}^*$, since $\sinh(0) = 0$.

Proposition 7.37. *We have:*

(1) $\sinh(x)' = \cosh(x)$

(2) $\cosh(x)' = \sinh(x)$

(3) $\tanh(x)' = \frac{1}{\cosh(x)^2}$

(4) $\coth(x)' = \frac{-1}{\sinh(x)^2}$

(5) $(x^a)' = ax^{a-1}$

(6) $(a^x)' = \log(a) \cdot a^x$

(7) $\log_a(x)' = \frac{1}{\log(a) \cdot x}$

The proof is left as an exercise.

7.2 One sided derivatives

Definition 7.38. If $f\colon E \to \mathbb{R}$ is a function and $x_0 \in E$ for which the then we say that the left (resp. right) derivative of f exists at x_0 if the function

$$\frac{f(x) - f(x_0)}{x - x_0} : E \setminus \{x_0\} \to \mathbb{R}$$

admits a left (resp. right limit). The value of this limit is then the left (resp. right) derivative.

Example 7.39. For $f(x) = |x|$ at $x = 0$ the left derivative is -1 and the right derivative is 1. In fact,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \frac{x}{x} = 1, \quad \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \frac{-x}{x} = -1.$$

As in the case of left and right limits, we can use left and right derivatives to decide when a function is differentiable at a given point.

Proposition 7.40. *Let $f\colon E \to \mathbb{R}$ be a function and $x_0 \in E$ a real number. Then f is differentiable at a point x_0 if and only if both its left and right derivatives exist and they agree. Furthermore, then the value of the derivative is the same as the common value of the left and the right derivatives.*

Proof. This is an immediate consequence of **??**, **??** and **??**. □

Example 7.41. (1) Let us consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}.$$

The function f is differentiable at 0 with derivative $f'(0) = 0$. Indeed,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2 - 0}{h} = \lim_{h\to 0^+} h = 0.$$

Similarly,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{h^3 - 0}{h} = \lim_{h\to 0^-} h^2 = 0.$$

Since the left derivative and the right derivative exist and agree at $x_0 = 0$, we can conclude that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

Thus, f is differentiable at 0 with $f'(0) = 0$.

(2) Let us consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x+1 & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases}.$$

The function f is not continuous at $x_0 = 0$, as $\lim\limits_{x\to 0^-} f(x) = 0$ whereas $\lim\limits_{x\to 0^+} f(x) = 1$. As a differentiable function is continuous, then f is not differentiable at 0. On the other hand, f is differentiable outside of 0, since on a sufficiently small neighborhood of any point $x_0 \neq 0$, f is given by a linear function and we know that linear functions are differentiable.

7.3 Higher derivatives

Given a function f, we may try to iterate inductively the process of taking the derivative of f, thus obtaining what we will call the second derivative of f, the third, derivative of f, etc.

Definition 7.42. Let $f\colon E \to \mathbb{R}$ be a function.

(1) The second derivative f'' of f is the function

$$\begin{aligned} f''\colon \{x \in E \mid f' \text{ is differentiable at } x\} &\to \mathbb{R} \\ x &\mapsto f''(x) := (f')'(x). \end{aligned}$$

(2) Assume that the n-th derivative $f^{(n)}$ of f has been defined. Then the $(n+1)$-st derivative $f^{(n+1)}$ of f is the function

$$\begin{aligned} f^{(n+1)}\colon \{x \in E \mid f^{(n)} \text{ is differentiable at } x\} &\to \mathbb{R} \\ x &\mapsto f^{(n+1)}(x) := (f^{(n)})'(x). \end{aligned}$$

The n-th derivative of f at $x \in E$ is denoted by $f^{(n)}(x)$. For the first, second and third derivative of f, we will adopt the notation f', f'', f''' rather than $f^{(1)}, f^{(2)}, f^{(3)}$.

Example 7.43. (1) The second derivative of $f(x) = \arctan(x)$ is $f''(x)\colon \mathbb{R} \to \mathbb{R}$,

$$f''(x) = (f')'(x) = (\frac{1}{1+x^2})' = \frac{-2x}{(1+x^2)^2}.$$

(2) Let us consider the function $f\colon \mathbb{R}^* \to \mathbb{R}$ defined by $f(x) = e^{\frac{1}{x}}$. Then,

$$\begin{aligned} f''(x) = (f')'(x) &= (e^{\frac{1}{x}} \cdot (-\frac{1}{x^2}))' \\ &= (e^{\frac{1}{x}})' \cdot (-\frac{1}{x^2}) + e^{\frac{1}{x}} \cdot (-\frac{1}{x^2})' = e^{\frac{1}{x}}(\frac{1}{x^4} + \cdot\frac{2}{x^3}). \end{aligned}$$

Example 7.44. Here we show an example of a function $f(x)$ such that $f(x)$ is differentiable two times, but not three times. That is, $f'(x)$ and $f''(x)$ exist for every $x \in \mathbb{R}$, but $f'''(0)$ does not exist.
Let us consider $f(x) := |x^3|$. Then, $f'(x)$ exists for all $x \in \mathbb{R}$, and:

$$f'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0. \end{cases}$$

This is immediate at $x \neq 0$ from the formula

$$f(x) = \begin{cases} x^3 & \text{for } x \geq 0 \\ -x^3 & \text{for } x \leq 0 \end{cases}$$

To conclude the above first claim we just have to compute the left and the right derivatives of $f(x)$ at $x = 0$, and show that both are 0. Indeed:

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^3 - 0}{x} = \lim_{x \to 0^+} x^2 = 0,$$

and

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^3 - 0}{x} = \lim_{x \to 0^-} -x^2 = 0.$$

This concludes our first claim.
Similarly, one can prove that $f''(x)$ exists for all $x \in \mathbb{R}$ and

$$f''(x) = \begin{cases} 6x & \text{for } x \geq 0 \\ -6x & \text{for } x < 0 \end{cases}$$

With other words, $f''(x) = 6|x|$. However, as $|x|$ is not differentiable at $x = 0$, we obtain that $f'''(0)$ does not exist.

Definition 7.45. $f\colon E \to \mathbb{R}$ is called a function of class C^n if its first n derivatives $f', f'', \dots, f^{(n)}$ exists and are all continuous at all points $x_0 \in E$.

Notation 7.46. To denote that a function $f\colon E \to \mathbb{R}$ is a C^n function, we will use the notation $f \in C^n(E, \mathbb{R})$. We will write $f \in C^\infty(E, \mathbb{R})$ if $f \in C^n(E, \mathbb{R})$, $\forall n \in \mathbb{N}$, and we will say that f is a C^∞ function.

Example 7.47. (1) According to **??**, x, x^2, etc. are C^n for all n, that is they are C^∞ function. More precisely, for $a \in \mathbb{N}$, defining $f(x) = x^a$ then

$$f^{(n)}(x) = \begin{cases} 0 & \text{if } n > a \\ a \cdot (a-1) \cdot \dots \cdot (a - n + 1)x^{a-n} & \text{for } n \leq a \end{cases}$$

(2) We can repeat the same computation for $f\colon [0, +\infty) \to \mathbb{R}$, $f(x) := x^\alpha = e^{\log(x)\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{N}$. Then $f^{(x)} = \alpha \cdot (\alpha - 1) \cdot (\alpha - 2) \cdot \dots \cdot (\alpha - n + 1)x^{\alpha - n}$, $x > 0$.

(3) $|x| : \mathbb{R} \to \mathbb{R}$ is not C^1, cf. **??**.

(4) $|x^3| : \mathbb{R} \to \mathbb{R}$ is C^2 but not C^3, cf. **??**.

7.4 Local and global extrema

Definition 7.48. Let $f\colon E \to \mathbb{R}$ be a function and let $x_0 \in E$.

(1) The function f admits a point of *local maximum* at x_0 if there is a real number $\delta > 0$ such that $]x_0 - \delta, x_0 + \delta[\subset E$ and for every $x \in E$ if $|x - x_0| < \delta$ then $f(x) \leq f(x_0)$.

(2) The function f has a point of *local minimum* at x_0 if there is a real number $\delta > 0$ such that $]x_0 - \delta, x_0 + \delta[\subset E$ and for every $x \in E$ if $|x - x_0| < \delta$ then $f(x) \geq f(x_0)$.

(3) We say that $x_0 \in E$ is a point of local extremum for f if it is either a point of local minimum or of local maximum.

(4) The function f has a point of global maximum at x_0 if $f(x_0) \geq f(x)$, for all $x \in E$.

(5) The function f has a point of global minimum at x_0 if $f(x_0) \leq f(x)$, for all $x \in E$.

Remark 7.49. We shall also say that f admits a local maximum (resp. local minimum, local extremum, global maximum, global minimum) at x_0 to indicate that property (1) (resp. (2), (3), (4), (5)) defined above is satisfied.

Remark 7.50. Let $f\colon E \to \mathbb{R}$ be a function and $x_0 \in E$. If x_0 is a point of global maximum (resp. global minimum) for f and E contains a neighborhood of x_0 of the form $]x_0 - \delta, x_0 + \delta[,\ \delta > 0$, then x_0 is also a point of local maximum (resp. local minimum) for f.

Example 7.51. Let us consider the function f

The following proposition shows that any point of local extremum for a function f coincides with a zero of the derivative f'.

Proposition 7.52. *If $f\colon E \to \mathbb{R}$ is differentiable at x_0, and f admits a local extremum at x_0, then $f'(x_0) = 0$.*

Proof. We present the local maximum case, as one just need to reverse a few signs, to modify the proof to obtain from it the case of local minimum.
Hence, let us assume that $x_0 \in E$ is a point of local by **??**, there is a real number $\delta > 0$ such that

$$|x - x_0| \leq \delta \Rightarrow f(x) \leq f(x_0). \tag{7.52.a}$$

However, then

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{(x - x_0)} \leq \underbrace{\lim_{x \to x_0^+} \frac{0}{(x - x_0)}}_{x > x_0, \text{ and } ??} = 0, \tag{7.52.b}$$

and

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{(x - x_0)} \geq \underbrace{\lim_{x \to x_0^-} \frac{0}{(x - x_0)}}_{x < x_0, \text{ and } ??} = 0. \tag{7.52.c}$$

As $f(x)$ is differentiable, at x_0 the two above limits agree (**??**). Hence the following stream of inequalities have to be all equalities, which conclude our proof:

$$\underbrace{0 \leq \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{(x - x_0)}}_{??} = f'(x_0) = \underbrace{\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{(x - x_0)} \leq 0.}_{??}$$

□

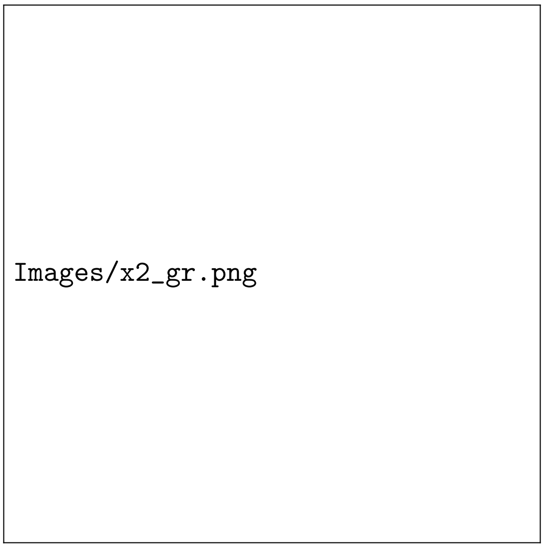

Figure 18: $f(x) = x^2$ has a global minimum at $x = 0$.

Example 7.53. (1) For $f(x) = x^2$, we have $f'(x) = 2x$. Hence, $f'(x) = 0$ if and only if $x = 0$. So, $x = 0$ is the only option for stationary point of f, thus also for a point of local extremum, and, indeed, f admits a local (and global) minimum at $x = 0$.

(2) For $f(x) = x^3$, we have $f'(x) = 3x^2$. So $f'(x) = 0$ if and only if $x = 0$ (as in the previous case). However, $f(0) = 0$ is not a local extremum. This underlines that **??** yields only a necessary, but not a sufficient condition for having a local extremum.

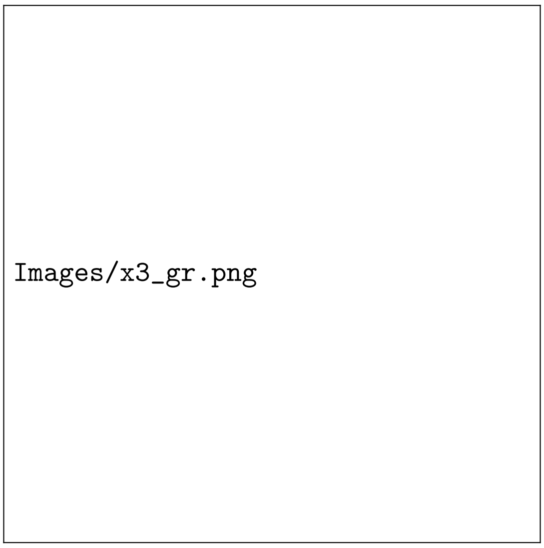

Figure 19: $f(x) = x^3$, $f'(0) = 0$, but 0 is not a point of local extremum for f.

Definition 7.54. Let $f : E \to \mathbb{R}$ be a function and let $x_0 \in E$. Assume that f is differentiable at x_0. Then, we say that x_0 a stationary point (for f) if $f'(x_0) = 0$.

We have just seen that when $f'(x_0) = 0$, we cannot necessarily conclude that x_0 is a point of local extremum for f. On the other hand, if the domain of f is a closed bounded interval $[a, b]$, then **??** implies that f admits both a global maximum and a global minimum in $[a, b]$. Therefore, using **??**, a point of global maximum for f can only be:

- either a stationary point, $x \in (a, b)$ such that $f'(x) = 0$, or

- $x = a$, or $x = b$.

Similarly, a point of global minimum for f can only be:

- either a stationary point, $x \in (a,b)$ such that $f'(x) = 0$, or
- $x = a$, or $x = b$.

Hence, to find the value of the global maximum and the global minimum of f over the interval $[a,b]$, it suffices to compute:

$$\max_{x\in[a,b]} f = \sup\{f(x) \mid x = a \text{ or } x = b \text{ or } f'(x) = 0\},$$
$$\min_{x\in[a,b]} f = \inf\{f(x) \mid x = a \text{ or } x = b \text{ or } f'(x) = 0\}.$$

This procedure guves an algorithmic approach to finding the values of global extrema.

Example 7.55. We compute the global minimum and the global maximum of

$$f(x) = \frac{4}{3}x^3 + \frac{3}{2}x^2 - x + 2$$

on the closed bounded interval $\left[-2, \frac{1}{2}\right]$. By the discussion in the paragraph before the example we have to compute:

$$f'(x) = 4x^2 + 3x - 1,$$

and then find the solutions of the equation $f'(x) = 0$. These are:

$$x = \frac{-3 \pm \sqrt{25}}{8} = \frac{-3 \pm 5}{8} = -1, \text{ and } x = \frac{1}{4}.$$

Then, we have to compute the function values at these two points, and at the endpoints of our interval. The point, where the function value is the maximal yields the maximum and where the function value is minimal yields the minimum of $f(x)$ on $\left[-2, \frac{1}{2}\right]$:

value of x	$f(x)$
-2	$-\frac{4}{3}8 + \frac{3}{2}4 + 2 + 2 = \frac{36-64}{6} + 4 = 4 - \frac{28}{6} = -\frac{4}{6}$
-1	$-\frac{4}{3} + \frac{3}{2} + 1 + 2 = \frac{9-8}{6} + 3 = 3 + \frac{1}{6}$
$\frac{1}{4}$	$\frac{4}{3\cdot 64} + \frac{3}{2\cdot 16} - \frac{1}{4} + 2 = \frac{4+18-48}{192} + 2 = \frac{-26}{192} + 2 = 2 - \frac{13}{96}$
$\frac{1}{2}$	$\frac{4}{3\cdot 8} + \frac{3}{2\cdot 4} - \frac{1}{2} + 2 = \frac{4+9-12}{24} + 2 = 2 + \frac{1}{24}$

So, $f(x)$ on $\left[-2, \frac{1}{2}\right]$ takes its minimum at $x = -2$ and its maximum at $x = -1$.

7.5 Rolle's and Mean Value theorem

We are ready to state and prove the two main results of this chapter: Rolle's theorem (**??**) and the Mean value theorem (**??**).

Theorem 7.56 (Rolle's Theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ is a continuous function, for $a, b \in \mathbb{R}$. Assume that f is differentiable on (a,b), and that $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If f is constant on $[a,b]$, then $f'(x) = 0$, $\forall x \in (a,b)$ and so we are done.
Hence, we can assume that f is not constant. Then, according to **??**, f has both a maximum and a minimum on $[a,b]$. However, as f is non-constant, one of these values have to be not equal to $f(a) = f(b)$. Formally, this means that there is a $c \in [a,b]$, such that $f(c) \neq f(a) = f(b)$. In particular, we must have $a < c < b$. Then, f is differentiable at c, and as f has a (local) extremum at c, we have $f'(c) = 0$ according to **??**. □

Remark 7.57. The differentiability assumption is needed for **??** to hold. For example, considering $f(x) := |x|$ on $[-1,1]$, then $f(-1) = f(1)$, but there is no point in $[-1,1]$ at the derivative of f is 0.

Theorem 7.58 (Mean value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, $a,b \in \mathbb{R}$. Assume that f is differentiable on (a,b). Then there exists $c \in (a,b)$ such that $f'(c)(b-a) = f(b) - f(a)$.*

Proof. Apply **??** to $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$. □

Example 7.59. Let f be a C^1 function (see **??**) on $[-1,1]$ such that $f(-1) = 2$, $f(0) = 4$ and $f(1) = 3$. We show using **??** that there is a $c \in]-1,1[$ such that $f'(c) = 1$:

(1) Applying **??** to $f|_{[-1,0]}$ we obtain that there is an $a \in]-1,0[$ such that $f'(a) = \frac{4-2}{0-(-1)} = 2$.

(2) Applying **??** to $f|_{[0,1]}$ we obtain that there is a $b \in]0,1[$ such that $f'(b) = \frac{3-4}{1-0} = -1$.

(3) As f is C^1, f' is continuous. Hence, **??** implies that there is a $c \in (a,b) \subseteq]-1,1[$ such that $f'(c) = 1$.

Corollary 7.60. *Let $f,g\colon [a,b] \to \mathbb{R}$ be continuous functions, $a,b \in \mathbb{R}$. Assume that f,g are differentiable over (a,b) and that $f'(x) = g'(x)$ for each $x \in (a,b)$. Then there exists a real number C such that $f(x) = g(x) + C$.*

Proof. By taking $h(x) := f(x) - g(x)$, it suffice to apply **??**. □

Lemma 7.61. *Let $h\colon [a,b] \to \mathbb{R}$ a continuous function which is differentiable on (a,b). Assume that $h'(x) = 0$, for all $x \in (a,b)$. Then, h is a constant function.*

Recall that by saying that h is a constant function we simply mean that $\forall x \in [a,b]$, $h(x) = C$ for some fixed real number $C \in \mathbb{R}$ (indipendent of x).

Proof. Assume that h is not constant. Then, there exists $c,d \in [a,b]$, $c < d$ such that $h(c) \neq h(d)$. Then, the Mean Value **??** implies that there exists $e \in (c,d)$, such that $h'(e) = \frac{h(d)-h(c)}{d-c} \neq 0$; nonetheless, this contradicts our assumption that $h'()$. □

7.5.1 Monotone functions and differentials

We can apply the Mean Value Theorem to characterize the derivative of monotone (differentiable) functions.

Corollary 7.62. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, $a,b \in \mathbb{R}$. Assume that f is differentiable on (a,b). Then,*

(1) f is increasing (resp. decreasing) if and only if $f'(x) \geq 0$ (resp. ≤ 0);

(2) if $f'(x) > 0$ (resp. < 0) for all $x \in (a,b)$, then f is strictly increasing (resp. strictly decreasing).

Proof. We only prove the increasing case of **??**, as the others are similar.

- First we assume that f is increasing. Then,

$$x \geq x_0 \Longrightarrow f(x) \geq f(x_0) \Longrightarrow \frac{f(x)-f(x_0)}{x-x_0} \geq 0,$$
$$x \leq x_0 \Longrightarrow f(x) \leq f(x_0) \Longrightarrow \frac{f(x)-f(x_0)}{x-x_0} \geq 0.$$

  Thus,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

Images/x3_gr.png

Figure 20: $f(x) = x^3$

- Second, let us assume by contradiction that $f'(x) \geq 0$ $\forall x \in (a, b)$ and that f is not increasing. Hence, there are $a \leq c < d \leq b$, such that $f(c) > f(d)$. However, then **??** tells us that then there esists $e \in \mathbb{R}$, $c < e < d$ such that $f'(e) = \frac{f(d)-f(c)}{d-c} < 0$.

$\square$

Example 7.63. Let $f\colon E \to \mathbb{R}$ be a strictly increasing (resp. strictly decreasing) function. Then, it does not necessarily follow that $f'(x) > 0$ (resp. $f'(x) < 0$). For example, $f(x) = x^3$ is strictly increasing, but $f'(0) = 3 \cdot 0^2 = 0$.

Example 7.64. Let us consider the function $f_a\colon \mathbb{R} \to \mathbb{R}$, $f(x) := \sin(x) + ax$, where $a \in \mathbb{R}$ is a fixed real number. Let us compute for what value of a f_a is monotone. As $f_a(x)$ is differentiable on $\mathbb{R}$, then f_a is monotone if and only if either $f'(x) \geq 0$, $\forall x \in \mathbb{R}$ or $f'(x) \leq 0$, $\forall x \in \mathbb{R}$. Thus, let us compute $f'(x)$:

$$f'(x) = \cos(x) + a.$$

Thus,

- f is increasing if and only if $a \geq 1$;
- f is decreasing if and only if $a \leq -1$.

Example 7.65. Using **??** and **??** we obtain that the all the functions of **??** are either monotone, or become monotone when restricted to $\mathbb{R}_+^*$ or to $\mathbb{R}_-^*$.

$f(x)$	$D(f)$	f'	monotonicity
$\sinh(x)$	$\mathbb{R}$	$\cosh(x)$	increasing over $\mathbb{R}$
$\cosh(x)$	$\mathbb{R}$	$\sinh(x)$	decreasing over $\mathbb{R}_-^*$ and increasing over $\mathbb{R}_+^*$
$\tanh(x)$	$\mathbb{R}$	$\frac{1}{\cosh(x)^2}$	increasing over $\mathbb{R}$
$\coth(x)$	$\mathbb{R}^*$	$\frac{-1}{\sinh(x)^2}$	decreasing over $\mathbb{R}_-^*$ and over $\mathbb{R}_+^*$
$x^a,\ a > 0$	$\mathbb{R}_+$	$ax^{a-1},\ x \neq 0$	increasing over $\mathbb{R}_+^*$
$x^a,\ a < 0$	$\mathbb{R}_+$	$ax^{a-1},\ x \neq 0$	decreasing over $\mathbb{R}_+^*$
$a^x, a > 1$	$\mathbb{R}$	$\log(a) \cdot a^x$	increasing over $\mathbb{R}$
$a^x, 0 < a < 1$	$\mathbb{R}$	$\log(a) \cdot a^x$	decreasing over $\mathbb{R}$
$\log_a(x),\ a > 1$	$\mathbb{R}_+^*$	$\frac{1}{\log(a)\cdot x}$	increasing over $\mathbb{R}_+^*$
$\log_a(x),\ 0 < a < 1$	$\mathbb{R}_+^*$	$\frac{1}{\log(a)\cdot x}$	decreasing over $\mathbb{R}_+^*$

7.5.2 L'Hôpital's rule

L'Hôpital rule gives a method to compute limits of fractions of function which are in the indeterminate forms

$$\frac{0}{0}, \frac{\infty}{\infty},$$

that is, either both values of the limit of the denominator and of the limit of the numerator approach 0, or they both approach $-\infty$ or $+\infty$ – in the latter case, the sign of ∞ does not really matter.

Example 7.66. How can we compute $\lim\limits_{x\to+\infty} \dfrac{e^x}{x}$? In this case,

$$\lim_{x\to+\infty} e^x = +\infty = \lim_{x\to+\infty} x.$$

In this example, we cannot answer using the algebraic rules of **??**.

Luckily, the following theorem provides us with new tools to carry out this kind of computations.

Theorem 7.67 (L'Hôpital rule)**.** *Let $f, g\colon (a, b) \to \mathbb{R}$ be differentiable functions, and let $a, b \in \overline{\mathbb{R}}$. Assume that the following conditions hold:*

(1) (exactly) one the following conditions hold for x_0:

- *(i) $x_0 \in (a, b)$;*
- *(ii) $x_0 = a \in \mathbb{R}$;*
- *(iii) $x_0 = b \in \mathbb{R}$;*
- *(iv) $x_0 = a = -\infty$;*
- *(v) $x_0 = b = +\infty$;*

(2) $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{x_0\}$;

(3) $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \alpha$ *for* $\alpha = 0$ *or* $\alpha = \pm\infty$.

Then, in the respective cases we have the following implications for any $\mu \in \overline{\mathbb{R}}$*:*

Cases **??**, **??** *and* **??**	*if* $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \mu \quad \Rightarrow \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \mu$
Case **??**	*if* $\lim_{x\to x_0^+} \frac{f'(x)}{g'(x)} = \mu \quad \Rightarrow \quad \lim_{x\to x_0^+} \frac{f(x)}{g(x)} = \mu$
Case **??**	*if* $\lim_{x\to x_0^-} \frac{f'(x)}{g'(x)} = \mu \quad \Rightarrow \quad \lim_{x\to x_0^-} \frac{f(x)}{g(x)} = \mu$

Proof. We prove only the $\alpha = 0$ and $x_0 \in (a,b)$ case, and we refer to page 121-122 of the book for the rest. As f and g are differentiable at x_0 they are also continuous there, and hence

$$f(x_0) = \lim_{x\to x_0} f(x) = \alpha = 0 \qquad \text{and} \qquad g(x_0) = \lim_{x\to x_0} g(x) = \alpha = 0. \tag{7.67.a}$$

So, by the mean value theorem for derivatives, there is a real number $c(x)$ between x and x_0 such that

$$f'(c(x)) = \frac{f(x) - f(x_0)}{x - x_0}. \tag{7.67.b}$$

In particular, $c(x)\colon E \setminus x_0 \to I \setminus x_0$ is a function such that $\lim_{x\to x_0} c(x) = x_0$. Then:

$$\mu = \underbrace{\lim_{x\to x_0} \frac{f'(x)}{g'(x)}}_{\text{definition of } \mu} = \underbrace{\lim_{x\to x_0} \frac{f'(c(x))}{g'(c(x))}}_{\textbf{??}} = \underbrace{\lim_{x\to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}}}_{\textbf{??}} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \underbrace{\lim_{x\to x_0} \frac{f(x)}{g(x)}}_{\textbf{??}}$$

$\square$

Remark 7.68. We show that the property (3) in the statement of **??** is a necessary one. Indeed, we show that if the limit $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ does not exist, then we cannot conclude anything about the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)}$.

(1) Let us take $f(x) = x + \sin(x)$, $g(x) = x$. Then f, g are differentiable over $\mathbb{R}$,

$$\lim_{x\to+\infty} f(x) = +\infty = \lim_{x\to 0} g(x), \quad g(x) \neq 0 \neq g'(x),\ \forall x \in \mathbb{R}^*.$$

Moreover,

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} 1 + \frac{\sin(x)}{x} = 1.$$

On the other hand,

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{1+\cos(x)}{1}$$

which is not defined since the limit $\lim_{x\to+\infty} \cos(x)$ does not exist. Hence, since the limit of the quotient of the derivatives of f, g does not exist, a priori, we cannot conclude anything about the limit of the quotient of f, g. Nonetheless, in this case we are lucky and we can still carry out the computation.

(2) Consider $f(x) = \sqrt{x} + \sin(x)$, $g(x) = x$. Then f, g are differentiable over $\mathbb{R}_+^*$,

$$\lim_{x\to+\infty} f(x) = +\infty = \lim_{x\to 0} g(x), \quad g(x) \neq 0 \neq g'(x),\ \forall x \in \mathbb{R}^*.$$

Moreover,

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{1}{\sqrt{x}} + \frac{\sin(x)}{x} = 0.$$

On the other hand,

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{\frac{1}{2\sqrt{x}} + \cos(x)}{1}$$

which is not defined since the limit $\lim_{x\to+\infty} \cos(x)$ does not exist.

(3) Consider $f(x) = x + \sin(x)$, $g(x) = x$. Then f, g are differentiable over $\mathbb{R}_+^*$,

$$\lim_{x\to+\infty} f(x) = +\infty = \lim_{x\to 0} g(x), \quad g(x) \neq 0 \neq g'(x),\ \forall x \in \mathbb{R}^*.$$

Moreover,

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{1}{\sqrt{x}} + \frac{\sin(x)}{x} = 0.$$

On the other hand,

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{\frac{1}{2\sqrt{x}} + \cos(x)}{1}$$

which is not defined since the limit $\lim_{x\to+\infty} \cos(x)$ does not exist.

Hence, if the limit of the quotient of the derivatives of f, g does not exist, a priori, we cannot conclude anything about the limit of the quotient of f, g. Nonetheless, in some cases, such as () here, we are lucky and we can still carry out the computation.

Example 7.69. Let us consider the limit $\lim_{x\to 0} \frac{\arcsin(x)}{\sin(x)}$. Then, $\lim_{x\to 0} \arcsin(x) = 0 = \lim_{x\to 0} \sin(x)$ and $\sin(x)' = \cos(x)$, $\arcsin(x)' = \frac{1}{\sqrt{1-x^2}}$.
Moreover, both $\sin(x)$ and $\cos(x)$ are non-zero over the pointed neighborhood $]-\frac{\pi}{2}, \frac{\pi}{2}[\setminus\{0\}$ of 0. Hence, we can apply **??** to get

$$\lim_{x\to 0} \frac{\arcsin(x)}{\sin(x)} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos(x)} = 1$$

Example 7.70. Let us consider the limit $\lim_{x\to+\infty} \frac{e^x}{x^n}$. Then, $f(x) = e^x$, $g(x) = x^n$, $n \in \mathbb{N}$. Let us start with the case $n = 1$. Then,

$$\lim_{x\to+\infty} \frac{e^x}{x} = \lim_{x\to+\infty} \frac{e^x}{1} = +\infty.$$

For $n = 2$,

$$\lim_{x\to+\infty} \frac{e^x}{x^2} = \lim_{x\to+\infty} \frac{e^x}{2x} = \lim_{x\to+\infty} \frac{e^x}{2} = +\infty.$$

Hence, inductively, one can prove that

$$\lim_{x\to+\infty} \frac{e^x}{x^n} = \lim_{x\to+\infty} \frac{e^x}{nx^{n-2}} = \lim_{x\to+\infty} \frac{e^x}{n(n-1)x^{n-2}} = \ldots = \lim_{x\to+\infty} \frac{e^x}{n!} = +\infty.$$

Hence, the exponential function e^x goes to $+\infty$ – as x goes to $+\infty$ – faster than any monomial x^n; a similar argument shows that it goes faster than any polynomial.

Images/arcsin_gr.png

Figure 21: $f(x) = \arcsin(x)$

Example 7.71. Similarly to the previous example,

$$\lim_{x\to 0^+} x^n \log(x) = \lim_{x\to 0^+} \frac{\log(x)}{\frac{1}{x^n}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-n}{x^{n+1}}} = \lim_{x\to 0^+} -\frac{x^n}{n} = 0,$$

while,

$$\lim_{x\to +\infty} \frac{\log(x)}{x^n} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{nx^{n-1}} = \lim_{x\to +\infty} \frac{1}{nx^n} 0$$

So, log goes to $-\infty$ as x goes to 0 and to $+\infty$ as x goes to $+\infty$ slower than $\frac{1}{x}$ and x, respectively.

7.5.3 Taylor expansion

Definition 7.72. Let $f\colon E \to \mathbb{R}$ be a function and let $x_0 \in E$ Assume that there is a neighborhood of $a \in E$ which is contained in the domain (so in E). We say that f admits an expansion to the n-th order x_0 if there is an equality of the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n \epsilon(x), \tag{7.72.c}$$

where a_i are real number, and $\epsilon_n(x)\colon E \to \mathbb{R}$ satisfies $\lim\limits_{x\to x_0} \epsilon_n(x) = 0$.

Proposition 7.73. *In the hypotheses of* **??**, *if a function f admits an n-th order expansion around a point $x_0 \in D(f)$, then the coefficients a_i in* **??** *are uniquely determined.*

Proof. Let

$$\begin{aligned} f(x) &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + (x-x_0)^n \epsilon_n(x), \\ f(x) &= a_0' + a_1'(x-x_0) + a_2'(x-x_0)^2 + \cdots + a_n'(x-x_0)^n + (x-x_0)^n \epsilon_n'(x), \end{aligned}$$

be two different expansions to order n of f around x_0. We show by induction on i that $a_i = a_i'$. For $i = 0$ this is given by passing to the limit as $x \to x_0$ of the two expansion:

$$\begin{aligned} a_0 &= \lim_{x\to x_0} a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + (x-x_0)^n \epsilon_n(x) \\ &= \lim_{x\to x_0} f(x) \\ &= \lim_{x\to x_0} a_0' + a_1'(x-x_0) + a_2'(x-x_0)^2 + \cdots + a_n'(x-x_0)^n + (x-x_0)^n \epsilon_n'(x) = a_0' \end{aligned}$$

Let us prove the induction step. Thus, let us assume that we know that $a_j = a'_j$ for $j = 0, \ldots, i-1$. Then,

$$\begin{aligned} f(x) =& a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + (x-x_0)^n \epsilon_n(x) \\ =& a'_0 + a'_1(x-x_0) + a'_2(x-x_0)^2 + \cdots + a'_n(x-x_0)^n + (x-x_0)^n \epsilon'_n(x) \\ =& a_0 + a_1(x-x_0) + \cdots + a_{i-1}(x-x_0)^{i-1} + a'_i(x-x_0)^i + \cdots + a'_n(x-x_0)^n + (x-x_0)^n \epsilon'_n(x) \end{aligned}$$

Hence, taking the expansions on the 1st and the 3rd line of the previous chain of equalities, and subtracting from both expansions $a_0 + a_1(x-a) + \cdots + a_{i-1}(x-a)^{i-1}$ and then dividing both by $(x-a)^i$, we obtain

$$\begin{aligned} & a_i + a_{i+1}(x-x_0) + \cdots + a_n(x-x_0)^{n-i} + (x-x_0)^{n-i}\epsilon_n(x) \\ =& a'_i + a'_{i+1}(x-x_0) + \cdots + a'_n(x-x_0)^{n-i} + (x-x_0)^{n-i}\epsilon'_n(x). \end{aligned}$$

Taking limit of this equality as $x \to x_0$ yields that $a_i = a'_i$, which concludes the induction step. Hence, $a_i = a'_i$ for each i. In particular, it also follows that $\epsilon(x) = \epsilon'(x)$ for each $x \in E$. □

When is it that we can find an expansion to order n for a function f around a point $x_0 \in D(f)$? The following theorem provides a first answer.

Theorem 7.74. *Let $n \geq 0$ be an integer. Let $f \colon E \to \mathbb{R}$ be a function defined on an open interval E, and let $x_0 \in E$. Assume that f is $n+1$ times differentiable over E. Then, for each $x \in E$ there exists $x' \in]x_0, x[$, if $x > x_0$ (resp. $x' \in]x, x_0[$, if $x < x_0$) and such that*

$$f(x) = \left(\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \right) + f^{(n+1)}(x') \frac{(x-a)^{n+1}}{(n+1)!}.$$

Remark 7.75. **??** not only tells us that, under the hypotheses posed in its statement, it is possible to find an order n expansion for a function f around a point x_0 but also that, when that is the case, we have a recipe to compute the coefficients which are given by the formula

$$a_j = \frac{f^{(j)}(x_0)}{j!}.$$

Moreover, we can also compute the error term in the

Proof. To understand the proof, note that the statement for $n = 0$ is just the Mean Value Theorem, cf. **??**. Indeed, that results implies that there exists $x' \in]x_0, x[$, if $x > x_0$ (resp. $x' \in]x, x_0[$, if $x < x_0$) such that $f'(x') = \frac{f(x)-f(x_0)}{x-x_0}$. Multiply by $x - x_0$, then

$$f(x) = f(x_0) + f'(x_0 + \theta_{x,x_0}(x-x_0))(x-x_0),$$

where $\theta_{x,x_0}(x-x_0) \in [0,1]$ and $x' = x_0 + \theta_{x,x_0}(x-x_0)$ – which is possible exactly because x' is contained between x and x_0. Let us recall that the proof of **??** was an application of Rolle's theorem to the function $g(y) := f(y) - f(x_0) - \frac{f(x)-f(x_0)}{x-x_0}(y-x_0)$. Furthermore, this techique was working since $g(x_0) = g(x)$, and $g'(y) = f'(y) - \frac{f(x)-f(x_0)}{x-x_0}$, so that $g'(y)$ being 0 yielded exactly the above equation.
Let us now define

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i,$$

and let us consider

$$g(y) = f(y) - P_n(y) + \frac{P_n(x) - f(x)}{(x-x_0)^{n+1}}(y-x_0)^{n+1}.$$

Then,

$$0 = g(x) = g(a) = g'(a) = \cdots = g^{(n)}(a),$$

which means that there exists y_1 between x_0 and x such that $g'(y_1) = 0$ by Rolle's theorem. But then applying Rolle's theorem again we obtain a y_2 between x_0 and y_1 such that $g^{(2)}(y_2) = 0$. Iterating this process we obtain a point y_{n+1} between x_0 and x such that $g^{(n+1)}(y_{n+1}) = 0$. In particular, by setting $x' := y_{n+1}$, then

$$0 = g^{(n+1)}(x') = f(x') + \frac{P_n(x) - f(x)}{(x - x_0)^{n+1}}(n+1)!.$$

Reorganizing the latter equation yields exactly the statement of the theorem. □

Corollary 7.76. *Let $n \geq 0$ be a real number. Let $f\colon E \to \mathbb{R}$ be a function defined on an open interval E. Assume that $f \in C^n(E, \mathbb{R})$, and let $x_0 \in E$. Then, the n-th order expansion of f around x_0 exists and is given by the formula*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + (x - x_0)^n \epsilon_n(x).$$

The idea behind the proof of the corollary is that by the previous theorem the error term is $f^{(n+1)}(x') - f^{(n+1)}(x)$, which converges to zero as x goes to a as x' is between a and x, and $f^{(n)}$ is continuous. For the precise proof we refer to page 126 of the book.

Example 7.77. Applying **??** to $f(x) = \frac{1}{1-x}$ and $x_0 = 0$ yields that the order n expansion takes the form

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \epsilon_n(x),$$

since

$$f^{(i)}(x) = \frac{i!}{(1-x)^{i+1}} \qquad \Rightarrow \qquad f^{(i)}(0) = i! \qquad \Rightarrow \qquad \frac{f^{(i)}(0)}{i!} = 1.$$

Example 7.78. Applying **??** to $f(x) = e^x$ and $x_0 = 0$ yields that the order n expansion takes the form

$$e^x = \sum_{i=0}^{n} \frac{x^i}{i!} + x^n \epsilon_n(x),$$

since

$$f^{(i)}(x) = e^x \qquad \Rightarrow \qquad f^{(i)}(0) = 1 \qquad \Rightarrow \qquad \frac{f^{(i)}(0)}{i!} = \frac{1}{i!}.$$

Example 7.79. Similarly, the $(2n+1)$-st order expansion of $\cos(x)$ around $x = 0$ is

$$\cos(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j}}{(2j)!} + x^{2n+1}, \epsilon(x)$$

while the $(2n+2)$-nd order expansion of $\sin(x)$ around $x = 0$ is

$$\sin(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j+1}}{(2j+1)!} + x^{2n+2} \epsilon_{2n+2}(x).$$

Example 7.80. One can also figure out expansions of products, sums, compositions, etc.

For example the 3-rd order expansion of $\sin(\cos(x))$ is as follows:

$$\begin{aligned}\cos(\sin(x)) =& \cos\left(x - \frac{x^3}{6} + x^3\epsilon_3(x)\right)\\ =& 1 - \frac{\left(x - \frac{x^3}{6} + x^3\epsilon_3(x)\right)^2}{2} + \left(x - \frac{x^3}{6} + x^3\epsilon_3(x)\right)^3 \eta_3(\sin(x)) = 1 - \frac{x^2}{2} + x^3\tau_3(x),\end{aligned}$$

where $x^3\tau_3(x)$ is the sum of all terms of the form $x^3h(x)$, where $\lim_{x\to 0} h(x) = 0$. In particular, $\lim_{x\to 0} \tau_3(x) = 0$ and hence the above is indeed the 3-rd order expansion.

In general, we can compute the expansion to order n of a composition $(f \circ g)(x)$ around a point $f(x_0)$ by substituting the expension to order n of g around x_0 into the expansion of f around $f(x_0)$ to order n and then re-ordering all the terms thus obtained up to order n. Let us highlight how one should be careful that the base-point of the expansion of the function f should be the value $g(x_0)$ of the function g at the base-point x_0. So, for example, $\sin(\cos(x))$ at 0 is not easy to compute this way, because one would need the expansion of sin around $\cos(0) = 1$, for which there is no nice formula.

Another example is by taking $h(x) = \frac{1}{1-(e^x-1)}$, $x_0 = 0$. Then we can rewrite h as the composition $h = f \circ g$ of $f(y) = \frac{1}{1-y}$ and $g(x) = e^x - 1$. Then,

$$\begin{aligned}\frac{1}{1-(e^x-1)} =& \frac{1}{1 - \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_3(x)\right)}\\ =& 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_3(x)\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_3(x)\right)^2\\ &+ \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_3(x)\right)^3 \eta_3(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_3(x))\\ =& 1 + x + \left(\frac{1}{2} + 1\right)x^2 + \left(\frac{1}{6} + 2\cdot 1\cdot\frac{1}{2} + 1\right)x^3 + x^3\tau_3(x)\\ =& 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + x^3\tau_3(x).\end{aligned}$$

Similarly, one can write the order 3 expansion of $\frac{1}{1-x}\cdot e^x$ around 0 as

$$\begin{aligned}\frac{1}{1-x}\cdot e^x =& (1 + x + x^2 + x^3 + x^3\epsilon_3(x))\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\eta_3(x)\right)\\ =& 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^2 + x^3\tau_3(x)\end{aligned}$$

You can find more examples in the book, pages 127-131.

Example 7.81. One can use expansions also to avoid using **??**. For example, to compute

$$\lim_{x\to 0} \frac{(e^x - 1 - x) + x\sin(x)}{\cos(x) - 1},$$

then we can try to compute the 2-nd order expansions first:

$$\begin{aligned}(e^x - 1 - x) + x\sin(x) =& \left(1 + x + \frac{x^2}{2} + x^2\epsilon_2(x) - 1 - x\right) + x(x + x^2\eta_2(x))\\ =& \frac{x^2}{2} + x^2 + x^2(\eta_2(x) + \epsilon_2(x)) = \frac{3}{2}x^2 + x^2 \qquad \underbrace{\gamma_2(x)}_{\gamma_2(x) := \eta_2(x) + \epsilon_2(x)} \qquad),\\ \cos(x) - 1 =& 1 - \frac{x^2}{2} + x^2\tau_2(x) - 1 = -\frac{x^2}{2} + x^2\tau_4(x)\end{aligned}$$

Then,

$$\lim_{x\to 0}\frac{(e^x-1-x)+x\sin(x)}{\cos(x)-1}=\lim_{x\to 0}\frac{\frac{3}{2}x^2+x^2\epsilon_3(x)}{-\frac{x^2}{2}+x^2\epsilon_4(x)}=\lim_{x\to 0}\frac{\frac{3}{2}+\epsilon_3(x)}{-\frac{1}{2}+\epsilon_4(x)}=-3.$$

7.5.4 Application of Taylor expansion to local extrema and inflection points

We have proven that if f has a point of local extremum at $x_0 \in D(f)$, then $f'(x_0)=0$, cf. **??**. However, we have also shown that the converse implication does not hold, cf. **??**. Nevertheless, we would like to know whether, for example, by imposing suitable conditions on the higher derivatives of a function f, we can still characterize when a stationary point is a point of local extremum for a function.

Let $f\colon E\to\mathbb{R}$ be a function, and let $x_0\in E$ be a stationary point for f. Moreover, let us assume that for some even natural number n, the first $n-1$ derivatives of f vanish at x_0

$$f'(x_0)=0=f''(x_0)=\cdots=f^{(n-1)}(x_0),$$

while the n-th derivative of f is non-zero and $f^{(n)}(x_0)>0$. Then, writing the n-th order expansion of f around x_0,

$$f(x)=f(x_0)+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+(x-x_0)^n\epsilon_n(x).$$

Thus, for x sufficiently close to x_0 it holds that $|\varepsilon_n(x)|<\frac{1}{2}\cdot\frac{f^{(n)}(x_0)}{n!}$ holds. In particular, for such values of x, then

$$f(x_0)<f(x_0)+\frac{1}{2}\cdot\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\le f(x)$$

This shows that x_0 is a point of local minimum for f. One can imitate this argument for the case where $f^{(n)}(x_0)<0$ to yield that x_0 is a point of local maximum for f. Thus, we can summarize the results obtained so far in the following theorem.

Theorem 7.82. *Let $n\ge 2$ be an even integer. Let $f\colon E\to\mathbb{R}$ be a function on an open interval E and let $x_0\in E$. Assume that f is differentiable n times on E and that $f^{(i)}(x_0)=0$, $\forall i=1,2,3,\dots,n-1$.*

(1) If $f^{(n)}>0$, then f has a point of local minimum at x_0.

(2) If $f^{(n)}<0$, then f has a point of local maximum at x_0.

Example 7.83. Consider the function $f(x)=\sin(x)+\frac{1}{2}x$ over the interval $[0,2\pi]$, cf. **??**.
Then, $f'(x)=0$ if and only if $\cos(x)=-\frac{1}{2}$, which is equivalent to $x=\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Whether or not we have a maximum or minimum at these points is decided by the sign of $f''(x)=-\sin(x)$.

- At $x=\frac{2\pi}{3}$, $f(x)''<0$, so $f(x)$ has a local maximum, and
- At $x=\frac{4\pi}{3}$, $f(x)''>0$, so $f(x)$ has a local minimum.

Question 7.84. What happens if we assume that n is an odd natural number in the statement of **??**?

In that case, the expansion to order n for f around x_0 takes the same expression as before

$$f(x)=f(x_0)+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+(x-x_0)^n\epsilon_n(x),$$

Images/sinxplushalfx_gr.png

Figure 22: $f(x) = \sin(x) + \frac{1}{2}x$ over the interval $[0, 2\pi]$

Images/hor_flex.png

Figure 23: $f(x) = x^3$ has a stationary point at $x = 0$ which is a flex, as the graph goes through the tangent line $y = 0$.

but this time the first non-leading term will be of the form $(x - x_0)^3$, $(x - x_0)^5$, or $(x - x_0)^7$, etc., depending on the precise value of n. But then for $x > x_0$, $(x - x_0)^n > 0$, while $x < x_0$, $(x - x_0)^n < 0$. This type of behavior characterizes what is called an inflection. That is to say, that for a stationary point x_0 to be an inflection point for f, we require that, on one side of x_0 the graph of the function is above the tangent line to the graph of f through $(x_0, f(x_0))$, and on the other side it is below it, cf. **??**

We can actually generalize this tentative definition, as follows, to comprise not just the case of stationary points.

Definition 7.85. Let $f\colon E \to \mathbb{R}$ be a function. Assume that f is differentiable at $x_0 \in E$. We say that f has an *inflection point* at x_0 if there exists $\delta > 0$ such that either one of the following two conditions is satisfied:

(1) $\{x \in E | a < x < a + \delta\} \Rightarrow f(x) - f(a) - f'(a)(x - a) > 0$, and
$\{x \in E | a - \delta < x < a\} \Rightarrow f(x) - f(a) - f'(a)(x - a) < 0$; or,

(2) $\{x \in E | a < x < a + \delta\} \Rightarrow f(x) - f(a) - f'(a)(x - a) < 0$, and
$\{x \in E | a - \delta < x < a\} \Rightarrow f(x) - f(a) - f'(a)(x - a) > 0$.

The reasoning contained in the paragraph before **??** immediately yields the following result.

Theorem 7.86. *Let $n \geq 3$ be an odd integer. Let $f : E \to \mathbb{R}$ be a function defined over an open interval E and let $x_0 \in E$. Assume that f is differentiable n times on E and that*

$$f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0,$$

while $f^{(n)}(x_0) \neq 0$. Then, f has an inflection point at x_0.

Example 7.87. Let us consider the function $f(x) = 2\sin(x) - x$. Then $f'(x) = 2\cos(x) - 1$, $f''(x) = -2\sin(x)$ and $f'''(x) = -2\cos(x)$. Hence, $f'(0) = f''(0) = 0$, and $f'''(0) \neq 0$. Hence $f(x)$ has an inflection point at $x = 0$ according to **??**.

Figure 24: The function $f(x) = 2\sin(x) - x$ has a flex at the point $x = 0$, which is non-stationary, as $f'(0) = 1$, through the tangent line $y = x$ to the graph of f at the point $(0, 0)$.

7.5.5 Convex and concave functions

Definition 7.88. Let $f : E \to \mathbb{R}$ be a function defined over an open interval E. We say that f is convex (resp. concave) if for every $a, b \in E$ and every $\lambda \in [0, 1]$ we have:

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

(resp. $f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b)$).

Remark 7.89. Let us maintain the same notation as in the above definition. We may assume that $a < b$. Then, for $\lambda \in [0, 1]$, $x := \lambda a + (1 - \lambda)b$ is a point between a and b. Geometrically, the above definition means the following:

(1) f is convex, if for any choice of $a, b \in E$, then between a and b, the graph of f lies completely below the line segment connecting $(a, f(a))$ and $(b, f(b))$;

(2) if f is concave, then between a and b, the graph of f lies completely above the line segment connecting $(a, f(a))$ and $(b, f(b))$.

We can characterize convexity and concavity of a function which is differentiable by means of the monotonicity of its first derivative.

`Images/convex.png`

Figure 25: The graph of the function $f(x) = (x+1)^2 - 3$ lies below the line segment connecting the points $(-4, f(-4)) = -4, 6$ and $(1, f(1)) = (1, 1)$.

`Images/concave.png`

Figure 26: The graph of the function $f(x) = -(x-1)^2 + 5$ lies below the line segment connecting the points $(-2, f(-2)) = (-2, 4)$ and $(1, f(1)) = (1, 1)$.

Theorem 7.90. *Let $f\colon E \to \mathbb{R}$ be a function defined on an open interval E. Assume that f is differentiable. Then f is convex (resp. concave) if and only if $f'\colon E \to \mathbb{R}$ is an increasing (resp. decreasing) function.*

Proof. We prove only the statements about convexity, as f is convex if and only if $-f$ is concave.

(1) First, let us assume that f is convex. Let $a < b$ be points of I. We want to prove that $f'(a) \le f'(b)$. By the above characterization of convexity we have

$$\frac{f(b) - f(\lambda a + (1-\lambda)b)}{b - (\lambda a + (1-\lambda)b)} \ge \frac{f(b) - f(a)}{b - a}, \text{ and } \frac{f(\lambda a + (1-\lambda)b) - f(a)}{(\lambda a + (1-\lambda)b) - a} \le \frac{f(b) - f(a)}{b - a}.$$

Now, as λ goes to 0, the left side of the first inequality converges to $f'(b)$, and as λ goes to 1 the left side of the second inequality converges to $f'(a)$. This yields:

$$f'(b) \geq \frac{f(b) - f(a)}{b - a} \geq f'(a)$$

(2) For, the other direction let us assume that f' is increasing. Fix $a < b \in E$. Set $x := \lambda a + (1 - \lambda)b$ for any $\lambda \in]0, 1[$ (for $\lambda = 0$ and 1 the convexity inequality is automatic). Then, the mean value theorem tells us that there are $a < x_1 < x < x_2 < b$ such that $\frac{f(x)-f(a)}{x-a} = f'(x_1)$ and $\frac{f(b)-f(x)}{b-x} = f'(x_2)$. In particular, by our assumption that the derivative is increasing it follows that $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x}$. But this shows that f is convex by the above characterization of convexity in terms of slopes.

□

If f' is differentiable – or, equivalently, f is twice differentiable – then f' being increasing (resp. decreasing) is equivalent to $f'' \geq 0$ (resp. $f'' \leq 0$). Thus, we can characterize convexity and concavity of a function which is twice differentiable by means of the sign of its second derivative.

Corollary 7.91. *Let $f : E \to \mathbb{R}$ be a two times differentiable function on an open interval. Then f is convex (resp. concave) if and only if $f''(x) \geq 0$ (resp. $f''(x) \leq 0$) for all $x \in E$.*

Example 7.92. (1) Let us consider $f(x) = e^x$. Then, $f''(x) = e^x$, thus $f : \mathbb{R} \to \mathbb{R}$ is convex, since the second derivative $e^x > 0$, $\forall x \in \mathbb{R}$.

(2) Let us consider $f(x) = \log(x)$. Then, $f''(x) = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$, so the function $\log(x)$, which is defined over $\mathbb{R}_+^*$ is concave over its entire domain.

Example 7.93. Here we explain why the graph of a differentiable convex function f must lie above the tangent line to the graph through a point $(a, f(a))$. That is to say, we show that for a differentiable convex function f,

$$f(x) \geq f(a) + f'(a)(x - a).$$

We take $x > a$, and leave the case $x < a$ to the reader. Thus, assuming that $x > a$, we wish to show that $f(x) \geq f(a) + f'(a)(x - a)$, or equivalently that

$$\frac{f(x) - f(a)}{x - a} \geq f'(a). \tag{7.93.d}$$

Indeed, by the Mean value theorem (**??**) there exists a $x' \in \mathbb{R}$, $a < x' < x$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(x'). \tag{7.93.e}$$

As f is convex, f' is increasing by **??**, hence, $f'(x') \geq f'(a)$. Adding this observation the equality in **??**, we have shown that **??** must hold.

7.6 Asymptotes

Definition 7.94. (1) If for some $c \in \mathbb{R}$, $\lim\limits_{x \to c^-} f(x) = \pm\infty$ or $\lim\limits_{x \to c^+} f(x) = \pm\infty$, then we say that the function f has a vertical asymptote at $x = c$.

(2) If for some $c \in \mathbb{R}$, $\lim\limits_{x \to +\infty} f(x) = c$ (resp. $\lim\limits_{x \to -\infty} f(x) = c$), then we say that the function f has a horizontal asymptote at $+\infty$ (resp. $-\infty$) at $y = c$.

(3) If for some $a \neq 0, b \in \mathbb{R}$, $\lim_{x \to +\infty} f(x) - ax = b$ (resp. $\lim_{x \to -\infty} f(x) - ax = b$), then we say that f has an oblique (or slant) asymptote at $+\infty$ (resp. $-\infty$) along the line $y = ax + b$.

Example 7.95. Here we give a few examples of the different notions introduce in the above definition.

(1) Vertical asymptote: $f(x) = \frac{1}{1-x}$ has a vertical asymptote at at $x = 1$, cf. **??**;

(2) Horizontal asymptote: the function $f(x) = 2 - e^{-x}$ has a horizontal asymptote at $+\infty$ of value $y = 2$, cf. **??**;

(3) Slant asymptote: the function $f(x) = 2 + 3x + \frac{1}{x^2}$ has a slant asymptote both at $+\infty$ and $-\infty$ along the line $y = 3x + 2$, cf. **??**.

`Images/1suxmeno1.png`

Figure 27: The function $f(x) = \frac{1}{x-1}$, and the line $x = 1$.

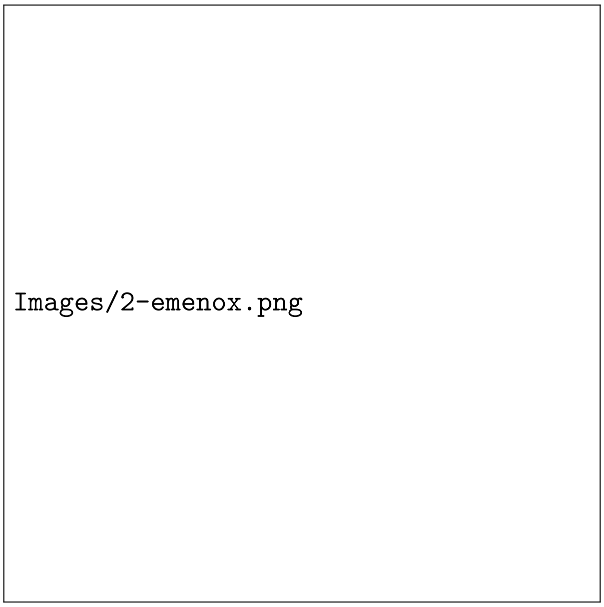

Figure 28: The function $f(x) = 2 - e^{-x}$, and the line $y = 2$.

8 Integration

8.1 Definition

The idea behind integration is that the integral $\int_a^b f(x)dx$ of a bounded function f on a closed interval $[a, b]$ should be the area under the graph of f. However, it is not that easy to say what this area means and when it is computable at all. If it is computable, we say that the function is integrable (**??**), and the value of this area is then called the integral $\int_a^b f(x)dx$ of f.

Now, the idea of trying to define the area under the graph of f is simple. We start with the only area that we can compute trustably, that is of rectangles, and then we try to approximate the area under the graph of f from above and from below using rectangles. These approximations are called upper and lower Darboux sums (**??**). We say that the area under the graph of f is computable, which as above means that the function is integrable, if these two approximations meet in the limit. This is spelled out in precise mathematical terms below.

Definition 8.1. (1) A *partition* $\sigma = (x_i)$ of a bounded interval $[a, b]$ is an ordered collection $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of points of $[a, b]$.

(2) The *norm* or *mesh* of σ is

$$\max\{x_i - x_{i-1} | 1 \leq i \leq n\}.$$

(3) A *refinement* $\sigma' = (x'_i)$ of σ is a partition such that each value of x_i shows up amongst x'_i. we indicate that σ' is a refinement of σ by writing $\sigma \succeq \sigma$.

(4) The *regular partition* of length n is $x_i := a + i\frac{b-a}{n}$, $i = 0, 1, 2, \ldots, n$.

Proposition 8.2. *Given a bounded interval $[a, b]$, any two partitions σ, σ' have a common refinement σ''. Moreover, each partition can be refined to a new one with arbitrarily small norm.*

Images/2piu3xpiu1sux2.png

Figure 29: The function $f(x) = 2 + 3x + \frac{1}{x^2}$, and the line $y = 3x + 2$.

Definition 8.3. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and $\sigma = (x_i)$ a partition of $[a,b]$. Then, the *upper Darboux sum* of f with respect to σ is

$$\overline{S}_\sigma = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$. The *lower Darboux sum* of f with respect to σ is

$$\underline{S}_\sigma = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

where $m_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$.

Example 8.4. Let us consider a constant function $f(x) = c$, $c \in \mathbb{R}$ over a closed bounded interval $[a,b]$. Then for any partition σ of $[a,b]$,

$$\underline{S}_\sigma = \overline{S}_\sigma = \sum_{i=1}^{n} c(x_i - x_{i-1}) = \underbrace{cx_n - cx_0}_{\text{telescopic sum}} = c(b-a).$$

Example 8.5. Let us consider the function $f(x) = x$ over a closed bounded interval $[a,b]$, and let $\sigma_n = \left(a + \frac{i(b-a)}{n}\right)$ be the regular partition of length n. Then,

$$\overline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right) \frac{b-a}{n} = a(b-a) + \frac{n(n+1)}{2}\frac{(b-a)^2}{n^2}$$

and

$$\underline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(a + (i-1)\frac{b-a}{n}\right) \frac{b-a}{n} = a(b-a) + \frac{(n-1)n}{2}\frac{(b-a)^2}{n^2},$$

where in both cases we used the following identity

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}.$$

Note that $\lim_{n\to\infty} \overline{S}_{\sigma_n} = \lim_{n\to\infty} \underline{S}_{\sigma_n} = a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2}{2} - \frac{a^2}{2}$.

Proposition 8.6. *Let $f : [a,b] \to \mathbb{R}$ be a function (not necessarily a continuous one). Assume that f admits an upper bound M (resp. a lower bound m) for its range $R(f)$. Then, for any partition σ of $[a,b]$, $m(b-a) \le \overline{S}_\sigma, \underline{S}_\sigma \le M(b-a)$. In particular, the sets*

$$\{\overline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\} \quad \text{and} \quad \{\underline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}$$

are bounded.

Proof. This follows immediately from the definitions, since for any interval $[x_i, x_{i+1}] \subset [a,b]$ then

$$m \le \inf_{[x_i,x_{i+1}]} f \le \sup_{[x_i,x_{i+1}]} f \le M.$$

Hence, for a partition $\sigma = \{x_i\}$ of $[a,b]$,

$$\begin{aligned} m(b-a) =& m\sum_{i=0}^{n-1} m(x_{i+1} - x_i) \le \sum_{i=0}^{n-1} \left(\inf_{[x_i,x_{i+1}]} f\right)(x_{i+1} - x_i) = \overline{S}_\sigma \\ \le& \sum_{i=0}^{n-1} \left(\sup_{[x_i,x_{i+1}]} f\right)(x_{i+1} - x_i) = \underline{S}_\sigma \le \sum_{i=0}^{n-1} M(x_{i+1} - x_i) = M(b-a) \end{aligned}$$

□

Definition 8.7. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

(1) The upper Darboux integral of f on $[a,b]$ is defined as

$$\overline{S} := \inf\{\overline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}.$$

(2) The lower Darboux integral of f on $[a,b]$ is defined as

$$\underline{S} := \sup\{\underline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}$$

Example 8.8. Using the above computation for the constant function, cf. **??**, we see that if f is the constant function on $[a, b]$, then $\overline{S} = \underline{S} = (b - a)c$.

Proposition 8.9. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(1) If σ is a partition of $[a, b]$ and σ' is a refinement of σ, then:

$$\underline{S}_\sigma \leq \underline{S}_{\sigma'}, \text{ and } \overline{S}_\sigma \geq \overline{S}_{\sigma'}.$$

(2) If σ is a partition of $[a, b]$, then:

$$\underline{S}_\sigma \leq \overline{S}_\sigma.$$

Corollary 8.10. *If $f : [a, b] \to \mathbb{R}$ is a bounded function, then $\underline{S} \leq \overline{S}$.*

Proof. It is enough to prove that $\underline{S}_{\sigma_1} \leq \overline{S}_{\sigma_2}$ for any partitions σ_1 and σ_2 of $[a, b]$. However, this follows straight from **??**. Indeed, if σ is a common refinement of σ_1 and σ_2, then **??** yields that

$$\underline{S}_{\sigma_1} \leq \underbrace{\underline{S}_\sigma}_{??.??} \leq \underbrace{\overline{S}_\sigma}_{??.??} \leq \underbrace{\overline{S}_{\sigma_2}}_{??.??}.$$

□

Definition 8.11. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We say that f is *integrable*, if $\overline{S} = \underline{S}$, in which case this common value is called the *integral of f between a and b*, and it is denoted by

$$\int_a^b f(x)dx.$$

Remark 8.12. Using **??**, f is integrable over a closed bounded interval $[a, b]$ if one exhibits a sequence (σ_n) of partitions such that $\lim_{n\to\infty} \overline{S}_{\sigma_n} = \lim_{n\to\infty} \underline{S}_{\sigma_n}$. Indeed, this follows immediately by the following chain of inequalities

$$\lim_{n\to\infty} \underline{S}_{\sigma_n} \leq \underline{S} \leq \overline{S} \leq \lim_{n\to\infty} \overline{S}_{\sigma_n}, \tag{8.12.a}$$

passing to the limit for $n \to \infty$.

Example 8.13. Using **??**, the constant functions are integrable on $[a, b]$, and

$$\int_a^b c\ dx = (b - a)c$$

Example 8.14. Using **??** and the computation of **??** for $f(x) := x$ over a closed bounded interval $[a, b]$ then f is integrable, and

$$\int_a^b xdx = \frac{b^2}{2} - \frac{a^2}{2}$$

Example 8.15. Consider the function $[0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 3 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, for all partition σ, $\overline{S}_\sigma = 6$, and $\underline{S}_\sigma = 0$. So, $\overline{S} = 6$, $\underline{S} = 0$, and hence f is not integrable.

Proposition 8.16. *If $f : [a, b] \to \mathbb{R}$ is continuous then it is integrable.*

Proof. As f is continuous over a closed bounded interval $[a, b]$, **??** implies the uniform continuity of f. Let us fix $\varepsilon > 0$. Let $\delta > 0$ be the constant in the definition of uniform continuity associated to $\frac{\varepsilon}{b-a}$ – that is,

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{b-a}.$$

Claim Let σ be a partition of $[a, b]$ with norm at most δ. Then $\overline{S}_\sigma - \underline{S}_\sigma \leq \varepsilon$.

Proof. In fact,

$$\begin{aligned}
\overline{S}_\sigma - \underline{S}_\sigma &= \sum_{i=1}^{n} (\max_{[x_i, x_{i+1}]} f - \min_{[x_i, x_{i+1}]} f)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) = \frac{\varepsilon}{b-a} \underbrace{(b-a)}_{\sum_{i=1}^{n}(x_i - x_{i-1}) = b-a} = \varepsilon
\end{aligned}$$

□

□

8.2 Basic properties

Proposition 8.17. *Let $f, g \colon [a, b] \to \mathbb{R}$ be integrable functions. Then,*

(1) *If f extends over $[b, c]$ for some $c \in \mathbb{R}$, $c > b$ and f is also integrable over $[b, c]$, then it is integrable over $[a, c]$, and*

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

(2) *Given $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable on $[a, b]$, and*

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(3) *If $f \leq g$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(4) *The function $|f|$ is integrable on $[a, b]$, and*

$$\int_a^b |f(x)|dx \geq \left| \int_a^b f(x)dx \right|$$

Proof. The proofs of all these statements follow the same pattern: one writes up the inequalities for lower (resp. upper) Darboux sums for a fixed partition σ. Then these inequalities remain valid when taking sup (resp. inf) of the lower (resp. upper) Darboux sums along all possible partitions of an interval $[a, b]$. This gives inequalities in both direction, which then implies equalities.
For example, let us show how this strategy works in the case of point **??** – we leave the other

cases to the reader. Let σ, and τ be partitions for $[a, b]$ and $[b, c]$, respectively. Then the union of σ and τ gives a partition ρ for $[a, c]$ and by definition we have

$$\overline{S}_\rho = \overline{S}_\sigma + \overline{S}_\tau, \tag{8.17.a}$$

$$\underline{S}_\rho = \underline{S}_\sigma + \underline{S}_\tau. \tag{8.17.b}$$

As both these equalities are true for all choice of partitions σ of $[a, b]$ and τ of $[b, c]$, by taking the inf (resp. the sup) of **??** (resp. of **??**) along all possible choices of partitions of $[a, b]$, $[b, c]$, $[a, c]$, then

$$\overline{S}^{[a,c]} \leq \overline{S}^{[a,b]} + \overline{S}^{[b,c]}, \text{ and } \underline{S}^{[a,c]} \geq \underline{S}^{[a,b]} + \underline{S}^{[b,c]}, \tag{8.17.c}$$

where $\overline{S}^{[a,c]}$ (resp. $\underline{S}^{[a,c]}$) denotes the upper (resp. lower) Darboux integral of partitions of $[a, c]$, and similarly for the other cases. As f is integrable both on $[a, b]$ and on $[b, c]$, we have $\int_a^b f(x)dx = \overline{S}^{[a,b]} = \underline{S}^{[a,b]}$ and $\int_b^c f(x)dx = \overline{S}^{[b,c]} = \underline{S}^{[b,c]}$. Thus, **??** yields

$$\int_a^b f(x)dx + \int_b^c f(x)dx \leq \underline{S}^{[a,c]} \leq \overline{S}^{[a,c]} \leq \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{8.17.d}$$

As the two ends of **??** are the same, we have everywhere equalities. This concludes both the integrability of f over $[a, c]$ as well as the statement of **??**. $\square$

Example 8.18.

$$\int_a^b (1+x)dx = \underbrace{\int_a^b 1dx + \int_a^b xdx}_{\text{point ?? of ??}} = \underbrace{(b-a)}_{??} + \underbrace{\frac{b^2-a^2}{2}}_{??}.$$

8.3 Fundamental theorem of calculus

In this section, we learn how to compute integrals using the anti-derivative, cf. **??**. We start by giving the definition of an anti-derivative.

Definition 8.19. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. A function $G : [a, b] \to \mathbb{R}$ is called an *anti-derivative* of f if

(1) G is continuous on $[a, b]$,

(2) G is differentiable on $]a, b[$, and

(3) $G'(x) = f(x)$ for all $x \in]a, b[$.

Remark 8.20. Given a continuous function $f\colon [a, b] \to \mathbb{R}$ admitting an anti-derivative $G\colon [a, b] \to \mathbb{R}$, then for any $C \in \mathbb{R}$, also $G_C\colon [a, b] \to \mathbb{R}$, $G_C(x) := G(x) + C$ is an anti-derivative for f. According to **??**, the vice versa is also true: namely, if G, H are anti-derivatives of f, then there exists $C \in \mathbb{R}$ such that $G(x) = H(x) + C$, $\forall x \in \mathbb{R}$.

Notation 8.21. The anti-derivatives of f are at times denoted by $\int f(x)dx + c$, where $c \in \mathbb{R}$ is a constant that is free to vary in $\mathbb{R}$. Also, sometimes the expressions $\int f(x)dx + c$ is also called the indefinite integral, while what we defined as the integral of f over $[a, b]$, $\int_a^b f(x)dx$ is then called the definite integral – where the definitiveness comes from the fact that we computed the integral over the closed bounded integral $[a, b]$. We use the integral/anti-derivative naming in this course.

Example 8.22. We collect here a few important functions together with their anti-dervatives.

function	e^x	$\cos(x)$	$\sin(x)$	$\frac{1}{x}$	$x^n,\ n \in \mathbb{N}$	...
anti-derivative	e^x	$\sin(x)$	$-\cos(x)$	$\log\lvert x\rvert$	$\frac{x^{n+1}}{n+1}$	...

Now that we have defined the notion of anti-derivative of a continuous function, there are two questions that arise spontaneously:

- Does an anti-derivative for a continuous function $f\colon [a,b] \to \mathbb{R}$ over a closed bounded interval $[a,b]$ always exist?
- If an anti-derivative exists for a continous function $f\colon [a,b] \to \mathbb{R}$, does it help us in any way in computing the value of the integral $\int_a^b f(x)dx$?

These two questions have some simple but very powerful answers provided by the following two theorems, that are usually called the first and second fundamental theorems of calculus.

Theorem 8.23. $\boxed{\text{FUNDAMENTAL THEOREM OF CALCULUS I}}$
Let $f : [a,b] \to \mathbb{R}$ be continuous. Then,

$$F(x) := \int_a^x f(t)dt$$

is an anti-derivative of f.

Theorem 8.24. $\boxed{\text{FUNDAMENTAL THEOREM OF CALCULUS II}}$
Let $f : [a,b] \to \mathbb{R}$ be continuous and let G be an anti-derivative of f. Then,

$$\int_a^b f(x)dx = G(b) - G(a).$$

Notation 8.25. In order for the statement of **??** to make full sense, we need to introduce some further notation: so far we defined $\int_a^b f(x)dx$ only for $a < b$. If $a = b$, then we define

$$\int_a^a f(x)dx := 0.$$

If $a > b$, then we also define

$$\int_a^b f(x) := -\int_b^a f(x)dx.$$

With these notations our previously proven rules give that if $f : [a,b] \to \mathbb{R}$ is continuous, and $c, d \in [a,b]$ are any points, then

$$\int_a^c f(x)dx + \int_c^d f(x)dx = \int_a^d f(x)dx.$$

The next statement is not too interesting in itself but it is needed in the proof of **??**.

Theorem 8.26. $\boxed{\text{MEAN VALUE THEOREM FOR INTEGRALS}}$
If $f : [a,b] \to \mathbb{R}$ is continuous, then there is a $c \in [a,b]$, such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof. As $[a,b]$ is closed and f is continuous, by **??**, f admits global maximum and minimum over $[a,b]$. Set $M := \max\limits_{x \in [a,b]} f(x)$ and $m := \min\limits_{x \in [a,b]} f(x)$. By **??**, f takes all values in $[m,M]$. However, by **??**, then

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M,$$

so there is a $c \in [a,b]$ such that $f(c)$ equals the above fraction, which is exactly the statement of the theorem. $\square$

Let us now give the proofs of the two fundamental theorems of calculus.

Proof of **??**. Fix $x_0 \in]a,b[$. Then, for any $x_0 \neq x \in]a,b[$:

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^{x} f(t)dt = \underbrace{f(c(x))}_{??},$$

for a real number $c(x)$ between x and x_0. Hence:

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0} f(c(x)) = \underbrace{\lim_{x\to x_0} f(x)}_{\lim\limits_{x\to x_0} c(x)=x_0} = \underbrace{f(x_0)}_{f \text{ is continuous}}.$$

□

Proof of **??**. We have already shown in **??** that $F(x) = \int_a^x f(t)dt$ is an anti-derivative of f. As both F and G are anti-derivatives, they differ by a constant $C \in \mathbb{R}$, that is, $(F-G)(x) = C$, $\forall x \in [a,b]$. Then:

$$\begin{aligned} G(b) - G(a) =&(G(b)+c) - (G(a)+c) = F(b) - F(a) \\ =& \int_a^b f(x)dx - \int_a^a f(x)dx = \int_a^b f(x)dx. \end{aligned}$$

□

Notation 8.27. The expression $G(b)-G(a)$ appearing in the statement of **??** is usually denoted by

$$G(x)|_a^b \qquad \text{or} \qquad G(x)|_{x=a}^{x=b}.$$

Example 8.28.

$$\int_{-5}^{-1}\frac{1}{x} = \big(\log|x|\big)\big|_{x=-5}^{x=-1} = \log 1 - \log 5 = -\log 5$$

8.4 Substitution

Theorem 8.29. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $\phi : [\alpha,\beta] \to [a,b]$ be a C^1 function. Then,*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt. \tag{8.29.a}$$

Proof. Define $G(x) := \int_a^x f(u)du$. By **??**, G is an anti-derivative of f, so that **??** tells us

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = G(\phi(\beta)) - G(\phi(\alpha)).$$

So, it is enough to show that the value of the right side of **??** is the same. To show that, let us just note that by the chain rule $G(\phi(t))' = G'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$. Then applying **??** to the integral $\int_\alpha^\beta f(\phi(t))\phi'(t)dt$ implies that

$$\int_\alpha^\beta f(\phi(t))\phi'(t)dt = G(\phi(\beta)) - G(\phi(\alpha)),$$

which concludes the proof. □

Example 8.30. In this example, we go from the right side of **??** to the left side.

$$\int_0^1 \sqrt{e^x} e^x dx = \underbrace{\int_1^e \sqrt{u} du}_{u=e^x \quad (e^x)'=e^x} = \left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{u=1}^{u=e} = \frac{2}{3}\left(e^{\frac{3}{2}} - 1\right)$$

Example 8.31. Let us consider the function $f\colon [0,1] \to \mathbb{R}$, $f(x) := \sqrt{1-x^2}$.
We want to compute the integral $\int_0^1 f(x)dx$. This integral computes the area of a quarter of

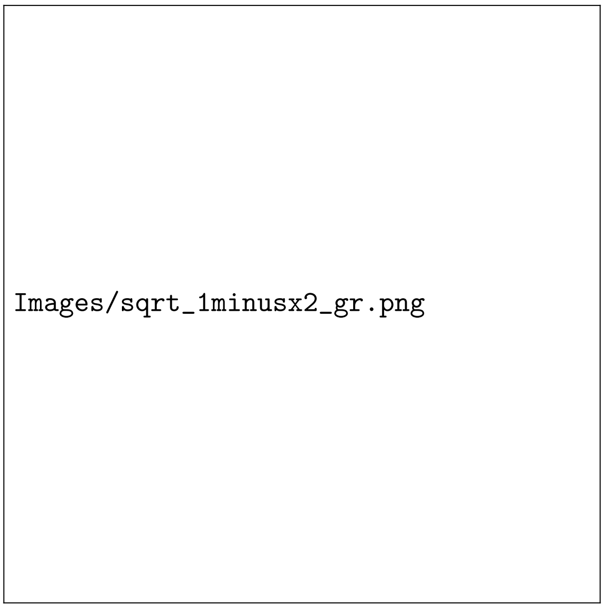

Figure 30: $f(x) := \sqrt{1-x^2}$.

a circle of radius 1, as shown by **??** so the result should be $\frac{\pi}{4}$. Indeed, the above computation shows that are train of thought is correct. Note that, opposite to the previous example, in this argument at our first substitution we go from the left side of **??** to the right side.

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2} dx &= \underbrace{\int_0^{\frac{\pi}{2}} \sqrt{1-(\sin(t))^2} \cos(t) dt}_{x=\sin(t) \quad \sin(t)'=\cos(t)} = \int_0^{\frac{\pi}{2}} \sqrt{\cos(t)^2} \cos(t) dt \\
&= \int_0^{\frac{\pi}{2}} |\cos(t)| \cos(t) dt = \underbrace{\int_0^{\frac{\pi}{2}} \cos(t)\cos(t) dt}_{t\in\left[0,\frac{\pi}{2}\right] \Rightarrow \cos(t)\geq 0 \Rightarrow |\cos(t)|=\cos(t)} \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos(2t)+1}{2} dt = \underbrace{\int_0^{\pi} \left(\frac{\cos(u)+1}{2}\right) \frac{1}{2} du}_{t=\frac{u}{2}} \\
&= \frac{1}{4} \int_0^{\pi} (\cos(u)+1) du = \frac{1}{4} (\sin(u)+u) |_{u=0}^{u=\pi} \\
&= \frac{1}{4} (\sin(\pi) + \pi - \sin(0) - 0) = \frac{\pi}{4}
\end{aligned}$$

Example 8.32. Recall that $\sinh(x) : \mathbb{R} \to \mathbb{R}$ is an odd function and it is strictly increasing (indeed, $\sinh(x)' = \cosh(x) > 0$). In particular, it has an inverse, which we denote by

$\sinh^{-1}(x)\colon \mathbb{R} \to \mathbb{R}$. With this we may compute similarly :

$$\int_0^1 \sqrt{1+x^2}dx = \underbrace{\int_0^{\sinh^{-1}(1)} \sqrt{1+(\sinh(t))^2}\cosh(t)dt}_{x=\sinh(t) \qquad \sinh(t)'=\cosh(t)} = \int_0^{\sinh^{-1}(1)} \sqrt{\cosh(t)^2}\cosh(t)dt$$

$$= \int_0^{\sinh^{-1}(1)} |\cosh(t)^2|\cosh(t)dt = \underbrace{\int_0^{\sinh^{-1}(1)} \cosh(t)\cosh(t)dt}_{\cosh(t)>0 \;\Rightarrow\; |\cosh(t)|=\cosh(t)} = \int_0^{\sinh^{-1}(1)} \frac{\cosh(2t)+1}{2}dt$$

$$= \underbrace{\int_0^{2\sinh^{-1}(1)} \frac{\cosh(u)+1}{2}\frac{1}{2}du}_{t=\frac{u}{2}} = \frac{1}{4}\int_0^{2\sinh^{-1}(1)} \cosh(u)+1du = \frac{1}{4}\left(\sinh(u)+u\right)\Big|_{u=0}^{u=2\sinh^{-1}(1)}$$

$$= \frac{\sinh(2\sinh^{-1}(1))+2\sinh^{-1}(1)}{4} = \frac{2\sinh(\sinh^{-1}(1))\cosh(\sinh^{-1}(1))+2\sinh^{-1}(1)}{4}$$

$$= \frac{2\sinh(\sinh^{-1}(1))\sqrt{1+\sinh(\sinh^{-1}(1))^2}+2\sinh^{-1}(1)}{4} = \frac{2\cdot 1\cdot\sqrt{1+1^2}+2\sinh^{-1}(1)}{4}$$

$$= \frac{2\sqrt{2}+2\sinh^{-1}(1)}{4}$$

Example 8.33. Substitution can be used the generally integrate $\cos(x)^n$ and $\sin(x)^n$ when n is a positive integer.

(1) The simplest case is when n odd. Here is an example of that:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(x)^5dx = \int_0^{\frac{\pi}{2}} \cos(x)(1-\sin(x)^2)^2dx = & \underbrace{\int_0^1 (1-u^2)^2du}_{u(x)=\sin(x) \quad u(x)'=\cos(x)} \\ =& \int_0^1 (1-2u^2+u^4)du = \left(u - \frac{2u^3}{3}+\frac{u^5}{5}\right)\Big|_{u=0}^{u=1} \\ =& 1-\frac{2}{3}+\frac{1}{5} = \frac{6}{15} = \frac{2}{5}.\end{aligned}$$

(2) On the other hand, when n is even, by reverse-engineering duplication formulas for sine and cosine, we can reduce again to the case of an odd power:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^4(x)dx = & \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{\cos(2x)}{2} + \frac{\cos(2x)^2}{4}dx \\ =& \int_0^{\frac{\pi}{2}} \frac{1}{4}dx - \int_0^{\frac{\pi}{2}} \frac{\cos(2x)}{2}dx + \int_0^{\frac{\pi}{2}} \frac{\cos(2x)^2}{4}dx \qquad (8.33.b) \\ =& \left(\frac{\pi}{8} - \frac{\sin(2x)}{4}\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos(4x)+1}{8}dx \\ =& \left(\frac{\pi}{8}+\frac{\pi}{16}+\frac{\sin(4x)}{32}\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}} = \frac{\pi}{8}+\frac{\pi}{16} = \frac{3\pi}{16}\end{aligned}$$

8.5 Integration by parts

Theorem 8.34. *If $f, g\colon : E \to \mathbb{R}$ be two C^1 functions on an open interval E, and let $a < b$ be elements of E. Then,*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx, \tag{8.34.a}$$

Proof. It is enough to show that

$$\int_a^b (f(x)g'(x) + f'(x)g(x))dx = f(x)g(x)|_a^b .$$

This follows immediately from the Leibniz formula, **??**,

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x).$$

□

Generally integration by parts are useful for products. The main question in applying **??** is how one chooses f and g. There is a rule which works in most cases (but not always!). Using the list below, when you encounter a product of two functions both of which belong to one of the categories in the list, the idea is that you should assign g' in the formula **??** to the function that belongs to the category appearing earlier in the list.

- E(xponential)
- T(rigonometric)
- A(lgebraic, that is, polynomial)
- L(ogarithm)
- I(nverse trigonometric).

We present now a few examples.

Example 8.35.

$$\begin{aligned}\underbrace{\int_a^b xe^x dx}_{g'(x)=e^x \quad g(x)=e^x \quad f(x)=x \quad f'(x)=1} &= (xe^x)|_{x=a}^b - \int_a^b e^x dx \\ &= (xe^x - e^x)|_{x=a}^b = ((x-1)e^x)|_{x=a}^b\end{aligned}$$

Example 8.36.

$$\begin{aligned}\underbrace{\int_a^b \sin(x)e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=\sin(x), f'(x)=\cos(x)} &= (\sin(x)e^x)|_{x=a}^b - \underbrace{\int_a^b \cos(x)e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=\cos(x), f'(x)=-\sin(x)} \\ &= (\sin(x)e^x - \cos(x)e^x)|_{x=a}^b + \int_a^b (-\sin(x))e^x dx\end{aligned}$$

Thus, we can rewrite the equality between the LHS of the first line and the second line as

$$\int_a^b \sin(x)e^x dx = \frac{(e^x(\sin(x) - \cos(x)))|_{x=a}^b}{2}.$$

Example 8.37.

$$\underbrace{\int_a^b \log(x)dx}_{f(x)=\log(x),f'(x)=\frac{1}{x},g'(x)=1,g(x)=x} = (x\log(x))|_{x=a}^b - \int_a^b 1dx = (x\log(x)-x)|_{x=a}^b$$

Example 8.38.

$$\begin{aligned}\underbrace{\int_a^b \arctan(x)dx}_{f(x)=\arctan(x),f'(x)=\frac{1}{1+x^2},g'(x)=1,g(x)=x} &= (x\arctan(x))|_{x=a}^b - \int_a^b \frac{x}{1+x^2}dx \\ &= (x\arctan(x))|_{x=a}^b - \underbrace{\frac{1}{2}\int \frac{1}{u}du}_{u(x)=1+x^2,u(x)'=2x} \\ &= \left(x\arctan(x) - \frac{1}{2}\log|u|\right)\Big|_{x=a}^b \\ &= \left(x\arctan(x) - \frac{1}{2}\log|1+x^2|\right)\Big|_{x=a}^b\end{aligned}$$

8.6 Integrating rational functions

A *rational function*, is a function of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with real coefficients.

We start by recalling how polynomials with real coefficients can be factorized.

Theorem 8.39 (Fundamental theorem of algebra over $\mathbb{R}$)**.** *Let $Q(x)$ be a polynomial with real coefficients. Then, $Q(x)$ can be factored as*

$$Q(x) = (x-a_1)^{k_1}\dots(x-a_n)^{k_n}(x^2+2b_1x+c_1)^{l_1}\dots(x^2+2b_mx+c_m)^{l_m}, \tag{8.39.a}$$

where the a_i, b_i and c_i are real numbers, $k_i, l_i > 0$ are positive integers, and the quadratic polynomials $x^2+2b_ix+c_i$, $1 \le i \le m$ are irreducible, that is, there is no real number x_0 such that $x_0^2+2b_ix_0+c_i = 0$, $1 \le i \le m$.

Remark 8.40. The Fundamental Theorem of Algebra is originally for polynomials $R(x)$ with complex coefficients. For those polynomial, the statement is even better: namely, polynomials with complex coefficients can be factored into linear terms. That is,

$$R(x) = (x-d_1)^{s_1}\dots\left(x-d_n'\right)^{s_n}, \quad d_i \in \mathbb{C},\ s_i \in \mathbb{N}^*. \tag{8.40.b}$$

This does work also for $R(x) := Q(x)$ a real polynomial,as $\mathbb{R} \subset \mathbb{C}$, however it may happen that some of the d_i that are complex and not real. Then, the expression cannot be used for integration because we did not learn integration of complex valued functions. Hence, the idea is to collect the d_i that are real numbers. These numbers provide the a_i in **??**. Furthermore, as we are working with a polynomial $R(x)$ with real coefficients,then **??** is invariant under conjugation, since $R(x)$ is. Hence, whenever d_i is not real, then also the conjugate of $(x-d_i)$ has to show up in **??** with the same power. That is, we have a factor of the right hand side of **??** of the form:

$$\begin{aligned}(x-d_i)^{s_i}\left(x-\overline{d_i}\right)^{s_i} &= \left((x-d_i)\left(x-\overline{d_i}\right)\right)^{s_i} \\ &= \left(x-2\left(d_i+\overline{d_i}\right)+d_i\overline{d_i}\right)^{s_i} = \left(x-2\operatorname{Re}(d_i)+|d_i|^2\right)^{s_i}\end{aligned}$$

Then, setting $b_j := -\operatorname{Re}(d_i)$, $l_j := s_i$ and $c_j := |b_i|^2$, we obtain one of the terms of the form $(x^2+2b_jx+c_j)^{l_j}$ in **??**.

Example 8.41. Take $Q(x) = x^3 + x^2 - 2$, and consider the factorization as in **??**. As the degree of Q is three, there must be a linear term (the product of the non-linear terms has even degree). This correspond to a real root of $Q(x)$, so let us search for it.

(1) *Finding the real root.*

1st try: $Q(0) = -2$. As $\lim\limits_{x\to+\infty} Q(x) = +\infty$, the Intermediate value theorem, **??**, implies that Q has a root greater than 0.

2nd try: $Q(1) = 0$. We found the root, great.

(2) *Factoring out the linear term.*

$$\begin{array}{ccccccc|c|c}
x^3 & + & x^2 & & & - & 2 & x-1 & x^2+2x+2 \\ \cline{8-9}
x^3 & - & x^2 & & & & & & \\ \cline{1-7}
 & & 2x^2 & & & - & 2 & & \\
 & & 2x^2 & - & 2x & & & & \\ \cline{1-7}
 & & & & 2x & - & 2 & & \\
 & & & & 2x & - & 2 & & \\ \cline{1-7}
 & & & & & & 0 & &
\end{array}$$

Hence, we have

$$(x^2 + 2x + 2)(x - 1) = x^3 + x^2 - 2$$

Remark 8.42. Unfortunately, for polynomials of degree ≥ 5 there is no algorithm for finding the roots; one just has to try to use the Intermediate Value theorem, hoping that the given polynomial yields a nice root.

Using **??**, we have the following nice factorization of rational functions.

Proposition 8.43. *Any rational function* $\frac{P(x)}{Q(x)}$ *can be written as*

$$\frac{P(x)}{Q(x)} = \alpha_1 R_1(x) + \cdots + \alpha_t R_t(x),$$

where the α_i *are real numbers, and* $R_i(x)$ *are of the form:*

(1) polynomial, or

(2) $\frac{1}{(x-r)^p}$*, or*

(3) $\frac{x+c}{(x^2+2rx+s)^p}$.

Instead of giving a proof, we explain the idea behind **??** in the following example.

Example 8.44. Given the rational function $\frac{4x^3+9x^2+11x+8}{(x^2+x+1)^2}$, let us try to factorize it as follows:

$$\begin{aligned}
\frac{4x^3 + 9x^2 + 11x + 8}{(x^2 + x + 1)^2} &= \frac{Ax + B}{(x^2 + x + 1)^2} + \frac{Cx + D}{x^2 + x + 1} \\
&= \frac{Ax + B + (Cx + D)(x^2 + x + 1)}{(x^2 + x + 1)^2} \\
&= \frac{Cx^3 + (C + D)x^2 + (A + C + D)x + (B + D)}{(x^2 + x + 1)^2},
\end{aligned}$$

which yields the following linear system

$$\begin{cases} C &= 4 \\ C + D &= 9 \\ A + C + D &= 11 \\ B + D &= 8 \end{cases}$$

for which the solutions are

$$\begin{aligned} C = 4 \Rightarrow 4 + D = 9 \Rightarrow D = 5 \\ \Rightarrow A + 4 + 5 = 11; B + 5 = 8 \\ \Rightarrow A = 2; B = 3 \end{aligned}$$

Thus,

$$\frac{4x^3 + 9x^2 + 11x + 8}{(x^2 + x + 1)^2} = \frac{2x + 3}{(x^2 + x + 1)^2} + \frac{4x + 5}{x^2 + x + 1}.$$

Having the decomposition stated in **??**, the question is how we integrate these terms separately.

Example 8.45. ◦ For $p > 1$, then

$$\int \frac{1}{(x - r)^p} dx = \frac{(x - r)^{1-p}}{1 - p}.$$

◦ For $p = 1$, then

$$\int \frac{1}{(x - r)} dx = \log |x - r|.$$

Example 8.46.

$$\begin{aligned} \int \frac{x + c}{(x^2 + 2rx + s)^p} &= \frac{1}{2} \int \frac{2(x + r)}{(x^2 + 2rx + s)^p} dx + \int \frac{c - r}{(x^2 + 2rx + s)^p} dx \\ &= \frac{1}{2} \int \frac{2(x + r)}{(x^2 + 2rx + s)^p} dx + (c - r) \int \frac{1}{(x^2 + 2rx + s)^p} dx. \end{aligned}$$

Hence, we need to compute the two integrals

$$\int \frac{2(x + r)}{(x^2 + 2rx + s)^p} dx, \quad \int \frac{1}{(x^2 + 2rx + s)^p} dx,$$

individually.

◦ Using the substitution $u = x^2 + 2rx + s$, then

$$\int \frac{2(x + r)}{(x^2 + 2rx + s)^p} dx = \begin{cases} \log |x^2 + 2rx + s| & \text{if } p = 1 \\ \frac{(x^2+2rx+s)^{1-p}}{1-p} & \text{if } p > 1 \end{cases}$$

◦ Hence, we now know how to integrate all the terms in **??**, except for the integral

$$\int \frac{1}{(x^2 + 2rx + s)^p} dx, \quad p > 0.$$

Hence,

$$\begin{aligned}
\int \frac{1}{(x^2+2rx+s)^p}dx &= \int \frac{1}{((x+r)^2+(s-r^2))^p}dx \\
&= \underbrace{\frac{1}{(s-r^2)^p}\int \frac{1}{\left(\left(\frac{x+r}{\sqrt{s-r^2}}\right)^2+1\right)^p}dx}_{s-r^2>0,\text{ as } x^2+2rx+s \text{ has no real roots}} \\
&= \underbrace{\frac{1}{(s-r^2)^{p-\frac{1}{2}}}\int \frac{1}{(u^2+1)^p}du,}_{u=\frac{x+r}{\sqrt{s-r^2}}}
\end{aligned}$$

and we are left to compute the integral

$$\int \frac{1}{(u^2+1)^p}du, \quad \text{for } p>0.$$

Setting

$$I_p := \int \frac{1}{(u^2+1)^p}du,$$

then $I_1 := \arctan(u)$ and furthermore, if $p \geq 1$, then we obtain a recursive formula as follows:

$$I_p = \int \frac{1}{(u^2+1)^p}du = \underbrace{\frac{u}{(u^2+1)^p} - \int \frac{(-p)u\cdot 2u}{(u^2+1)^{p+1}}du}_{\text{integrating by parts with } f(u)=\frac{1}{(u^2+1)^p},\, g'(u)=1}$$

$$= \frac{u}{(u^2+1)^p} + 2p\int \frac{u^2+1-1}{(u^2+1)^{p+1}}du = \frac{u}{(u^2+1)^p} + 2pI_p - 2pI_{p+1}$$

So, by looking at the two ends of the equation, we obtain the recursive equality:

$$I_{p+1} = \frac{\frac{u}{(u^2+1)^p} + (2p-1)I_p}{2p}.$$

Remark 8.47. Be careful, there is an error on page 201 of the book where they prove the formulas above: intead of $2p-1$, they wrote $2(p-1)$!!

Example 8.48. Let us compute I_2 for example:

$$I_2 = \frac{\frac{u}{u^2+1} + I_1}{2} = \frac{1}{2}\left(\frac{u}{u^2+1} + \arctan(u)\right)$$

Example 8.49. Let us get back to **??**:

$$\int \frac{4x^3+9x^2+11x+8}{(x^2+x+1)^2}dx = \underbrace{\int \frac{2x+3}{(x^2+x+1)^2}dx + \int \frac{4x+5}{x^2+x+1}dx}_{\text{by ??}}$$

$$= \underbrace{\int \frac{2x+3}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dx + \int \frac{4x+5}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx}_{\text{completing the square in the denominators}}$$

$$= \underbrace{\int \frac{(2x+1)+2}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dx + \int \frac{(4x+2)+3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx}_{\text{writing the numerators in terms of a multiple of } x+\frac{1}{2}}$$

$$= \underbrace{\int \left(\frac{2}{\sqrt{3}}\right)^3 \frac{\frac{2}{\sqrt{3}}(2x+1)+\frac{4}{\sqrt{3}}}{\left(\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right)^2}dx + \int \frac{2}{\sqrt{3}} \frac{\frac{2}{\sqrt{3}}(4x+2)+\frac{6}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}dx}_{\substack{\text{We multiply the numerators and the denominators by adequate multiples of } \frac{2}{\sqrt{3}}, \\ \text{to make them of the form } u^2+1 \text{ or } (u^2+1)^2}}$$

$$= \underbrace{\frac{4}{3}\int \frac{2u+\frac{4}{\sqrt{3}}}{(u^2+1)^2}du + \int \frac{4u+\frac{6}{\sqrt{3}}}{u^2+1}du}_{u=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \quad \Rightarrow \quad x=\frac{\sqrt{3}}{2}u-\frac{1}{2} \quad \Rightarrow \quad x(u)'=\frac{\sqrt{3}}{2}}$$

$$=\frac{4}{3}\int \frac{2u}{(u^2+1)^2}du + \frac{16}{3\sqrt{3}}\int \frac{1}{(u^2+1)^2}du$$

$$+2\int \frac{2u}{u^2+1}dx + \frac{6}{\sqrt{3}}\int \frac{1}{u^2+1}du$$

$$=\frac{4}{3}\frac{-1}{u^2+1} + \frac{8}{3\sqrt{3}}\left(\frac{u}{u^2+1}+\arctan(u)\right) + 2\log|u^2+1| + \frac{6}{\sqrt{3}}\arctan(u)$$

$$=\frac{4}{3}\frac{-1}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \frac{8}{3\sqrt{3}}\left(\frac{\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right)$$

$$+2\log\left|\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right| + \frac{6}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

8.6.1 Rational functions in exponentials

There is a method of integrating functions obtained by plugging in e^x into a rational function. We explain it via the next example:

Example 8.50.

$$\begin{aligned}\int \frac{1}{e^x+1}dx = \int \frac{1}{(e^x+1)e^x}e^x dx = & \underbrace{\int \frac{1}{(t+1)t}dt}_{t(x)=e^x \quad t(x)'=e^x} \\ = & \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \log|t| - \log|t+1| \\ = & \log|e^x| - \log|e^x+1| = x - \log|e^x+1|\end{aligned}$$

8.6.2 Rational functions in roots

There is a method of integrating functions obtained by plugging in $\sqrt{x}$ into a rational function. We explain it via the next example.

Example 8.51.

$$\begin{aligned}\int \frac{1}{\sqrt{x}+1}dx = & \int \left(\frac{1}{\sqrt{x}+1}2\sqrt{x}\right)\frac{1}{2}\frac{1}{\sqrt{x}}dx \\ = & \underbrace{\int \frac{2t}{t+1}dt}_{t(x)=\sqrt{x} \quad t(x)'=\frac{1}{2}\frac{1}{\sqrt{x}}} = \int 2dt - \int \frac{2}{t+1}dt \\ = & 2t - 2\log|t+1| = 2\sqrt{x} - 2\log|\sqrt{x}+1|.\end{aligned}$$

8.7 Improper integrals

So far we have studied integrals of functions (mostly continuous ones) that are defined over a closed bounded interval.

But how can we make sense of integrating a function over an unbounded interval, for example, $\int_1^{+\infty} \frac{1}{x^2}dx$? Or more generally, we have a real valued function f that is continuous on an interval I of the form $[a,b[$, $]a,b]$ or $]a,b[$, where $a, b \in \overline{\mathbb{R}}$, but either f does not extend continuously to $[a,b]$, or the interval $[a,b]$ does not exist at all – such as in the case when a or b are $\pm\infty$.

Definition 8.52. Let $f \colon I \to \mathbb{R}$ be a continuous function.

(1) If $I = [a,b[$, $a \in \mathbb{R}$, and either $b \in \mathbb{R}$ or $b = +\infty$, then we define the *improper integral* of f on I to be the limit

$$\int_a^{b-} f(t)dt := \lim_{x\to b-}\left(\int_a^x f(t)dt\right),$$

provided that the above limits exist.

(2) If $I =]a,b]$, $b \in \mathbb{R}$, and either $a \in \mathbb{R}$ or $a = -\infty\infty$, then we define the *improper integral* of f on I to be the limit

$$\int_{a^+}^b f(t)dt := \lim_{x\to a+}\left(\int_x^b f(t)dt\right),$$

provided that the above limits exist.

(3) If $I =]a, b[$, $a, b \in \overline{\mathbb{R}}$, then we define the *improper integral* of f on I to be the limit

$$\int_{a^+}^{b^-} f(t)dt := \int_{a^+}^{c} f(t)dt + \int_{c}^{b-} f(t)dt$$

for any chosen $c \in I$, provided that the improper integrals

$$\int_{a^+}^{c} f(t)dt, \quad \int_{c}^{b-} f(t)dt$$

exist.

If the limits above exist and are finite, then we say that the improper integrals converge. If the above limits diverge, we say that the corresponding improper integral is divergent.

Remark 8.53. (1) It is an easy exercise to verify that part (3) of the above the definition does not depend on the choice of $c \in I$.

(2) By abuse of notation many times the $+$ and the $-$ is forgotten from the lower and upper limits.

Example 8.54.

$$\int_{0^+}^{1} \frac{1}{\sqrt{t}} dt = \lim_{x \to 0^+} 2t^{\frac{1}{2}}\Big|_{t=x}^{t=1} = \lim_{x \to 0^+} 2 - 2\sqrt{x} = 2$$

Example 8.55.

$$\int_{0^+}^{1} \frac{1}{t} dt = \lim_{x \to 0^+} \log(t)|_{t=x}^{t=1} = \lim_{x \to 0^+} -\log(x) = +\infty$$

So, $\int_{0^+}^{1} \frac{1}{t} dt$ is divergent.

Example 8.56.

$$\int_{0^+}^{1} \log(t)dt = \lim_{x \to 0^+} (\log(t)t - t)|_{t=x}^{t=1} = -1 - \lim_{x \to 0^+} (\log(x)x - x) = -1 - \lim_{x \to 0^+} (\log(x)x)$$

Here, we may compute $\lim_{x \to 0^+}(\log(x)x)$ using L'Hospital's rule:

$$\lim_{x \to 0^+} (\log(x)x) = \lim_{x \to 0^+} \frac{\log(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} -x = 0.$$

Hence, $\int_{0^+}^{1} \log(t)dt = -1$.

Improper integrals enjoy many of the basic features of standard integrals.

Proposition 8.57. *Let $f, g \colon I \to \mathbb{R}$ be continuous functions defined over an interval I, where I is either one of the following intervals*

$$[a, b'[,]a', b],]a', b'[, \qquad a, b \in \mathbb{R}, \quad a', b' \in \overline{\mathbb{R}}.$$

Then,

(1) If $I = [a, b[$ (resp. $I =]a, b]$, $I =]a, b[$), $a, b \in \mathbb{R}$ and f extends to a continuous function defined over the interval $[a, b]$, then the improper interval

$$\int_{a}^{b-} f(x)dx \text{ (resp. } \int_{a+}^{b} f(x)dx, \ \int_{a+}^{b-} f(x)dx).$$

converges and it is equal to $\int_a^b f(x)dx$.

(2) If $c \in I$ and $c \neq \sup I, \inf I$ then if the improper interval of f on I converges, we have that

$$\int_a^b f(x)dx + \int_a^c f(x)dx = \int_c^b f(x)dx.$$

(3) Given $\alpha, \beta \in \mathbb{R}$, if the improper integral of f, g over I converge, then also $\alpha f + \beta g$ is integrable on I, and

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(4) If $0 \leq f \leq g$, then

(i) if the improper integral $\int_a^b g(x)dx$ converges then also the improper integral $\int_a^b f(x)dx$ does;

(ii) if the improper integral $\int_a^b f(x)dx$ diverges then also the improper integral $\int_a^b g(x)dx$ does.

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Remark 8.58. In part (2-4) of the previous proposition, we have used the simplified notation for improper integrals that was introduced in **??**(2).

Definition 8.59. In the hypotheses of **??**, we say that an improper integral is absolutely convergent if the improper integral defined by integrating the function $|f|$ instead of f is convergent.

The following is an an analogue, for improper integrals, of **??**.

Proposition 8.60. *If an improper integral is absolutely convergent, then it is also convergent.*

Example 8.61. The backwards implication of **??** does not hold, as shown by the next example.

$$\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt = \underbrace{\lim_{x \to +\infty} \frac{-\cos(x)}{x} - \frac{-\cos\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} - \int_{\frac{\pi}{4}}^{+\infty} \frac{-\cos(t)}{-t^2} dt}_{g'=\sin(t) \quad f=\frac{1}{t} \quad g=-\cos(t) \quad f'=\frac{1}{-t^2}} = \frac{\sqrt{2}}{\pi} - \int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(t)}{t^2} dt. \tag{8.61.a}$$

So, $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt$ is convergent if so is $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(t)}{t^2} dt$. However, the latter is convergent because it is absolute convergent:

$$\int_{\frac{\pi}{4}}^{+\infty} \left|\frac{\cos(t)}{t^2}\right| dt \leq \int_{\frac{\pi}{4}}^{+\infty} \frac{1}{t^2} dt = \lim_{x \to +\infty} \left(\frac{-1}{t} \Big|_{\frac{\pi}{4}}^{x} \right) = \frac{4}{\pi} + \lim_{x \to +\infty} \frac{1}{x} = \frac{4}{\pi}.$$

This yields that $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt$ is convergent.

However, be careful, the fact that $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(t)}{t^2}dt$ is absolute convergent, does not mean that so is $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t}dt$. That is, equation **??** does not work for $\frac{\sin(t)}{t}$ replaced by $\left|\frac{\sin(t)}{t}\right|$. And, in fact, $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t}dt$ is not absolute convergent, because

$$\int_{\frac{\pi}{4}}^{n\pi} \left|\frac{\sin(t)}{t}\right| dt \geq \sum_{k=1}^{n} \int_{k\pi-\frac{3\pi}{4}}^{k\pi-\frac{\pi}{4}} \left|\frac{\sin(t)}{t}\right| dt \geq \sum_{k=1}^{n} \frac{\pi}{2} \left(\min_{t\in\left[k\pi-\frac{3\pi}{4},k\pi-\frac{\pi}{4}\right]} \frac{|\sin(t)|}{t} \right)$$

$$\geq \sum_{k=1}^{n} \frac{\pi}{2} \left(\frac{\min_{t\in\left[k\pi-\frac{3\pi}{4},k\pi-\frac{\pi}{4}\right]} |\sin(t)|}{\max_{t\in\left[k\pi-\frac{3\pi}{4},k\pi-\frac{\pi}{4}\right]} t} \right) = \sum_{k=1}^{n} \frac{\pi}{2} \frac{\frac{1}{\sqrt{2}}}{k-\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} \sum_{k=1}^{n} \frac{1}{k}.$$

As $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, $\displaystyle\lim_{n\to\infty} \int_{\frac{\pi}{4}}^{n\pi} \left|\frac{\sin(t)}{t}\right| dt$ does not exist. Therefore, $\displaystyle\int_{\frac{\pi}{4}}^{\infty} \left|\frac{\sin(t)}{t}\right| dt$ is divergent.

Example 8.62. A typical application of improper integral is to give an upper bound on infinite sums. For example

$$\sum_{k=10}^{\infty} \frac{1}{k^2} \leq \int_{9}^{+\infty} \frac{1}{x^2}dx = \left.\frac{-1}{x}\right|_{x=9}^{x\to+\infty} = \left(\lim_{n\to\infty x\to+\infty} \frac{-1}{x} \right) - \frac{-1}{9} = \frac{1}{9}$$