EPFL

Lecturer: Z. Patakfalvi
Analysis I - (n/a)
13th January 2020
3 hours

n/a

n/a

SCIPER: **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 16 pages, the last ones possibly blank. TOTAL: 31 questions. Do not unstaple.

- Place your student card on your table.
- **The only papers you are allowed to use are the booklet of the exam and the scratch paper provided by the proctors.**
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  - $+3$ points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - -1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  - $+1$ points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - -1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Set $I = \displaystyle\int_0^2 e^{(x^2)}\,\mathrm{d}x$. Then,

- ☐ $2 \leq I \leq 200$.
- ☐ $I = e^{\frac{8}{3}} - 1$.
- ☐ $0 \leq I < \frac{14}{3}$.
- ☐ $I \geq 200$.

Question 2 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x \sin\left(\mathrm{e}^{\frac{1}{x}} - 1\right) & \text{if} \quad x \neq 0\,, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Then,

- ☐ f is continuous over $\mathbb{R}$, and it is differentiable from the right, but not from the left at $x = 0$.
- ☐ f is C^1 over $\mathbb{R}$, that is, it is differentiable and its derivative is continuous over $\mathbb{R}$.
- ☐ f is differentiable over $\mathbb{R}$, but f' is not continuous over the entire $\mathbb{R}$.
- ☐ f is continuous over $\mathbb{R}$, and it is differentiable from the left but not from the right at $x = 0$.

Question 3 : Define $a_n = \left(\sqrt{n+2} - \sqrt{n+1}\right) \sin\left(\frac{1}{n}\right)$, for every $n \in \mathbb{N}^*$. Then,

- ☐ the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, but it is not absolutely convergent.
- ☐ the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.
- ☐ both the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ are convergent.
- ☐ the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ is divergent.

Question 4 : Let R be the radius of convergence of the power series $f(x) = \displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)^{(n^b)} x^n$.

- ☐ If $b = 2$, then $R = 1$.
- ☐ If $b = 1$, then $R = e^{-1}$.
- ☐ If $b = 3$, then $R = e$.
- ☐ If $b = 4$, then $R = e^2$.

Question 5 : Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = |x \cos(x)|$.

- ☐ There exists $u \in \left]0, \frac{\pi}{4}\right[$ such that $f'(u) = \frac{\sqrt{2}}{2}$.
- ☐ f is increasing on $\left]0, \frac{\pi}{2}\right[$.
- ☐ There is a single local minimum of f on the entire $\mathbb{R}$.
- ☐ There exists $u \in \left]-\frac{\pi}{8}, \frac{\pi}{8}\right[$ such that $f'(u) = 0$.

Question 6 : Among the functions $f, g, h : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x}\sin(\frac{1}{x}) & \text{if } x > 0 \\ -\sqrt{-x} & \text{if } x \leq 0 \end{cases}, \qquad g(x) = \begin{cases} x \operatorname{sh}(\frac{1}{x}) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

$$h(x) = \begin{cases} \sqrt{x}\operatorname{Arctg}(\frac{1}{x}) & \text{si } x > 0 \\ x \operatorname{Log}(|x|) & \text{si } x < 0 \\ 0 & \text{si } x = 0 \end{cases},$$

find those that are continuous at $x = 0$:

- ☐ g and h.
- ☐ f and g.
- ☐ all three.
- ☐ f and h.

Question 7 : Consider the integral $I = \displaystyle\int_1^2 x \operatorname{Log}(1+x)\,\mathrm{d}x$. Then,

- ☐ $I = 2\operatorname{Log}(3) - \frac{1}{2}\operatorname{Log}(2)$.
- ☐ $I = \frac{1}{2}\operatorname{Log}(2) + \frac{1}{4}$.
- ☐ $I = 2\operatorname{Log}(3) + \frac{1}{2}\operatorname{Log}(2)$.
- ☐ $I = \frac{3}{2}\operatorname{Log}(3) - \frac{1}{4}$.

Question 8 : Let $f\colon [1, +\infty[\to \mathbb{R}$ be the function defined by $f(x) = \sin\left(\operatorname{Arctg}(\sqrt{x}\,)\right)$. Then the range of f is

- ☐ $]0, 1]$.
- ☐ $[-1, 1]$.
- ☐ $\left[\frac{\sqrt{2}}{2}, 1\right[$.
- ☐ $\left[0, \frac{\sqrt{2}}{2}\right[$.

Question 9 : For which numbers $a, b \in \mathbb{R}$ is the function $f\colon \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} (ax+1)(bx-1) & \text{if } x \geq 0, \\ \sin(a^2 x) - b & \text{if } x < 0, \end{cases}$$

differentiable at $x = 0$?

- ☐ $a = \dfrac{1 \pm \sqrt{5}}{2}$ and $b = -1$
- ☐ $a = \dfrac{-1 \pm \sqrt{5}}{2}$ and $b = 1$
- ☐ $a = \pm 1$ and $b = -1$
- ☐ $a = \pm 1$ and $b = 1$

Question 10 : The imaginary part of $\left(-1 + i\sqrt{3}\,\right)^5$ is

- ☐ $16\sqrt{3}$.
- ☐ $-16\sqrt{3}$.
- ☐ $32\sqrt{3}$.
- ☐ $32\sqrt{3}\,i$.

Question 11 : The limit $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{5n + \sqrt{3n - \sqrt{2n}}}}$

- ☐ exists, and it is $\frac{1}{\sqrt{5+\sqrt{3-\sqrt{2}}}}$.
- ☐ exists, and it is $\frac{1}{\sqrt{5}}$.
- ☐ exists, and it is $\frac{1}{\sqrt{6}}$.
- ☐ does not exist.

Question 12 : Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \dfrac{1}{1+\mathrm{e}^{-x}}$. The Taylor expansion of order 3 of f around $x_0 = 0$ is

- ☐ $f(x) = \frac{1}{2} - \frac{x}{4} - \frac{x^3}{48} + x^3\varepsilon(x)$.
- ☐ $f(x) = \frac{1}{2} + \frac{x}{4} + \frac{x^3}{48} + x^3\varepsilon(x)$.
- ☐ $f(x) = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + x^3\varepsilon(x)$.
- ☐ $f(x) = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{24} + x^3\varepsilon(x)$.

Question 13 : Let $(a_n)_{n\geq 1}$ be the sequence defined as follows: for all $n \geq 1$,

$$a_n = \sin\left(\frac{\pi}{4} + n\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{4} + n\frac{\pi}{2}\right).$$

Then,

- ☐ $\limsup_{n\to\infty} a_n = 2$ and $\liminf_{n\to\infty} a_n = -2$.
- ☐ $\limsup_{n\to\infty} a_n = \sqrt{2}$ and $\liminf_{n\to\infty} a_n = -\sqrt{2}$.
- ☐ $\limsup_{n\to\infty} a_n = 0$ and $\liminf_{n\to\infty} a_n = -\sqrt{2}$.
- ☐ $\limsup_{n\to\infty} a_n = \sqrt{2}$ and $\liminf_{n\to\infty} a_n = 0$.

Question 14 : The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^{\frac{2}{\alpha}}\left(n^{2\alpha}+1\right)}}$ converges if

- ☐ $\frac{1}{2} < \alpha < 1$.
- ☐ $\alpha = \frac{1}{2}$.
- ☐ $0 < \alpha < \frac{1}{2}$.
- ☐ $1 < \alpha < 2$.

Question 15 : Set $x_0 \in \mathbb{R}$, and let $x_{n+1} = x_n - \dfrac{1}{3^n}$ for every $n \in \mathbb{N}$. Then

- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ converges to x_0.
- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ converges to 0.
- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ converges to $x_0 - \frac{3}{2}$.
- ☐ for all $x_0 \in \mathbb{R}$, the sequence $(x_n)_{n\geq 0}$ is divergent.

Question 16 : The improper integral $\displaystyle\int_1^{\infty} \frac{x^{3/2}+3}{x^3}\,dx$

- ☐ diverges.
- ☐ converges, and its value is $-\frac{7}{2}$.
- ☐ converges, and its value is $\frac{8}{3}$.
- ☐ converges, and its value is $\frac{7}{2}$.

Question 17 : Set $A = \left\{x \in \mathbb{R}_+^* \setminus \{1\} \,:\, \dfrac{1}{\mathrm{Log}(x)} < 1\right\}$. Then,

- ☐ $\mathrm{Inf}\, A = 0$.
- ☐ A is not bounded from below.
- ☐ $\mathrm{Sup}\, A = \mathrm{e}$.
- ☐ $\mathrm{Inf}\, A = \mathrm{e}$.

Question 18 : Define $f\colon \mathbb{R}^* \to \mathbb{R}$ by $f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$, and define the sequence $(x_n)_{n\geq 1}$ by setting $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$, and for some fixed $x_0 \in \mathbb{R}^*$.

- ☐ If $x_0 = -2$, then the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.
- ☐ If $x_0 = \frac{1}{\sqrt{2}}$, then the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.
- ☐ If $x_0 = 1$, then the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.
- ☐ There does not exist any $x_0 \in \mathbb{R}^*$ for which the sequence $(x_n)_{n\geq 1}$ converges to $-\sqrt{2}$.

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Then f is continuous at exactly two points.

☐ TRUE ☐ FALSE

Question 20 : A strictly increasing function $f : [0,1] \to [0,1]$ is always bijective.

☐ TRUE ☐ FALSE

Question 21 : The radius of convergence of the power series $f(x) = \sum_{k=0}^{+\infty} (3x)^k$ is 3.

☐ TRUE ☐ FALSE

Question 22 : Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, and let $x_0 \in \mathbb{R}$ be such that

$$\lim_{x \to x_0-} f(x) = f(x_0)\,.$$

Then f is differentiable from the left at x_0.

☐ TRUE ☐ FALSE

Question 23 : Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers. If $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

☐ TRUE ☐ FALSE

Question 24 : Consider a function $f\colon \mathbb{R} \to \mathbb{R}$. If f is differentiable at x_0, then the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sin(f(x))$ is also differentiable at x_0.

☐ TRUE ☐ FALSE

Question 25 : Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and increasing function, and for all $n \in \mathbb{N}$, let a_n be the real number defined by $a_n = f(n)$. Then $(a_n)_{n\geq 0}$ is a Cauchy sequence.

☐ TRUE ☐ FALSE

Question 26 : For all $\omega \in \mathbb{C}$, $\omega \neq 0$, there exist infinitely many complex numbers $z \in \mathbb{C}$ such that $\operatorname{Im}(\omega z) = 0$.

☐ TRUE ☐ FALSE

Question 27 : Let $f\colon [-1,1] \to [-1,1]$ be a continuous and bijective function, such that $f(0) = 0$. Then $\displaystyle\int_{-1}^{1} f(x)\,dx = 0$.

☐ TRUE ☐ FALSE

Question 28 : Let $f\colon\ ]-1,1[\ \to \mathbb{R}$ be a C^3 function, that is, the third derivative of f exists and it is continuous. Assume that the Taylor expansion of order 2 of f around $x_0 = 0$ is given by $f(x) = 1 + 2x + x^2 + x^2\varepsilon_1(x)$. Then the Taylor expansion of the function $(f(x))^2$ around $x_0 = 0$ is $(f(x))^2 = 1 + 4x^2 + x^2\varepsilon_2(x)$.

☐ TRUE ☐ FALSE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 29: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 *Do not write here.*

Show that there is a single $x \in [2, +\infty[$ such that $\mathrm{Log}(x) - x + 2 = 0$.

Question 30: *This question is worth 6 points.*

0 1 2 3 4 5 6 *Do not write here.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the first and the second derivatives of f exist and are continuous. Show that if f is convex and for some real numbers $a < b$, $f(a) = f(b)$, then the global minimum of f is taken on the interval $[a, b]$. That is, there is a real number $c \in [a, b]$ such that f has global minumum at c.

Question 31: *This question is worth 5 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 *Do not write here.*

Show that if $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ for some real numbers $x, y \in \mathbb{R}$, then $\lim_{n\to\infty}(x_n + y_n) = x + y$.