Lecturer: R. Svaldi
Analysis I - (n/a)
11th January 2021
3 hours

n/a

n/a

SCIPER : **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 8 pages, the last ones possibly blank. TOTAL: 34 questions. Do not unstaple.

- Place your student card on your table.
- **The only papers you are allowed to use are the booklet of the exam and the scratch paper provided by the proctors.**
- Using a **calculator** or any electronic device is not permitted during the exam.
- For the **multiple choice** questions, we give :
  - $+3$ points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - -1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  - $+1$ points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - -1 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

Part 1: this part of the exam contains 23 multiple choice questions.

For each question, mark the box corresponding to the answer that you wish to select as the correct one.
Each question has a unique correct answer.

Question 1 : Given any function $f\colon \mathbb{R} \to \mathbb{R}$ which is twice differentiable on $\mathbb{R}$ and that admits a point of local minimum at $x = 0$, then,

- ☐ $f'(0) = 0$ and $f''(0) \neq 0$
- ☐ $f'(0) \neq 0$ and $f''(0) \neq 0$
- ☐ $f'(0) = 0$ and $f''(0) \geq 0$
- ☐ $f'(0) = 0$ and $f''(0) \leq 0$

Question 2 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{for } x \leq 0\,, \\ \sqrt{1-x^2} & \text{for } 0 < x \leq 1\,, \\ 0 & \text{for } x > 1\,. \end{cases}$$

Then,

- ☐ f is differentiable at $x = 0$ and f is continuous at $x = 1$
- ☐ the left derivative of f exists at $x = 0$ and f is differentiable at $x = 1$
- ☐ f is continuous at $x = 0$ and f is differentiable at $x = 1$
- ☐ the right derivative of f exists at $x = 0$ and f is differentiable at $x = 1$

Question 3 :
Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) := x^3$. We define the function $f_1 := f$ and, for any natural number $n \geq 2$, the function $f_n := f \circ f_{n-1}$. Then, for all natural number $m \geq 1$,

- ☐ $f_m(x) = mx^3$
- ☐ $f_m(x) = x^{(3m)}$
- ☐ $f_m(x) = (3x)^m$
- ☐ $f_m(x) = x^{(3^m)}$

Question 4 : Let $c \in \mathbb{R}$, and let $(a_n)_{n\geq 1}$ be the sequence

$$a_n := \begin{cases} \sum_{k=0}^{n} \frac{2^k}{k!} & \text{for } n \text{ even,} \qquad \left(\sum_{k=0}^{n} \frac{1}{k!}\right)^c \quad \text{for } n \text{ odd.} \end{cases}$$

Then,

- ☐ $\lim\limits_{m\to\infty} a_{2m} = +\infty$
- ☐ the sequence $(a_n)_{n\geq 1}$ converges for exactly one value of c
- ☐ $\lim\limits_{m\to\infty} a_{2m+1} = c$
- ☐ The sequence $(a_n)_{n\geq 1}$ diverges for any value of c

Question 5 : Let $f\colon\,]-\pi,\pi[\,\setminus\{0\}\to\mathbb{R}$ be the function defined as $f(x)=\dfrac{\operatorname{Arctg}(x^2)}{x\sin(x)}$. Then,

- ☐ f can be extended to a function $\hat{f}(x)$ which is continuous at $x=0$ and $\hat{f}(0)=0$.
- ☐ f can be extended to a function $\hat{f}(x)$ which is continuous at $x=0$ and $\hat{f}(0)=1$.
- ☐ f can be extended to a function $\hat{f}(x)$ which is continuous at $x=0$ and $\hat{f}(0)=\frac{\pi}{2}$.
- ☐ f cannot be extended to a function $\hat{f}(x)$ which is continuous at $x=0$, since $\lim\limits_{x\to 0^+} f(x)\neq \lim\limits_{x\to 0^-} f(x)$.

Question 6 : Let $f\colon\mathbb{R}\to\mathbb{R}$ be the function $f(x):=x^2\sin(x^2)$ and let $R(f)\subset\mathbb{R}$ be the range of f,

$$R(f):=\{y\in\mathbb{R}\,|\,\exists\, x\in\mathbb{R} \text{ such that } f(x)=y\}\,.$$

Then,

☐ $R(f)=[0,+\infty[$ ☐ $R(f)=[0,1]$ ☐ $R(f)=\mathbb{R}$ ☐ $R(f)=[-1,1]$

Question 7 : Let $A\subset\mathbb{R}$ be the set $A:=\left\{x\in\mathbb{R}\colon 0<\operatorname{Arctg}\left(\frac{1}{x}\right)<\frac{\pi}{4}\right\}$. Then,

☐ $A=\,]0,1[$ ☐ A is not bounded
☐ $A=\,]1,\frac{\pi}{2}[$ ☐ $\operatorname{Inf} A=\frac{\pi}{2}$

Question 8 : For what values of the parameter $b\in\mathbb{R}$ does the series

$$\sum_{k=1}^{\infty}\left(b+\frac{1}{k}\right)^k$$

converge?

☐ $b\le 1$ ☐ $b\in\,]-1,1[$ ☐ $b<1$ ☐ $b\in\,]-1,1]$

Question 9 : Let a,b be two real numbers for which the function $f\colon\mathbb{R}\to\mathbb{R}$,

$$f(x):=\begin{cases} ax+b & \text{for } x\le 0\\ \dfrac{\sqrt{1+x}-1}{x} & \text{for } x>0\end{cases}$$

is differentiable at $x=0$. Then,

☐ $f(-3)=\frac{3}{8}$ ☐ $f(-3)=-\frac{3}{8}$ ☐ $f(-3)=\frac{7}{8}$ ☐ $f(-3)=\frac{1}{4}$

Question 10 : Which of the functions $f,g\colon\mathbb{R}\to\mathbb{R}$, defined as

$$f(x):=\begin{cases}\sqrt{|x|}\,\sin(\frac{1}{x}) & \text{for } x\neq 0,\\ 1 & \text{for } x=0,\end{cases}\qquad g(x):=\begin{cases}\frac{1}{x}\operatorname{Arctg}(x) & \text{for } x\neq 0,\\ 1 & \text{for } x=0,\end{cases}$$

are continuous at $x=0$?

☐ Both f and g ☐ Neither f nor g ☐ Only g ☐ Only f

Question 11 : The value of the integral $\int_{-\pi}^{\pi} x \sin(x)\,\mathrm{d}x$ is

- [ ] $\frac{3\pi}{2}$
- [ ] $-\pi$
- [ ] 2π
- [ ] 0

Question 12 : Consider the sets $A, B \subseteq \mathbb{C}$,

$$A := \left\{z \in \mathbb{C} \colon z^2\,(|z|-2) = 0\right\}, \qquad B := \{z \in \mathbb{C} \colon \operatorname{Re}(z) = 1\}.$$

Then,

- [ ] $A \cap B$ consists exactly of two distinct points.
- [ ] $A \cap B$ consists exactly of one point.
- [ ] $A \cap B$ contains a line.
- [ ] $A \cap B$ is the empty set.

Question 13 : The derivative of the function $f(x) := (1+x^2)^{1+x^2}$ at the point $x = 1$ is

- [ ] $8\,(\operatorname{Log}(2)+1)$
- [ ] 8
- [ ] 4
- [ ] $4\,(\operatorname{Log}(2)+1)$

Question 14 : Let $(a_n)_{n\geq 0}$ be the sequence $a_n := \dfrac{(n+3)^{1/2} - n^{1/2}}{(n+1)^{-1/2}}$. Then,

- [ ] $\lim_{n\to\infty} a_n = \frac{3}{2}$
- [ ] $\lim_{n\to\infty} a_n = 0$
- [ ] $\lim_{n\to\infty} a_n = +\infty$
- [ ] $\lim_{n\to\infty} a_n = 3$

Question 15 : Let z be the complex number $z := 1 + \sqrt{3}\mathrm{i}$. Then,

- [ ] $z^{14} \in \mathbb{R}$
- [ ] $\operatorname{Re}(z^8) > 0$
- [ ] $|z^6| > 73$
- [ ] $\operatorname{Im}(z^{11}) < 0$

Question 16 : Let $(a_n)_{n\geq 1}$ be the sequence $a_n := (-1)^n \sin\left(\frac{1}{n^2}\right)$. Then,

- [ ] $\sum_{n=1}^{\infty} (a_n)^2$ converges, but $\sum_{n=1}^{\infty} a_n$ diverges
- [ ] $\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} a_n$ diverges
- [ ] $\sum_{n=1}^{\infty} a_n$ converges, but it does not converge absolutely
- [ ] $\sum_{n=1}^{\infty} a_n$ converges absolutely

Question 17 : Let $f \colon [0,4] \to \mathbb{R}$ be a function which is continuous on $[0,4]$, differentiable on $]0,4[$ and such that $f'(x) \geq 2$ for all $x \in]0,4[$. Then,

- [ ] $0 \leq f(3) - f(2) \leq 1$
- [ ] $f(4) - f(1) \leq 4$
- [ ] $f(4) - f(0) \leq 1$
- [ ] $f(2) - f(0) \geq 4$

Question 18 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) := \sin(\sin(x))$ and take $x_0 = 0$. Then, the expansion of f to order 5 around x_0 is given by

- ☐ $f(x) = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + x^5\varepsilon(x)$
- ☐ $f(x) = x - \frac{1}{3}x^3 + \frac{1}{120}x^5 + x^5\varepsilon(x)$
- ☐ $f(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + x^5\varepsilon(x)$
- ☐ $f(x) = x - \frac{1}{6}x^3 + \frac{1}{10}x^5 + x^5\varepsilon(x)$

Question 19 : Let $f\colon\ ]-1,1[\ \to \mathbb{R}$ be the function $f(t) := \dfrac{1}{4+3t}$, and take $t_0 = 0$. Then the expansion of f to order 2 around t_0 is given by

- ☐ $f(t) = \frac{1}{4} - \frac{3}{16}t + \frac{9}{32}t^2 + t^2\varepsilon(t)$
- ☐ $f(t) = \frac{1}{4} + \frac{3}{16}t - \frac{9}{64}t^2 + t^2\varepsilon(t)$
- ☐ $f(t) = \frac{1}{4} - \frac{3}{16}t + \frac{9}{64}t^2 + t^2\varepsilon(t)$
- ☐ $f(t) = \frac{1}{4} - \frac{3}{16}t + \frac{9}{128}t^2 + t^2\varepsilon(t)$

Question 20 : The generalised integral $\displaystyle\int_1^{2^-} \frac{x+1}{\sqrt{2-x}}\,\mathrm{d}x$

- ☐ converges and its value is $\frac{8}{3}$
- ☐ converges and its value is 4
- ☐ converges and its value is $\frac{16}{3}$
- ☐ diverges

Question 21 : Let $I \subset \mathbb{R}$ be the domain of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}\ (x+1)^n$. Then,

- ☐ $I = \left]-2, 0\right[$
- ☐ $I = \left[-\frac{5}{2}, \frac{1}{2}\right]$
- ☐ $I = \left]-1, 1\right[$
- ☐ $I = \mathbb{R}$

Question 22 : The value of the integral $\displaystyle\int_0^1 x\sqrt{x^2+1}\,\mathrm{d}x$ is

- ☐ $3(2\sqrt{2}-1)$
- ☐ $\dfrac{\sqrt{2}-1}{2}$
- ☐ $\dfrac{2\sqrt{2}-1}{3}$
- ☐ $\dfrac{\sqrt{2}}{2}$

Question 23 : Let $(a_n)_{n\geq 1}$ be the sequence $a_n := \dfrac{(5n+1)^n}{n^n 5^n}$. Then,

- ☐ $\lim\limits_{n\to\infty} a_n = 5$
- ☐ $\lim\limits_{n\to\infty} a_n = \mathrm{e}^{\frac{1}{5}}$
- ☐ $\lim\limits_{n\to\infty} a_n = 0$
- ☐ $\lim\limits_{n\to\infty} a_n = +\infty$

Part 2: this part of the exam contains 11 true/false questions.

For each question, mark the box next to "TRUE" if you think the statement of the question is correct or mark the box next to "FALSE" if you think the statement of the question is incorrect.

Question 24 : Let $f\colon\ ]0,1[\ \to \mathbb{R}$ be a monotone function which is non-constant and differentiable over $]0,1[$. Then, either $f'(x) \geq 0$ for all $x \in\ ]0,1[$ or $f'(x) \leq 0$ for all $x \in\ ]0,1[$.

☐ TRUE ☐ FALSE

Question 25 : For all given $y \in \mathbb{R}$, $y \neq 0$, the equation $z^4 = iy$, in the indeterminate z, has exactly 4 distinct solutions in $\mathbb{C}$.

☐ TRUE ☐ FALSE

Question 26 : Let $(a_k)_{k\geq 0}$ be a sequence of real numbers such that for all $k \geq 0$, $a_k \geq 0$, and $\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = 0$. Then the power series $\sum\limits_{k=0}^{\infty} a_k\, x^k$ converges for all $x \in \mathbb{R}$.

☐ TRUE ☐ FALSE

Question 27 : Let $A \subset \mathbb{R}$ be a subset of the real numbers. If $\operatorname{Inf} A \in A$ and $\operatorname{Sup} A \in A$, then A is a closed interval.

☐ TRUE ☐ FALSE

Question 28 : Let $g, h\colon]-1,1[\to \mathbb{R}$ be two functions which are differentiable on $]-1,1[$ Assume that $g(0) = h(0) = 0$, and $h'(x) \neq 0$ for all $x \in\ ]-1,1[$. If $\lim\limits_{x\to 0} \dfrac{g'(x)}{h'(x)}$ does not exist, then also $\lim\limits_{x\to 0} \dfrac{g(x)}{h(x)}$ does not exist.

☐ TRUE ☐ FALSE

Question 29 : Let $f\colon \mathbb{N} \to \mathbb{R}$ be a function such that for all $n \geq 1$, $f(n) > n$. Then the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{f(n)}$ converges.

☐ TRUE ☐ FALSE

Question 30 : Let $(x_n)_{n\geq 0}$ be a sequence of real numbers for which $\lim_{n\to\infty} x_n = +\infty$. Then, the sequence $(x_n)_{n\geq 0}$ does not contain any bounded subsequence.

☐ TRUE ☐ FALSE

Question 31 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. If there exists $a \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = +\infty$, then f is not continuous on $\mathbb{R}$.

☐ TRUE ☐ FALSE

Question 32 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and take $x_0 = 0$. Assume that f admits an expansion to order 2 around x_0 given by $f(x) = a + bx + cx^2 + x^2\varepsilon(x)$. Then, the expansion to order 2 around x_0 for the function $g(x) := f(x)^3$ is given by $g(x) = a^3 + b^3x + c^3x^2 + x^2\varepsilon(x)$.

☐ TRUE ☐ FALSE

Question 33 : Let $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ be two converging sequences. Assume that $b_n \neq 0$ for all $n \geq 0$. Then, the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists.

☐ TRUE ☐ FALSE

Question 34 : Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 1$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $(b_n)_{n\geq 1}$ be the sequence

$$b_n = \int_0^{a_n} f(x)\,\mathrm{d}x\,, \qquad n \geq 1\,.$$

Then, the sequence $(b_n)_{n\geq 1}$ converges.

☐ TRUE ☐ FALSE