Analysis I (English)
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Analysis 1 - Exercise Set 9

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Find the local and global maximum/minimum of the function $f(x) = |x^2 - x| + |x|$, by sketching the graph of the function.

2. Compute the following limits if they exist.

   (a) $\lim_{x \to 1} \frac{x^2 - x}{x^2 - 2x + 1}$

   (b) $\lim_{x \to +\infty} \left(\sqrt[3]{x+1} - \sqrt[3]{x} \right)$

   (c) $\lim_{x \to 0} \frac{(-1)^{[x]}}{\sin(x)^3} + \frac{1}{\sin(x)^2}$

3. Consider the function
$$f(x) = \frac{x(x-1)\tan(x-1)}{x^3 - 3x + 2},$$
   whose domain is $\mathbb{R} \setminus \{-2, 1\}$.

   (a) Study its continuity at $x_0 = 0$.

   (b) Find, if it exists, a continuous extension of the function f in $x_0 = 1$, or otherwise show that f cannot have a continuous extension at $x_0 = 1$.

4. (a) Prove or disprove that a function is continuous if and only if it is uniformly continuous.

   (b) Prove or disprove that $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ is a uniformly continuous function.

   (c) Show that the function $f \colon]0, b[\to \mathbb{R}$ defined by $f(x) = x^2$ is continuous and also uniformly continuous for $b < +\infty$. Show that f is not uniformly continuous when $b = +\infty$.

5. Let I be an interval, $f \colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

   (a) $f(I)$ is an interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).

   (b) If I is a bounded and closed interval, then $f(I)$ is a bounded and closed interval (where here we also admit the degenerate case $f(I) = [m, m] = \{m\}$).

   (c) If I is open, then $f(I)$ is an open interval.

   (d) If $I = [a, b[$ with $a, b \in \mathbb{R}$, $a < b$, then f attains its maximum and minimum in I. That is, there exists $m, M \in R(f)$ such that $R(f) = [m, M]$.

6. Find, if it exists, continuous extension of the function $f :]0, 1] \to \mathbb{R}$ given by $f(x) = \frac{\tan(\sqrt{1+x} - 1)}{x^{3/2}}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0. (*Note: you have to care just about the limit from the right, that is:* $x \to 0^+$)

7. Use the intermediate value theorem to show that the following equations have at least one solution in $\mathbb{R}$:

(a) $e^{x-1} = x + 1$

(b) $x^2 - \dfrac{1}{x} = 1$

8. State if the following functions are continuous and differentiable at $x = 0$.

    (a) $|\sin(x)|$

    (b) $|x^3|$

9. Check if the following functions are uniformly continuous

    (a) $\sqrt{x}$ with domain $[0, +\infty)$

    (b) x^3 with domain $[0, \pi]$

    (c) x^3 with domain $\mathbb{R}$

10. Let f and g be two continuous functions in $[a, b]$, such that $f(a) > g(a)$ and $f(b) < g(b)$. Show that there is $c \in]a, b[$ such that $f(c) = g(c)$. (*Hint: use the function $h = f - g$ and the intermediate value theorem.*)

11. Find the inverse of the following functions if they exist. Give the domain of both functions.

    (a) $f(x) = \sqrt{(2x+4)^3 - 7}$

    (b) $f(x) = \frac{2x+3}{3x+5}$

    (c) $f(x) = \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x}$

12. Let I be an interval, $f\colon I \to \mathbb{R}$ be a continuous function and $f(I)$ the image of I by f. Say if the following statement are true or false.

    (a) If I is bounded, then $f(I)$ is bounded.

    (b) If $I = [a, \infty[$ with $a \in \mathbb{R}$, then f attains its maximum and minimum in I.

    (c) If f is strictly increasing and I is open, then $f(I)$ is open.

13. Find, if it exists, continuous extension of the function $f :]2, \infty[\to \mathbb{R}$ give by $f(x) = \frac{\sqrt{x} - \sqrt{2} + \sqrt{x-2}}{\sqrt{x^2-4}}$ at $x_0 = 2$, or otherwise show that f cannot have a continuous extension at x_0.

14. **The Bisection Algorithm:** Using the intermediate value theorem and successive bisection of the interval $[0, 1]$, find an interval of the length $L \leq \frac{1}{8}$ that contains a solution of the equation

$$x^3 + x - 1 = 0.$$

15. Let the function $f\colon [0, \infty) \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \dfrac{3x^2 - 10x + 3}{x^2 - 2x - 3}\,, & x > 3 \\ \alpha\,, & x = 3 \\ \beta x - 4\,, & x < 3 \end{cases}$$

Find $\alpha, \beta \in \mathbb{R}$ such that the function is continuous at $x = 3$.

16. Show that if $f(x)$ is continuous on $[-1, 1]$ and $f(-1) = f(1)$, then there exists $\delta \in [0, 1]$ such that $f(\delta) = f(\delta - 1)$.

17. Find, if it exists, continuous extension of the function $f : [-\pi/4, 0[\cup]0, \pi/4] \to \mathbb{R}$ given by $f(x) = \frac{1 - \cos x}{\tan^2 x}$ at $x_0 = 0$, or otherwise show that f cannot have a continuous extension at x_0.

18. Let us define the functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

(a) Find domain and range for each of the 3 functions.

(b) Show that

$$\cosh(x)^2 - \sinh(x)^2 = 1.$$

(c) Find a suitable domain, for each of the 3 functions, over which the function is invertible.

(d) Compute

$$\lim_{x\to+\infty} \cosh(x), \quad \lim_{x\to-\infty} \cosh(x),$$
$$\lim_{x\to+\infty} \sinh(x), \quad \lim_{x\to-\infty} \sinh(x),$$
$$\lim_{x\to+\infty} \tanh(x), \quad \lim_{x\to-\infty} \tanh(x).$$