

Analysis 1 - Exercise Set 8

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Study the continuity of the function $f: \mathbb{R} \rightarrow \mathbb{R}$ in $x = 0$, where

$$f(x) = \begin{cases} \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

2. Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^n$ is continuous using the definition with ε and δ .

NB: A function $f: E \rightarrow \mathbb{R}$ is continuous at a point $x_0 \in E$ if and only if, by definition, $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. Use the (ε, δ) definition of limit by plugging in $f(x_0)$ as the value l of the limit.

(*Hint:* $(a^n - b^n) = (a - b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$)

3. Consider the function

$$f(x) = \frac{ax + b}{cx + d}, \quad ad - bc \neq 0$$

Find the largest sets $D, R \subset \mathbb{R}$ such that that $f: D \rightarrow R$ is invertible and find f^{-1} . What happens when $ad - bc = 0$?

4. State if the following statements are true or false:

- (a) If $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = c \in \mathbb{R}$ then f is continuous at 0.
- (b) If $f \circ g$ is continuous, then f is continuous.
- (c) For every $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$, there exists at most one value $a \in \mathbb{R}$ such that the function $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$

$$\hat{f}(x) := \begin{cases} f(x) & x \neq 0, \\ a & x = 0, \end{cases}$$

is continuous.

- (d) If $|f(x)|$ is continuous everywhere then $f(x)$ is also continuous.
- (e) If $f \circ g$ is continuous, then g is continuous.
- (f) If functions f and g are continuous everywhere then f/g is also continuous everywhere.
- (g) If $f(x)$ is continuous everywhere, then $|f(x)|$ is continuous everywhere.
- (h) If the composition $f \circ g$ is not continuous at $x = a$, then g is not continuous at $x = a$ or f is not continuous at $g(a)$.

5. Let $f: S \rightarrow T$ be a bijective function. Show that $f^{-1} \circ f = Id_S$, and $f \circ f^{-1} = Id_T$.

6. Show that:

- (a) $\lim_{x \rightarrow +\infty} e^x = +\infty$;
- (b) $\lim_{x \rightarrow +\infty} \arctan(x) = \frac{\pi}{2}$, $\lim_{x \rightarrow -\infty} \arctan(x) = -\frac{\pi}{2}$;
- (c) $\lim_{x \rightarrow +\infty} \arctan(e^x) = +\frac{\pi}{2}$.
7. Make an example of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is continuous only at one point. Can you generalize this to make an example of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which is continuous only at n points for fixed $n \in \mathbb{N}$?
8. Use the ϵ, δ definition of continuity to show that $f(x) := \sin(x)$ is continuous everywhere. (Hint: Using geometry, show $\sin(\theta) \leq \theta \quad \forall \theta \in [0, \frac{\pi}{2}]$. Then, using this result show that $|\sin(\theta)| \leq |\theta| \quad \forall \theta \in \mathbb{R}$.)
9. Compute the following limits if they exist.
- (a) $\lim_{x \rightarrow 1} \left(\frac{1}{1-x} - \frac{3}{1-x^3} \right)$
- (b) $\lim_{x \rightarrow a} \frac{\cos(x) - \cos(a)}{x-a}$ with $a \in \mathbb{R}$
- (c) $\lim_{x \rightarrow +\infty} (x|\sin(x)| - x^2 + 4)$
10. We say that a function f defined on a pointed neighborhood E of $x_0 \in \mathbb{R}$ has a continuous extension at the point x_0 if there exists a number $a \in \mathbb{R}$ such that the function

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in E, \\ a & \text{if } x = x_0, \end{cases}$$

is continuous at x_0 .

Find, if it exists, a continuous extension of the function $f: [0, 1[\cup]1, \infty[\rightarrow \mathbb{R}$,

$$f(x) = \frac{\sqrt{x+1} - \sqrt{2x}}{\sqrt{1+2x} - \sqrt{3}}$$

in $x_0 = 1$, or otherwise show that f cannot have a continuous extension at x_0 . Finally, compute $\lim_{x \rightarrow +\infty} f(x)$.

11. Find the values $\alpha \in \mathbb{R}$ such that the limit $\lim_{x \rightarrow \alpha} \frac{\tan(x - \alpha)^2}{(x - \alpha)^2}$ exists in $\mathbb{R}$.
12. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$f(x) = \begin{cases} \sin(x) & \text{if } x \in \mathbb{Q} \\ \cos(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

State whether the following sentences are true or false.

- (a) f is bounded.
- (b) $\max\{f(x) : x \in [0, 2\pi[\} = 1$.
- (c) $\min\{f(x) : x \in [0, 2\pi[\} = -1$.
- (d) $\lim_{x \rightarrow \frac{\pi}{2}} f(x) = 1$.
- (e) f is continuous.
- (f) f is continuous at $x_0 = \frac{\pi}{4}$.

13. Compute the following limits if they exist.

(a) $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - 2}{x - 1}$

(b) $\lim_{x \rightarrow 2} \frac{(-1)^{[x]} x^2 + 3}{x - 2}$

(c) $\lim_{x \rightarrow 0} \frac{\sin(x)^2}{\sin(x^2)}$

14. Let $A = \left\{ \left(\frac{\pi}{2} + n\pi \right)^{-1} : n \in \mathbb{N} \right\}$. Find, if it exists, a continuous extension of the function $f:]0, 1] \setminus A \rightarrow \mathbb{R}$,

$$f(x) = \tan\left(\frac{1}{x}\right) \left(1 - \sin\left(\frac{1}{x}\right)^2 \right)$$

at the points $x_0 \in A \cup \{0\}$, or otherwise show that f cannot have a continuous extension at x_0 .

15. Find the values $\alpha \in \mathbb{R}$ such that the limit $\lim_{x \rightarrow \alpha} \frac{x^4 - 2\alpha x^3 + 4x^2}{(x - \alpha)^2}$ exists in $\mathbb{R}$.

16. Study the continuity of the function $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{1 + 2^{\frac{1}{x}}}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

at $x = 0$.

17. Find the values $\alpha, \beta \in \mathbb{R}$ such that the limit $\lim_{x \rightarrow 0} \frac{x^2 \sin(\frac{1}{x}) + \alpha|x|}{\sqrt{x^2 + \beta} \left| \cos(\frac{1}{x}) \right|}$ exists in $\mathbb{R}$.

18. Study the continuity of the function $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} \frac{1 - \cos(x)}{x^2}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

in $x = 0$.