Analysis I (English)
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Analysis 1 - Exercise Set 7

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Show that for every $n \in \mathbb{N} \setminus \{0\}$

$$\sum_{k=1}^{n} (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}} = 1 + \frac{(-1)^{n+1}}{2n+1}.$$

   (b) Use the previous part to compute $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 - \frac{1}{4}}$.

   (c) Is the series absolutely convergent?

2. Determine for which values of $a \in ((0, +\infty) \setminus \{1\})$, the series

$$\sum_{n=1}^{\infty} \frac{a^n}{a^{2n} - 1}$$

   converges. (*Hint: use the Cauchy criterion.*)

3. Compute $\lim_{n \to \infty} \sqrt[n]{n!}$

4. Let (x_n) be a sequence.

   (a) Show that if $\lim_{n \to \infty} \sqrt[n]{|x_n|} = \rho$, with $\rho > 1$, then (x_n) is unbounded. In particular, it diverges.

   (b) Show that if $\lim_{n \to \infty} \sqrt[n]{|x_n|} = \rho$, with $0 \le \rho < 1$, then (x_n) converges to 0. In particular, it is bounded.

   (c) Provide two sequences (y_n) and (z_n) with the following properties: (y_n) converges and $\lim_{n \to \infty} \sqrt[n]{|y_n|} = 1$, and (z_n) diverges to $+\infty$ and $\lim_{n \to \infty} \sqrt[n]{|z_n|} = 1$

5. Let $(t_n) \subset \mathbb{R}^*$ be a sequence. Assume that $\lim_{n \to \infty} t_n = 0$. Show that $\lim_{n \to \infty} \frac{\sin(t_n)}{t_n} = 1$.
   (*Hint: recall that for* $x \in [0, \frac{\pi}{2}]$,

$$0 \le \sin(x) \le x \le \tan(x) \Rightarrow 1 \le \frac{x}{\sin(x)} \le \frac{1}{\cos(x)} \Rightarrow \cos(x) \le \frac{\sin(x)}{x} \le 1$$
$$\Rightarrow \cos(x)^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1 \Rightarrow 1 - \sin(x)^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1$$
$$\Rightarrow 1 - x^2 \le \left(\frac{\sin(x)}{x}\right)^2 \le 1 \Rightarrow \sqrt{1 - x^2} \le \frac{\sin(x)}{x} \le 1.)$$

6. Show that if $\lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|} = 1$, then anything can happen for $\sum_{n=0}^{\infty} x_n$. That is, it is possible to find sequences (x_n) such that:

(a) (x_n) is unbounded;

(b) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ diverges;

(c) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges absolutely;

(d) (x_n) is bounded and $\sum_{n=0}^{\infty} x_n$ converges but not absolutely.

For each item above, provide an example.

7. For each of the following, determine whether the series is convergent or divergent.

   (a) $\sum_{k=1}^{\infty} \frac{\pi^k}{k \cdot 2^k}$

   (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+4}-\sqrt{n+2}}{n}$

   (c) $\sum_{k=2}^{\infty} \frac{k^2-1}{(k-1)^3}$

8. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$

   (b) $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$

9. For the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}, \qquad g(x) = \begin{cases} 2x-3 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

10. State if the following are true or false.

    (a) The function $f = \sqrt{1 - \cos x}$ is even.

    (b) There is no function which is both even and odd.

    (c) Let f be an odd function. If f is bijective, then f^{-1} is also odd.

11. Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, Determine the monotonicity (increasing or decreasing) of the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ in the following cases:

    (a) if f and g are both increasing.

    (b) if f and g are both decreasing.

    (c) if f is increasing and g is decreasing. What can we say about $f \circ g$?

12. Using the definition, state if the following functions are injective, surjective or bijective. If the function is bijective, find the inverse function.

    (a) $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin x$

    (b) $f : [0, \pi] \to [-1, 1]$, $f(x) = \cos x$

13. For the two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ below, find $g \circ f$ and $f \circ g$.

$$f(x) = \begin{cases} |2x-1| & \text{if } x \geq -1 \\ -x(x+2) & \text{if } x < -1 \end{cases}, \qquad g(x) = \begin{cases} -\sqrt{x-4} & \text{if } x \geq 4 \\ 1 - x/2 & \text{if } x < 4 \end{cases}$$

14. State if the following are true or false.

    (a) If f is an even function and g is an odd function, then $h = f \cdot g$ is an odd function.
    (b) If f is an even function and g is an odd function, then $h = f \circ g$ is an odd function.
    (c) A function is either even or odd or both.

15. Calculate the following limits.

    (a) $\lim_{x \to 0} \frac{x^3+4x}{2x}$
    (b) $\lim_{x \to 0} \frac{\sqrt{9+x}-3}{x}$
    (c) $\lim_{x \to 0} \frac{\cos(x)-1}{x^2}$

16. Calculate the following limits.

    (a) $\lim_{x \to 4} \frac{x^2+5x-36}{x^2-16}$
    (b) $\lim_{x \to 1} \frac{x^n-1}{x-1}$ (*Hint: Try to factorize* $x-1$ *from the numerator.*)

17. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^{500}}{(1.0001)^n}$$

    (a) converges absolutely.
    (b) converges, but not absolutely.
    (c) approaches $+\infty$.
    (d) approaches $-\infty$.

18. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(\frac{n}{\sqrt{n}+1} - \frac{n+1}{\sqrt{n+1}+1} \right)$$

    (a) diverges.
    (b) converges to $\frac{1}{2} - \frac{2}{\sqrt{2}+1}$.
    (c) converges to $\frac{1}{2}$.
    (d) converges to 0.

19. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$$

    is

    (a) divergent.
    (b) converges to e.
    (c) converges to e^{-1}.
    (d) converges to 1.

20. (Multiple choice) The limit

$$\lim_{n\to\infty} \frac{\sqrt[3]{1-\frac{1}{n}}-1}{\sqrt[4]{1-\frac{1}{n}}-1}$$

is

(a) $\frac{3}{4}$
(b) $\frac{4}{3}$
(c) ∞
(d) 0

21. (Multiple choice) The limit

$$\lim_{n\to\infty} \left(\frac{n+2}{n}\right)^n \frac{n+2}{n+1}$$

is

(a) e^2
(b) e
(c) ∞
(d) 0

22. (Multiple choice) The limit

$$\lim_{n\to\infty} n^2 \cdot \sin\left(\frac{2n+3}{n^3}\right)$$

is

(a) 0
(b) $\frac{1}{2}$
(c) 2
(d) ∞