Analysis I (English)
Roberto Svaldi and Stefano Filipazzi
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Analysis 1 - Exercise Set 6

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Compute, if they exists, the limits of the following sequences

   (a) $\sqrt[n]{\frac{3}{n}}$

   (b) $(-1)^n \left(\frac{n^2+1}{n-1}\right)$

   (c) $\frac{1}{n^2}\left(\sqrt{1+n+\pi n^2+\frac{\sin(n)}{n}}-1\right)$

   (d) $\sqrt[n]{n\log(n)}$ (*Hint*: $1 < \log(n) < n$ for $n > 3$)

   (e) $n^2\left(\sqrt{1+\frac{1}{n}+\pi\frac{1}{n^2}+\frac{\sin(n)}{n^5}}-1\right)$

   (f) $\left(\frac{n-1}{n}\right)^{n^2}$

   (g) $\sqrt[n]{\frac{2n}{3n^2-1}}$

   (h) $\frac{4n^2-2\pi}{-n^3+\sqrt{7}n}$

   (i) $\frac{(n+1)!}{n!-(n+1)!}$

   (j) $\frac{\sqrt{\frac{cos(n)}{n^2}+1}-1}{\sqrt{e-\frac{1}{n}}-\sqrt{e}}$

2. Let $a, b \in \mathbb{R}_+$ and (x_n) be a sequence defined by the recurrence relation

$$x_{n+1} = ax_n^2 \quad x_0 = b.$$

   (a) Show by induction that every element in the sequence (x_n) is given by

$$x_n = a^{2^n-1}b^{2^n}.$$

   (b) Use part (a) to compute

$$\lim_{n\to+\infty} x_n.$$

3. Show that the following recursive sequence is convergent and calculate the limit

$$a_n = \frac{7}{3} - \frac{1}{1+a_{n-1}}, \quad a_1 = 1.$$

4. This question is going to show that, whenever we have a sequence that is defined recursively, we need to show that it converges, and that computing the candidates for the limit is not enough.

   Consider the sequence defined as $a_1 = 10$, $a_{n+1} = a_n^2$ for $n \geq 1$.

(a) Show that, if the limit of (a_n) exists, then it is either 0 or 1.

(b) Show that (a_n) diverges to $+\infty$.

5. Compute the limit of $a_n = \left(\frac{n+3}{n+1}\right)^n$ using subsequences. (*Hint: first, manipulate the definition of a_n so that it looks more to the sequence of a previous exercise, then use the subsequence with odd indices.*)

6. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

   (a) If a sequence is not bounded above, it must be increasing.

   (b) Any monotone sequence has a convergent subsequence.

   (c) If (a_n) has no divergent subsequence, then (a_n) is convergent.

   (d) If (a_n) is Cauchy convergent, then also $(|a_n|)$ is Cauchy convergent.

   (e) If (a_n) is a Cauchy sequence, then the sequence $b_n = c \cdot a_n$, $c \neq 0$ is a Cauchy sequence.

   (f) If (a_n) is Cauchy, there exists $\varepsilon > 0$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \in \mathbb{N}$.

   (g) Any sequence has a convergent subsequence.

   (h) If (a_n) and (b_n) are Cauchy sequences, then the sequence $c_n = a_n + b_n$ is a Cauchy sequence.

7. Show if the sequence

$$a_n = \frac{\sin(a_{n-1}) + 1}{2} \qquad a_1 = 0$$

   satisfies the definition of Cauchy sequence. (*Hint: Use the trigonometric formulas from Exercise Sheet 1*)

8. Let (a_n) and (b_n) be two sequences. Show the following facts.

   (a) Assume that (a_n) and (b_n) are bounded. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

   (b) Provide an example of sequences (a_n) and (b_n) such that the inequality in part (a) is strict.

   (c) Assume that $\liminf a_n = 5$. Show that there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, $a_n \geq 4$.

   (d) Assume (b_n) is defined as follows:

$$b_n = \begin{cases} \frac{100}{n} \text{ if } 3|n \\ 2 - \frac{1}{n} \text{ if } 3|n-1 \\ \frac{1}{2} \text{ if } 3|n-2 \end{cases}$$

   Compute $\limsup b_n$, $\liminf b_n$, and exhibit a subsequence of (b_n) converging to $\limsup b_n$ and a subsequence converging to $\liminf b_n$.

9. State if the following statements are true or false. If you think the statement is true, then prove that; otherwise, provide a counterexample.

   (a) If (x_n) is a sequence that converges to 0, then the series $\sum_{n=0}^{\infty} x_n$ converges.

   (b) Let (x_n) and (y_n) be two sequences such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.

   (c) Let (x_n) and (y_n) be sequences such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. If the series $\sum_{n=0}^{\infty} x_n$ diverges, then the series $\sum_{n=0}^{\infty} y_n$ diverges.

(d) Let (x_n) and (y_n) be sequences. If the series $\sum_{n=0}^{\infty} x_n$ converges and the sequence (y_n) converges, then the series $\sum_{n=0}^{\infty} x_n y_n$ converges.

10. For each of the following, determine whether the series is convergent or divergent.

    (a) $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$

    (b) $\sum_{n=0}^{\infty} \frac{2n^2+1}{3n^2+2}$

    (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

11. For each of the following, determine whether the series is convergent or divergent.

    (a) $\sum_{n=0}^{\infty} \frac{\sin\left(2n^2\right)}{n^2+3}$

    (b) $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

    (c) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{n+3}$

12. For each of the following, determine whether the series is convergent or divergent.

    (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+3)}}$

    (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^5}}{n^3+1}$

    (c) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

13. (Multiple choice) The series

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$$

    is

    (a) divergent.

    (b) converges to $2 + \sqrt{2}$.

    (c) converges to $2 - \sqrt{2}$.

    (d) cannot be determined.

14. (Multiple choice) The series

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

    (a) converges absolutely.

    (b) converges, but not absolutely.

    (c) diverges to $+\infty$.

    (d) diverges to $-\infty$.

15. Terminate the proof that we started in class showing the convergence of $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$. This is what we have proven in class and that you can assume:

    (a) the subsequence (y_k) of (s_n),

$$y_k := s_{2k+1} = \sum_{i=0}^{2k+1} \frac{(-1)^i}{i}$$

    is strictly increasing;

    (b) (y_k) is bounded; in particular (y_k) converges to a limit $y \in \mathbb{R}$.

    (Hint: Show that (s_n) is a Cauchy sequence. Use the fact that since (y_k) converges, then it is Cauchy, and that $s_{2k} - \frac{1}{2k+1} = s_{2k+1}$.)