Analysis I (English)
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Analysis 1 - Exercise Set 5

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. If a sequence (x_n) converges, then its limit is unique.

2. Assume that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. Prove the following fact: for any $l \in \mathbb{N}$, $\lim_{n\to\infty} x_{n+l}$ exists and $\lim_{n\to\infty} x_{n+l} = x$.

3. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

   (a) If
   $$\lim_{n\to\infty} a_n = 0,$$
   then
   $$\lim_{n\to\infty} (a_n \sin(n)) = 0.$$

   (b) If (a_n) is bounded, then
   $$\lim_{n\to\infty} (a_n e^{-n}) = 0.$$

   (c) If
   $$\lim_{n\to\infty} a_n = 0,$$
   then the sequence $b_n := a_n e^n$ is unbounded.

4. Compute the following limits:

   (a) $\lim_{n\to\infty} \frac{2^n - 3^n}{3^n + 1}$

   (b) $\lim_{n\to\infty} n^3 \left(1 - \cos(\frac{1}{n})\right) \sin(\frac{1}{n})$

   (*Hint: Use the fact that* $\lim_{m\to\infty} \frac{\sin(\frac{1}{m})}{\frac{1}{m}} = 1$ *and* $\lim_{m\to\infty} \cos\left(\frac{1}{m}\right) = 1$.)

   (c) $\lim_{n\to\infty} \frac{\sin^2(n)}{2^n}$

   (d) $\lim_{n\to\infty} n(\sqrt{n^4 + 6n + 3} - n^2)$

5. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

   (a) If
   $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$
   then (a_n) converges.

(b) If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$$

then (a_n) diverges.

6. Determine if the sequence (a_n) is convergent or not in the following cases.

   1. $a_n = \frac{n}{e^n}$.
   2. $a_n = \frac{10^n}{n!}$
   3. $a_n = \frac{n^n}{e^n}$
   4. $a_n = \frac{n!}{n^n e^{\frac{n}{2}}}$

7. Determine if the following sequences converge or not. If the sequence is convergent, determine its limit.

   (a) $a_n = \frac{3n^2-1}{10n+5n^2}$

   (b) $a_n = \frac{3^{2n}}{n}$

   (c) $a_n = \frac{(-1)^n n^2}{2^n}$

8. Compute the following limits:

   (a) $\lim\limits_{n\to\infty} \frac{n^3}{7^n}\cos\left(n^2\right)$

   (b) $\lim\limits_{n\to\infty} \dfrac{\sin(n+1)-\sin(n-1)}{\cos(n+1)+\cos(n-1)}$
   (*Hint: Use trigonometric formulas from Exercise Sheet 1*)

   (c) $\lim\limits_{n\to\infty} \dfrac{\sin\left(\sqrt{n^3+n^2+1}\right)}{n^3+n^2+1}$

9. Give an example of a sequence (x_n) such that the sequence $y_n = x_{n+1} - x_n$ converges to 0 but (x_n) itself is divergent.

10. Prove that if $\lim\limits_{n\to\infty} x_n = +\infty$ and (y_n) bounded from below, then $\lim\limits_{n\to\infty}(x_n+y_n) = +\infty$. Show that this is true also if $+\infty$ is replaced with $-\infty$ and (y_n) is assumed to be bounded from above.

11. Prove that if $x_n \neq 0$, for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} |\frac{x_n}{x_{n-1}}| = +\infty$ then (x_n) is unbounded and, thus it diverges. Construct examples of sequences (x_n) satisfying the conditions above and such that $\lim\limits_{n\to\infty} x_n = +\infty$ (resp. $\lim\limits_{n\to\infty} x_n = -\infty$).

12. Consider the recursive sequence $a_{n+1} = 7 - \frac{10}{a_n}$, with initial datum $a_1 = 4$. Compute the first three values. Then, show that it is bounded by 2 and 5, that it is increasing, and then compute the limit.

13. Consider the recursive sequence $a_{n+1} = \sqrt{8a_n - 7}$, with initial datum $a_1 = 4$. Show that it is bounded by 1 and 7, that it is increasing, and then compute the limit.

14. Find the limit for $x_n = \frac{\sin(x_{n-1})}{2}, x_0 = 1$. [*Hint:* use the fact $|\sin(x)| \leq |x|$ for all x]

15. Let $(a_n), (b_n)$ be sequences. State if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

(a) If (a_n) is monotone, then $\lim\limits_{n\to\infty} a_n$ exists or $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$.

(b) If (a_n) and (b_n) are monotone, then the sequence $c_n = a_n + b_n$ is monotone.

(c) If $\lim\limits_{n\to\infty} |a_{n+1} - a_n| = 0$, then (a_n) is a bounded sequence.

(d) An unbounded sequence can have a convergent subsequence.

(e) If (a_n) has no convergent subsequence, then (a_n) is unbounded.

16. Compute the following limits:

(a) $\lim\limits_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$

(*Hint:* $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim\limits_{n\to\infty} \frac{(n+1)^n}{n^n}$, *and* $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^{n+1} = \lim\limits_{n\to\infty} \frac{(n+2)^{n+1}}{(n+1)^{n+1}}$)

(b) $\lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$

(*Hint:* $e^{-1} = \left(\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right)^{-1} = \lim\limits_{n\to\infty} \left(\left(1 + \frac{1}{n}\right)^{-n}\right) = \lim\limits_{n\to\infty} \frac{n^n}{(n+1)^n}$)

(c) $\lim\limits_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^n$

(*Hint:* $\left(1 - \frac{1}{n^2}\right) = \left(1^2 - \left(\frac{1}{n}\right)^2\right)$)

17. Let $a_n = \left(1 + \frac{2}{n}\right)^n$. We are going to compute its limit using subsequences

(a) Compute $\lim\limits_{n\to\infty} a_{2n}$.

(b) Show that $a_n \leq a_{n+1}$.

(c) Use subsequences, squeeze theorem and monotone convergence to show that the sequence (a_n) is convergent and its limit.