Analysis I (English)
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Analysis 1 - Exercise Set 4

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Prove that for all $n \in \mathbb{N}$, $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$.
   (b) Find all the solutions to the equation $x^n = 1$, for $n \in \mathbb{N}$.

2. Find all the solutions of the following equations in $\mathbb{C}$.
   (a) $z^2 + 6z + 12 - 4i = 0$
   (b) $(z^3 - 1)^2 = -1$

3. (a) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Prove that $\{a_n\}$ is bounded if and only if so is $\{b_n\}$.
   (b) Prove that if $q < -1$, then the sequence $x_n = aq^n$, $a \in \mathbb{R}$, is unbounded.
   (c) Provide an example of a sequence $\{a_n\}$ that is bounded above and such that $\{|a_n|\}$ is not bounded above.

4. Let $\{x_n\}$ be the recursive sequence defined as
$$\begin{cases} x_n = x_{n-1} + (-1)^n n^2 \\ x_0 = 0. \end{cases}$$
   Prove that for all $m \in \mathbb{N}$
$$\begin{cases} x_{2m} = (2m+1)m \\ x_{2m+1} = -(2m+1)(m+1). \end{cases}$$
   [*Hint*: use induction on m].
   Is $\{x_n\}$ bounded from above? Is it bounded from below?

5. Check if the sequence starting from $n = 1$ defined as $a_n = \frac{\sin(\frac{1}{n})}{n}$ is monotone, and if it converges or diverges.

6. Check if the sequence
   (a) $a_n = \frac{n}{4n-1}$
   (b) $a_n = (-1)^n \frac{n^2+\pi}{n}$ starting from $n = 1$

   is monotone, and if it converges or diverges.

7. Find the limit of the following sequences, if they exist:
   (a) $a_n = \dfrac{5n^2 - 3n + 2}{3n^2 + 7}$
   (b) $a_n = (-1)^n \dfrac{\sqrt[4]{n}}{\sqrt[3]{n}}$

(c) $a_n = \frac{\sqrt{n}-n+n^2}{2n^2+n^{\frac{3}{2}}+n}$

(d) $a_n = \sin(\frac{1}{n}) + \frac{n-2}{n\sqrt{2}+77}$

8. Let (a_n) be a sequence. Specify if the following statements are true or false. If you think that the statement is true, you should prove it, otherwise, provide a counterexample to the statement.

   (a) If $\{a_n\}$ is bounded then $\{a_n\}$ is convergent.

   (b) If $\{a_n\}$ is bounded and $a_n \geq 0$, $\forall n \in \mathbb{N}$, then $\{a_n\}$ is convergent.

   (c) If $\{a_n\}$ is monotone and unbounded, then it is bounded from above.

   (d) If $\{a_n\}$ is monotone and unbounded, then it is bounded from below.

   (e) If $\{a_n\}$ is bounded and monotone then $\{a_n\}$ is convergent.

   (f) If $\{a_n\}$ is convergent, then there exists $\epsilon > 0$ such that $|a_n| \leq \epsilon$ for all $n \in \mathbb{N}$.

   (g) Let $\{a_n\}$ be a sequence and let $\{b_n\}$ be the sequence defined as $b_n := |a_n|$. Then, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} b_n = 0$.

9. Let $p > q$ be natural numbers. Show that if $P(x) = \sum_{i=0}^{p} c_i x^i$ is a polynomial with real coefficients of degree p (that is, $c_p \neq 0$), and $Q(x) = \sum_{j=0}^{q} b_j x^j$ is a polynomial with real coefficients of degree q (that is, $b_q \neq 0$), then the sequence (a_n) defined as

$$a_n := \frac{P(n)}{Q(n)} \quad \text{is unbounded.}$$

10. Find the limit of the following sequences, if they exist:

   (a) $a_n = \sqrt{2n^2+3} - \sqrt{(2n+1)(n+4)}$

   (b) $a_n = \sqrt{n}(\sqrt{n^3+2n} - \sqrt{n^3+4})$

11. Find the limit of the following sequences:

   (a) $a_n = \sin\left(\frac{1}{n}\right)$

   (b) $a_n = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

   (c) $a_n = n \cdot \sin\left(\frac{2n+3}{n^3}\right)$

   Hint: remember that for $0 < x < \pi/2$ we have the inequalities:

$$0 \leq \sin(x) \leq x \leq \tan(x) \quad \Rightarrow \quad 1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)} \quad \Rightarrow \quad \cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

$$\Rightarrow \quad \cos(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \quad \Rightarrow \quad 1-\sin(x)^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1$$

$$\Rightarrow \quad 1-x^2 \leq \left(\frac{\sin(x)}{x}\right)^2 \leq 1 \quad \Rightarrow \quad \sqrt{1-x^2} \leq \frac{\sin(x)}{x} \leq 1 \, .$$

12. Show that the sequence given by

$$a_1 = 2$$
$$a_n = \frac{1}{2}(a_{n-1} + 6)$$

is increasing and bounded above by 6. (*Hint: Use induction for both*)

13. Let $\{a_n\}$ be a sequence. Specify if the following statements are true or false. If you believe that the statement is true, you should give a proof, otherwise, provide a counterexample to the statement.

    (a) If
    $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$
    then $\{a_n\}$ converges.

    (b) If
    $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$
    then $\{a_n\}$ diverges.

14. Consider two sequences of real numbers a_n and b_n. Assume that $0 < a_n < 3$ and $-4 < b_n < 0$ for every n. Which of the following claims is true? (Only one choice is correct)

    (a) The sequence $\frac{1}{a_n}$ is bounded.
    (b) The sequence $a_n b_n$ is bounded below by -4.
    (c) The sequence $a_n + b_n$ has to be negative.
    (d) The sequence $a_n b_n$ is bounded below by -12.
    (e) The sequence $\frac{a_n}{b_n}$ is bounded.

15. Let z and w be two complex numbers. Which of the following statements is true? (Only one choice is correct)

    (a) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{w}$
    (b) $|z| = z \cdot \overline{z}$
    (c) $\mathrm{Im}(z + w) = \mathrm{Re}(i(z + w))$
    (d) $\mathrm{Re}(z + w) = \mathrm{Im}(i(z + w))$
    (e) $i\mathrm{Re}(z + w) = \mathrm{Im}(z + w)$

16. Prove the following properties of the binomial coefficients:

    (a) Symmetry: $\binom{n}{k} = \binom{n}{n-k}$;
    (b) Binomial formula: Assuming the recurrence formula that you find in (c) below, prove that $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. [*Hint*: use induction on n. You may use the result from (c).]
    (c)* Recurrence: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Deduce from this that $\binom{n}{k} \in \mathbb{N}$; [*Hint*: use induction on n.]