Analysis I (English)
Roberto Svaldi and Stefano Filipazzi
Fall Semester 2021–2022

Analysis 1 - Exercise Set 3

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Let $[x]$ denote the integral part of a number $x \in \mathbb{R}$.
   Prove that, for every $x \in \mathbb{R}$, $[x] = -[-x]$.

2. Let $a, b \in \mathbb{R}$. Prove that $||a| - |b|| \leq |a - b|$ and $||a| - |b|| \leq |a + b|$

3. Compute $\sup S$ and $\inf S$ where $S \subseteq \mathbb{R}$ is defined as

   (a) $S := \bigcup_{n=1}^{\infty} [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$. Does S admit maximum and/or minimum?

   (b) $S := \bigcap_{n=1}^{\infty} (-1 - \frac{1}{n}, 1 + \frac{1}{n})$. Does S admit maximum and/or minimum?

4. Compute $\min S$ where $S \subseteq \mathbb{N}$ is defined as

   (a) $S := \{n \in \mathbb{N} : \sqrt{n} > 17\}$

   (b) $S := \{n \in \mathbb{N} : \sum_{i=1}^{n} i \geq 17\}$

   (c) $S := \{n \in \mathbb{N} : \sum_{i=1}^{n} 2^{-i} > 1.7\}$

5. Compute $\max S$ where $S \subseteq \mathbb{Z}$ is defined as

   (a) $S = \{n \in \mathbb{Z} \mid n \neq 0 \text{ and } n + \frac{20}{n} < 9\}$

   (b) $S = \{n \in \mathbb{Z} \mid (\sqrt{3})^n \leq 10^{17}\}$.

   (c) $S = \{n \in \mathbb{Z} \mid \alpha^n \leq C\}$ where $\alpha > 1$ and $C > 1$ are constants. [You must discuss how $\max S$ varies, when α and C vary.]

6. For the following complex numbers z compute the real and imaginary part, the complex conjugate $\bar{z}$, the absolute value $|z|$, the argument (also called phase) $\arg(z)$ and the inverse z^{-1}:
$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i; \quad z = 16i; \quad z = 2 + 3i - 3e^{i\frac{\pi}{2}}; \quad z = e^{-5\pi i} + i.$$

7. Write the following complex numbers in the form $x + iy$.

   (a) i^{17}

   (b) $\frac{4-i}{3-2i}$

   (c) $2i(i-1) + \left(\sqrt{3} + i\right)^3 + (1+i)\overline{(1+i)}$

8. Compute

(a) $(1+i\sqrt{3})^{1980}$

(b) $(1+i\sqrt{3})^{1988}$

9. Find all the solution of the following equations in $\mathbb{C}$. [The unknown is $z = x + iy$, or, if you prefer you could use polar form.]

   (a) $z^2 = i$

   (b) $z^5 = 1$.

   (c) $z^2 = -3 + 4i$.

10. In the context of complex numbers, state if the following statements are true or false.

    (a) There exists a $n \in \mathbb{N}$ such that $(1+i\sqrt{3})^n$ is pure imaginary.

    (b) There exists a positive $n \in \mathbb{N}$ such that $(1-i\sqrt{3})^n$ is real.

11. Show that for all $\theta \in \mathbb{R}$ and for all $n \in \mathbb{N}$

$$(\cos(\theta) + i\sin(\theta))^n = (\cos(n\theta) + i\sin(n\theta)).$$

12. Prove that for all $z_1, z_2 \in \mathbb{C}$,

    (a) $z_1 = 0$ if and only if $|z_1| = 0$.

    (b) $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{i(\alpha_1-\alpha_2)}$, where $z_2 \neq 0$ and

$$z_1 = |z_1|e^{i\alpha_1}, \quad z_2 = |z_2|e^{i\alpha_2}, \quad \alpha_1,\ \alpha_2 \in \mathbb{R}$$

    are the polar forms of the z_i using Euler's formula.

    (c) $|\frac{z_1}{|z_1|}| = 1$.

13. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\mathrm{Im}(z(2-i)) = 1$ is

    (a) A point.

    (b) A line.

    (c) A circle.

    (d) Empty.

14. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $\bar{z} = i(z-1)$ is

    (a) A point.

    (b) A line.

    (c) A circle.

    (d) Empty.

15. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $z^2 \cdot \bar{z} = z$ is

    (a) A point.

    (b) A circle.

    (c) A point and a circle.

    (d) A disk.

16. (Multiple choice) The set of all $z \in \mathbb{C}$ that satisfy the equation $|z + 3i| = 3|z|$ is
    (a) A point.
    (b) A line.
    (c) A circle.
    (d) Empty.

17. Given the function $f : \mathbb{C} \to \mathbb{C}$ defined as $f(z) = \frac{1+iz}{iz+i}$
    (a) find the domain of the function f. That is, determine the set $\mathrm{Dom}(f) \subseteq \mathbb{C}$ such that $z \in \mathrm{Dom}(f)$ if and only if $f(z)$ is defined;
    (b) find all complex numbers z such that $f(z) = z$;
    (c) find the preimages of $3 + i$.