Analysis I (English)
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Analysis 1 - Exercise Set 2

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Let $p \in \mathbb{N}$ be a prime number. Prove that $\sqrt{p}$ is not rational.
   (b) Show that $\sqrt{7+\sqrt{17}}$ is irrational. (*Hint: Use part (a) to prove that $\sqrt{17}$ is irrational. Now assume that $\sqrt{7+\sqrt{17}}$ is rational and show that it contradicts the fact that $\sqrt{17}$ is irrational.*)
   (c) Show that $\sqrt{2}+\sqrt[3]{3}$ is irrational. (*Hint: Let $r=\sqrt{2}+\sqrt[3]{3}$ and assume it is rational. Compute $(r-\sqrt{2})^3$ and use the result that you obtained plus the assumption on the rationality of r to find a contradiction.*)

2. Let S be a subset of $\mathbb{R}$. Let a be a lower bound (respectively an upper bound) for S. Show that any real number b such that $b<a$ (respectively $b>a$) then b is also a lower bound (resp. an upper bound) for S.

3. Let A be a bounded interval in $\mathbb{R}$, i.e., A is a subset of $\mathbb{R}$ of either one of the following forms: $[a,b]$, or $]a,b[$, or $[a,b[$, or $]a,b]$, with $a,b\in\mathbb{R}$ and $a<b$. State if the following statements are true or false. If you true, explain why. If false, find an example of an interval that contradicts that statement.
   (a) $\sup(A)\in A$ and $\inf(A)\in A$.
   (b) If $\sup(A)\in A$ and $\inf(A)\in A$ then A is closed.
   (c) If A is closed then $\sup(A)\in A$ and $\inf(A)\in A$.
   (d) If $\sup(A)\notin A$ and $\inf(A)\notin A$ then A is open.
   (e) If A is open then $\sup(A)\notin A$ and $\inf(A)\notin A$.

4. Let A be a bounded interval in $\mathbb{R}$, i.e., A is a subset of $\mathbb{R}$ of either one of the following forms: $[a,b]$, or $]a,b[$, or $[a,b[$, or $]a,b]$, with $a,b\in\mathbb{R}$ and $a<b$. Show that $\inf A=a$, $\sup A=b$. When is the infimum (resp. maximum) of A a minimum (resp. a maximum)?

5. Let S be a subset of $\mathbb{R}$. Show that if $\sup(S)$, $\inf(S)$, $\max(S)$, $\min(S)$ exist, then they are unique.

6. Let $S\subseteq\mathbb{R}$ be the subset of the real numbers defined as $S:=\{x\in\mathbb{R}\,|x\in\mathbb{Q}\text{ and }x^3\geq 5\}$.
   (a) Show that S is not empty (i.e., exhibit an element of S).
   (b) Show that $\sqrt[3]{5}$ is a lower bound for S.
   (c) Show that $\inf(S)=\sqrt[3]{5}$. *(Hint: you should use the denseness of $\mathbb{Q}$ in this step).*

   Hint: you can use the fact that every real number has a unique real cubic root, and that $a^3\leq b^3$ if and only if $a\leq b$.

7. For each of the following sets, check if they are bounded or unbounded. When the set is bounded from above or below, give a few examples of lower and upper bounds, then compute the supremum and infimum and check if maximum and minimum exist.

(a) $A = \{x \in \mathbb{R} \mid x^2 \leq 2\}$.
(b) $B = \{x \in \mathbb{R} \mid x \in \mathbb{Q} \text{ and } x^2 \leq 2\}$.
(c) $C = \{(-1)^n + \frac{1}{n+1} | n \in \mathbb{N}\}$.

8. Let a be a real number. Assume that $a \geq 0$. Prove that $a = 0$ if and only if for any $\varepsilon > 0, a \leq \epsilon$.

9. Let L and L' be two real numbers. Prove that the following are equivalent:

   (a) $L = L'$;
   (b) for every $\epsilon > 0$, $|L - L'| \leq \epsilon$.

10. Let $S \subseteq \mathbb{R}$ be a non-empty subset. Assume that S is bounded from above and that $\sup(S) \notin S$. Show that the following fact holds: for every $\epsilon > 0$, $S \cap]\sup(S) - \epsilon, \sup(S)[$ is infinite (i.e., there are infinitely many elements of S in $]\sup(S) - \epsilon, \sup(S)[$).
Hint: In this problem, you can freely use that a finite non-empty set has both maximum and minimum.

11. Let S be a non-empty and bounded subset of $\mathbb{R}$. We define

$$S' := \{x \in \mathbb{R} | -x \in S\}.$$

Show that

   (a) If M is an upper bound of S, then $-M$ is a lower bound of S'.
   (b) If m is a lower bound of S, then $-m$ is an upper bound of S'.
   (c) $\sup(S) = -\inf(S')$.
   (d) $\inf(S) = -\sup(S')$.

12. Let S be the subset of $\mathbb{R}$ defined as

$$S := \bigcap_{n=1}^{\infty} [0, 1 + \frac{1}{n}]$$

Compute $m := \sup S$. Is m the maximum of S? (*Hint*: $x \in S \iff \forall n \in \mathbb{N}, x \in [0, 1 + \frac{1}{n}]$)

13. (Multiple choice) The subset S of $\mathbb{R}^2$ defined as[1]

$$S := \{(x, y) \in \mathbb{R}^2 \,| x = -y, -y = x - 1\}$$

is:

   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

14. (Multiple choice) The subset S of $\mathbb{R}^2$ defined as

$$S := \{(x, y) \in \mathbb{R}^2 \,| \sqrt{x^2 + (y + 3)^2} = 3\sqrt{x^2 + y^2}\}$$

is:

   (a) A point.
   (b) A line.
   (c) A circle.
   (d) Empty.

[1] In this exercise (x, y) does not denote an open interval between x and y, but it instead denotes the point of coordinates x and y in $\mathbb{R}^2$