Analysis I (English)
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Analysis 1 - Exercise Set 1

1. Real Numbers.

   (a) Explain the difference between a rational and an irrational number.

   (b) Classify the following numbers as rational, irrational, natural, integer. (A number may belong to more than one set).

   (i) -2

   (ii) $4\frac{1}{3}$

   (iii) $\sqrt{10}$

   (iv) 0

   (v) π

   (c) Show that $\sqrt{7}$ is an irrational number. (*Hint: assume that you can write* $\sqrt{7} = \frac{a}{b}$ *where* a *and* b *are integers where their greatest common divisor is* $\gcd(a, b) = 1$*. Then try to find a contradiction.*)

2. Trigonometry.

   Show that the following identities hold:

   (a) $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x \cos^2 x$

   (b) $\sin x + \sin y = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$

   (c) $\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$

   (d) $\cos x + \cos y = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$

   (e) $\cos x - \cos y = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$

   (*Hints: For (a) use the identities:* $(a^3+b^3) = (a+b)(a^2-ab+b^2)$, $(a+b)^2 = a^2+2ab+b^2$, $\sin^2 x + \cos^2 x = 1$. *For (b)-(e) use the identities*

   $$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

   $$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

   and compute $\sin(\alpha+\beta) + \sin(\alpha-\beta)$, $\sin(\alpha+\beta) - \sin(\alpha-\beta)$ *etc. then try to find* x *and* y *in terms of* α *and* β)

3. Trigonomentry.

   Simplify the following trigonometric expressions to obtain algebraic expressions (i.e., only involving sums, ratios, roots, etc.).
   Example: if $-1 \leq x \leq 1$, we have $\cos(\arcsin x) = \sqrt{1-x^2}$.

   (a) $\sin(\arcsin x)$, where $-1 \leq x \leq 1$

   (b) $\sin(\arccos x)$, where $-1 \leq x \leq 1$

   (c) $\tan(\arccos x)$, where $-1 \leq x \leq 1$

4. Arithmetic manipulations.

   Prove the following identities.

   (a)
   $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad n \geq 1.$$
   (*Hint: try to add the elements of the two finite sequences* $(1, 2, \ldots, n)$ *and* $(n, n-1, n-2, \ldots, 1)$ *term by term*)

   (b) Give an alternative proof, to the one given in the first lecture, for the equality
   $$a + a^2 + a^3 + \cdots + a^n = a \cdot \frac{1-a^n}{1-a}, \quad a \neq 1,\ n > 1.$$
   (*Hint: use the identity*
   $$(b^n - a^n) = (b-a)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}),$$
   and replace b with 1)

   (c)* For any $n \in \mathbb{N}$,
   $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$
   (*Hint: start by computing* $k^3 - (k-1)^3$ *for a natural number* k)

5. Equations.

   Solve the following equations:

   (a) $\frac{2x}{x+1} = \frac{2x-1}{x}$;
   (b) $x^4 - 3x^2 + 2 = 0$;
   (c) $3|x-4| = 10$.

6. Inequalities.

   Determine the solutions to the following inequalities.

   (a) $x^2 < 2x + 8$;
   (b) $x(x-1)(x-2) > 0$;
   (c) $\frac{2x-3}{x+1} \leq 1$;
   (d) $|x^2 - 1| \leq 1$.

7. Functions.

   Let
   $$f(x) = \frac{1}{1 - \frac{2}{1 + \frac{1}{1-x}}}$$
   (a) Find x, such that $f(x) = 3$.
   (b) Find the domain of f.

8. Functions.

   Do there exist functions f and g defined on $\mathbb{R}$ such that
   $$f(x) + g(y) = xy$$
   for all real numbers x and y? (*Hint: Try to evaluate for* $(x,y) = (0,0),\ (x,y) = (1,0),\ (x,y) = (0,1),\ (x,y) = (1,1)$)

9. Functions.

   Recall that a function $F: X \to Y$ is called injective if for every pair of elements a and b in X, $F(a) = F(b)$ implies that $a = b$; in other words, it is injective if distinct elements have distinct images.
   Consider now three functions $f, g, h : \mathbb{R} \to \mathbb{R}$. For each of the following statements, say whether that is true or false. If you think it is true, then provide a proof of that, or, else, if false, provide a counterexample.

   (a) $f \circ (g + h) = f \circ g + f \circ h$;
   (b) $(f + g) \circ h = (f \circ h) + (g \circ h)$;
   (c) $f \circ g = g \circ f$ if and only if $f = g$;
   (d) if $f \circ f$ is injective then f is injective;
   (e) if $f \circ g$ is injective then g is injective;
   (f) if $f \circ g$ is injective then f is injective.

10. Functions.

    Let $f: \mathbb{N} \to \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$ and $g(n) = n + 1$, respectively.

    (a) Compute $f \circ g$;
    (b) compute $g \circ f$;
    (c) compute g^m for every $m \in \mathbb{N}$.

11. * Functions.

    Consider the following set of $n + 1$ points in $\mathbb{R}^2$:

    $$S := \{(x_i, y_i) | i = 0, 1, \cdots, n\},$$

    where $x_i \neq x_j$ for $i \neq j$.
    Find a polynomial p of degree at most n such that $p(x_i) = y_i$ for any $i = 0, 1, \cdots, n$.
    You may use the following fact: If p is a sum of polynomials of degree n, then p is a polynomial of degree at most n.
    (*Hint: Try to first find degree n polynomials φ_i for $i = 0, 1, \cdots, n$ s.t. $\varphi_i(x_j) = \delta_{ij}$, where δ_{ij} is defined as follows:*

    $$\delta_{ij} = \begin{cases} 0 & \textit{for } i \neq j, \\ 1 & \textit{for } i = j. \end{cases}$$

    Using the polynomials φ_i, can you construct p?)

12. Sets.

    For sets E, F and D prove the following:

    (a) Commutativity: $E \cap F = F \cap E$ and $E \cup F = F \cup E$;
    (b) Associativity: $D \cap (E \cap F) = (D \cap E) \cap F$ and $D \cup (E \cup F) = (D \cup E) \cup F$;
    (c) Distributivity: $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$ and $D \cup (E \cap F) = (D \cup E) \cap (D \cup F)$;
    (d) De Morgan laws: $(E \cap F)^c = E^c \cup F^c$ and $(E \cup F)^c = E^c \cap F^c$.

13. Representations of numbers.

    Prove that a real number is a rational number if and only if the decimal representation becomes periodic.