Analysis I (English)
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Analysis 1 - Exercise Set 14

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State on which closed intervals the following function is integrable and compute the antiderivatives:
$$f(x) = \frac{1}{1-x^2}.$$

2. (a) Compute $\int_0^1 \frac{3x^2+3}{x^6+6x^4+9x^2+1}dx$. (*Hint: recall* $\arctan(t)'$)
   (b) Compute $\int_0^1 \frac{x^2+x-2}{x^3+2x^2+x+2}dx$.
   (c) Compute $\int_0^1 \frac{x^4+8x^3-4x^2-4x}{x^4-x^2-12}dx$.

3. Calculate the following formal integrals
   (a) $\int \frac{x(x^2+x-2)}{(x^2-x+2)(x-2)^2}dx$.
   (b) $\int \frac{3x+4}{1+x^2}\, dx$ (*Hint: recall* $(\arctan(x))'$).

4. Calculate the following integrals.
   (a) $\int_2^3 \frac{x^2+1}{x^2-1}\, dx$
   (b) $\int_2^3 \frac{\sqrt{x+1}}{x}\, dx$

5. Let $f\colon E \to \mathbb{R}$, $E =]a, b[$ an open interval. Assume that f is differentiable on E and that there exists a positive real number $C > 0$ such that $|f'(x)| \leq C$, $\forall x \in E$. Show that f is Lipschitz.

6. Compute the following improper integrals
   (a) $\int_1^{+\infty} \frac{x}{\sqrt{(x^2+5)^3}}dx$
   (b) $\int_0^{+\infty} \frac{\operatorname{arccot} x}{1+x^2}dx$
   (c) $\int_0^{+\infty} \left(x^3(8+x^4)^{-5/3} + 2xe^x\right) dx$

7. Find a recursive formula for $T_n = \int \cos^{2n}(x)dx \quad \forall n \in \mathbb{N}$.

8. Find the radius and interval of convergence for each series.
   (a) $\sum_{k=1}^{\infty} \frac{7k-22}{k^2(55k+94)}(x+2)^k$
   (b) $\sum_{k=0}^{\infty} \frac{4}{(k+5)!}(x+6)^k$
   (c) $\sum_{k=1}^{\infty} \frac{k\cdot k!}{k^4+3k^2}(x-1)^k$

9. State if the following integrals converge or diverge.
   (a) $\int_7^{+\infty} \frac{x}{(\sqrt{x^2-33})^2}dx$

(b) $\int_0^{+\infty} xe^{-x^2} dx$

10. For each of the following functions compute the Taylor series at 0, compute the radius of convergence and show that the function equals its Taylor series.

 (a) e^x

 (b) $\sin(x)$

11. Compute the Taylor series of the following functions at 0 and determine the radius of convergence. (*Hint: Recall the following Taylor series* $\log(y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(y-1)^k$ *and use the uniqueness of the Taylor series.*)

 (a) $f(x) = \log(1+x)$

 (b) $f(x) = \frac{1}{1+x}$

 (c) $f(x) = \frac{1}{(1-x)^2}$

12. Compute the following integrals using Taylor series. Your answer should be in the form of an explicit numerical series, but you do not need to compute the sum of the series.

 (a) $\int_0^{\frac{1}{2}} \log(1+2x^2)dx$

 (b) $\int_0^1 \sin(x^3)dx$

13. Compute the following improper integrals

 (a) $\int_{1/2}^{+\infty} \frac{1}{\sqrt{2x}(2x+1)}dx$

 (b) $\int_0^{+\infty} \frac{9x+8}{(x+2)(x^2+1)}dx$

14. Find the radius and interval of convergence for each series.

 (a) $\sum_{n=0}^{\infty} n(x-5)^n$

 (b) $\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^n (x+3)^n$

 (c) $\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$ (without checking the boundary points)

15. State if the following integrals converge or diverge. Motivate your answer.

 (a) $\int_1^{+\infty} \frac{\log x}{x^2} dx$

 (b) $\int_1^{+\infty} \frac{1}{\sqrt{1+x}} dx.$

16. For each of the following functions compute the Taylor series at 0, compute the radius of convergence and show that the function equals its Taylor series.

 (a) $\cos(x)$

 (b) $\sinh(x)$

17. Compute the Taylor series of the following functions at 0 and determine the radius of convergence.

 (a) $f(x) = \frac{15}{6-8x}$

 (b) $f(x) = 2xe^{4x^2}$

 (c) $f(x) = \operatorname{arccot} x$

18. Compute the following integrals using Taylor series. Your answer should be in the form of an explicit numerical series, but you do not need to compute the sum of the series.

    (a) $\int_0^{\frac{1}{3}} \frac{1}{1+x^3} dx$

    (b) $\int_0^1 \cosh(\sqrt{x}) dx$

    (c) $\int_0^1 x^4 \sin(x^3) dx$

Review Exercises

19. (Multiple Choice Question) The integral $\int_1^2 x^2 \log x \; dx$ is

    (a) $\frac{8}{3}\log(2) + \frac{7}{9}$

    (b) $\frac{8}{3}\log(2) - \frac{7}{9}$

    (c) $8\log(2) - \frac{7}{9}$

    (d) $8\log(2) + \frac{7}{9}$

20. (Multiple Choice Question) The integral

$$\int_{2/\pi}^{3/\pi} \frac{\sin\left(\frac{1}{x}\right)}{x^2} \; dx$$

is

    (a) $\frac{1}{2}$

    (b) $\frac{\sqrt{3}}{2}$

    (c) $-\frac{1}{2}$

    (d) $-\frac{\sqrt{3}}{2}$

21. The integral

$$\int_0^1 x^2 e^x \; dx$$

is

    (a) e

    (b) $2 - e$

    (c) $e + 2$

    (d) $e - 2$

22. For each of the following sequences defined by recursion, show the convergence and find the limit $\lim_{n\to\infty} a_n$.

    (a) $a_{n+1} = \frac{1+a_n}{2+a_n}, \quad a_0 = 1.$

    (b) $a_{n+1} = 1 + \frac{1}{2}a_n^2 - \frac{1}{2}a_n, \quad a_0 = \frac{3}{2}.$

23. **True/False** If the statement is true, you should prove it. If the statement is false, you should provide a counterexample.

    (a) If $(a_n) \subset \mathbb{R}$ is a Cauchy sequence, then (a_n^2) is also a Cauchy sequence.

    (b) If (a_n^2) is a Cauchy sequence then (a_n) is also a Cauchy sequence.

24. Check the convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$.

(b) $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n+3}$

(d) $\sum_{n=1}^{\infty} e^{-n}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$

(f) $\sum_{n=1}^{\infty} \frac{n(\sqrt{n}+1)}{\sqrt{n^3+2n^2-1}}$

(g) $\sum_{n=1}^{\infty} \frac{n^3}{(\log 3)^n}$

(h) $\sum_{n=1}^{\infty} \frac{(-5)^n}{4^{2n+1}(n+1)}$

(i) $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(n+2)!}$

25. **True/False.** If the statement is true, you should prove it. If the statement is false, you should provide a counterexample. Let (a_n) and (b_n) be numerical sequences.

(a) If $\lim_{n\to\infty} |a_n| = a$, then $\limsup_{n\to\infty} a_n = a$ and $\liminf_{n\to\infty} a_n = -a$.

(b) If $\limsup_{n\to\infty} |a_n| = 0$, then (a_n) converges to zero.

(c) If $\limsup_{n\to\infty} a_n = 0$, then $a_n \leq 0$ for all $n \in \mathbb{N}$.

(d) If $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} b_n = 0$, then $\limsup_{n\to\infty}(a_n - b_n) = 0$.

26. Check if the limit of the following sequences exist.

(a) $a_n = \frac{(-1)^n}{n+1}$

(b) $a_n = (-1)^n + (-1)^{n+2}$

(c) $a_n = \sin n\pi + \cos n\pi$

(d) $a_n = 2(-1)^n + \frac{n}{n+1}$