Analysis I (English)
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Analysis 1 - Exercise Set 13

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. State on which closed intervals the following functions are integrable and compute the antiderivatives.
   (a) $f(x) = e^x$;
   (b) $f(x) = \sinh(x)$;
   (c) $f(x) = (ax+b)^s$ with $s \in \mathbb{Z}$ and $a, b \in \mathbb{R} \setminus \{0\}$;
   (d) $f(x) = \cos(x)^3$;
   (e) $f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0; \end{cases}$
   (f) $f(x) = \cot(x)$, $\cot(x) := \frac{\cos(x)}{\sin(x)}$.
   (g) $f(x) = |x|^s$, $s > 0$.

2. Determine the number c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^2 + 3x + 2$ on the interval $[1, 4]$.

3. Let
$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \leq x \leq 3 \end{cases}$$
   Compute $\int_0^3 f(x)dx$.

4. **True/False:** If the statement is true you should prove it. If it is false you should give a counterexample. Let F be an anti-derivative of f on $[a, b]$.
   (a) If $f(x) \leq 0$ for all $x \in [a, b]$, then $F(x) \leq 0$ for all $x \in [a, b]$.
   (b) For all $x \in [a, b]$, we have $F(x) = \displaystyle\int_a^x f(t)\, dt$.

5. Show that:
   (a) if $f : [-a, a] \to \mathbb{R}$ is an integrable odd function then $\int_{-a}^a f(x)dx = 0$;
   (b) if $f : [-a, a] \to \mathbb{R}$ is an integrable even function then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

6. Calculate the following formal integrals.
   (a) $\int \sin(x)^2 dx$
   (b) $\int \arcsin(x)\, dx$
   (c) $\int \frac{\sinh(x)}{e^x+1}\, dx$

(d) $\int e^{ax}\cos(bx)\,dx$ $\quad (a \neq 0)$, (*Hint: apply integration by parts multiple times until you see a pattern.*)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with a period $T > 0$. Let F be defined by

$$F(x) = \int_0^x f(t)\,dt.$$

Show that F is periodic with period T if and only if

$$\int_0^T f(t)\,dt = 0.$$

8. Calculate the following integrals.

(a) $\int_{\pi^2/16}^{\pi^2/9} \cos(\sqrt{x})\,dx$

(b) $\int_0^{\pi^{1/2017}} \sin(\sin(x^{2017}))\cos(x^{2017})\,x^{2016}\,dx$

9. **True/False:** Let $I \subset \mathbb{R}$ be an open non-empty and bounded interval and let $f\colon I \to \mathbb{R}$ be a continuous function. Let $[a,b] \subseteq I$. If the statement is true you should prove it. If the statement is false you should give a counter example.

(a) If $\displaystyle\int_a^b f(x)\,dx = 0$, then f has a zero $[a,b]$.

(b) If $\displaystyle\int_a^b f(x)\,dx \geq 0$, then $f(x) \geq 0$ for all $x \in [a,b]$.

(c) If $f(x) < 0$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,dx < 0$.

10. Calculate the following formal integrals.

(a) $\int \frac{\sin(x)}{\cos(x)^3}\,dx$

(b) $\int x^2\cos(x)\,dx$

(c) $\int x\log x\ dx$

(d) $\int \frac{1}{\sqrt{4-3x^2}}\,dx$ (*Hint: recall* $(\arcsin x)'$)

(e) $\int (2x+2)e^{x^2+2x+3}\,dx$

(f) $\int \frac{x^2+1}{x^3+3x}\,dx$

(g) $\int \frac{x^3}{(1+x^4)^{\frac{1}{3}}}\,dx$

(h) $\int \frac{\sin(\log(x))}{x}\,dx$

(i) $\int (2x+5)(x^2+5x)^7\,dx$

11. Let f be a continuous function on a closed interval $[a,b]$. Show that $|f|$ is integrable and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

12. Calculate the following integral:

$$\int_0^{\pi/2} \sin(x)^5\,dx$$

(*Hint: remember* $\cos^2(x) + \sin^2(x) = 1$ *and try to make a substitution of the form* $u = \cos x$).

13. Prove that if $f, g : I \to \mathbb{R}$ are square-integrable continuous functions over I[1], then

$$\left|\int_I f(x)g(x)dx\right| \leq \left(\int_I f(x)^2 dx\right)^{\frac{1}{2}} \left(\int_I g(x)^2 dx\right)^{\frac{1}{2}}$$

This is known as the Cauchy–Schwarz inequality. (*Hint: If at least one of the functions is zero, then there is nothing to prove. Suppose both are non-zero. Evaluate* $\int_I (f(x) - \lambda g(x))^2 dx$ *and choose* $\lambda \in \mathbb{R}$ *carefully.*)

Revision Exercises

Questions 14-17 are multiple choice questions. In each of the questions you should explain why your choice is correct.

14. The equation $x(e^x - e^{-x}) - e^x = 0$

    (a) has no solution belonging to the interval $[0, +\infty[$.

    (b) has exactly one real solution.

    (c) has no solution belonging to the interval $]-\infty, 0[$.

    (d) has at least two real solutions.

15. Let the one-to-one function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sinh(\sinh(x))$ and let $a = f(1)$. Then the derivative of the inverse function $g = f^{-1}$ at a is

    (a) $g'(a) = \frac{1}{\cosh(\sinh(1))}$.

    (b) $g'(a) = \frac{1}{\cosh(\sinh(a))\cosh(a)}$.

    (c) $g'(a) = \frac{1}{\cosh(\sinh(1))\cosh(1)}$.

    (d) $g'(a) = \frac{1}{\cosh(\sinh(a))\cosh(1)}$.

16. The limit $\lim\limits_{x\to 0} \frac{e^{|x|}-1-|x|}{x^2}$ is

    (a) 0.

    (b) 1.

    (c) $\frac{1}{2}$.

    (d) Does not exist.

17. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{e^x - 1}$. The truncated expansion of order 2 of f around $x = 0$ is

    (a) $f(x) = 1 + x + x^2 + x^2\epsilon(x)$

    (b) $f(x) = 2x + x^2 + x^2\epsilon(x)$

    (c) $f(x) = 1 + x + \frac{1}{2}x^2 + x^2\epsilon(x)$

    (d) $f(x) = 1 + x + 2x^2 + x^2\epsilon(x)$

    where $\lim\limits_{x\to 0} \epsilon(x) = 0$.

Questions 18-22 are true or false questions. If the statement is true, you should prove it. If it is false, you should give a counter example.

18. For $a < b$ in $\mathbb{R}$, let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on $]a, b[$. If $f(a) = f(b) = 0$, then there exists $c \in]a, b[$ such that $f''(c) = 0$.

[1] Meaning that f^2 and g^2 are integrable.

19. Define $f : \mathbb{R} \to \mathbb{R}, f(x) = \int_0^x |t| dt$. Then $f'(x) = x \quad \forall x \in \mathbb{R}$.

20. Let $f : I \to \mathbb{R}$ be differentiable over an open interval $I \subset \mathbb{R}$. Then the derivative of f at the point $y \in I$ satisfies

$$f'(y) = \lim_{x \to 0} \frac{f(y+x) - f(y)}{x}.$$

21. Let $f :]0,1[\to \mathbb{R}$ be a differentiable function on $]0,1[$. Then the function $f' :]0,1[\mapsto \mathbb{R}$ is differentiable on $]0,1[$.

22. A function $f : \mathbb{R} \to \mathbb{R}$ s.t. $\forall \varepsilon > 0$ and $\forall x, y \in \mathbb{R}$ with the following property

$$|x - y| \leq 2\varepsilon \Rightarrow |f(x) - f(y)| \leq \varepsilon$$

is continuous on $\mathbb{R}$.