Analysis I (English)
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Fall Semester 2021–2022

Analysis 1 - Exercise Set 12

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Find the Taylor expansion of order n at $x = 0$ of the following functions.

   (a) $f(x) = e^{\sin(x)}, \quad n = 4$

   (b) $f(x) = \sqrt{1 + \sin(x)}, \quad n = 3$

2. For each one of the following functions, determine whether the funciton is differentiable at $x = 0$. If yes, also compute the derivative at $x = 0$:

   (a) $f(x) = \begin{cases} x+1, & x \geq 0 \\ x, & x < 0 \end{cases};$

   (b) $f(x) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases};$

   (c) $f(x) = \begin{cases} \frac{\sin(x)-x}{x}, & x > 0 \\ 0, & x = 0 \\ \frac{\cos(x)-\frac{x^2}{2}}{x^4}, & x < 0 \end{cases}.$

3. Find the vertical and horizontal asymptotes of the function $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

4. State if the following statements are true or false. Let $f, g : I \to \mathbb{R}$ be two convex functions, where $I \subset \mathbb{R}$ is some interval. If it is true, prove it. If not, give a counter example.

   (a) The function $f + g$ is convex.

   (b) The function $h = f \cdot g$ is convex.

   (c) If g is increasing then the function $h = g \circ f$ is convex.

5. Consider $f\colon]a, b[\mapsto \mathbb{R}$. Let $g\colon]c, d[\mapsto \mathbb{R}$ be the restriction to f to the interval $]c, d[\subset]a, b[$, i.e., $f(x) = g(x) \quad \forall x \in]c, d[$. Show that

   (a) If $f \in C^n(]a, b[, \mathbb{R})$ then $g \in C^n(]c, d[, \mathbb{R})$.

   (b) If f is Lipschitz continuous, then g is Lipschitz continuous.

6. Find the local extrema and the absolute maximum and minimum of $f(x) = x^2 - \left|x + \frac{1}{4}\right| + 1$ in $[-1, 1]$.

7. Let $a, b \in \overline{\mathbb{R}}$, $a < b$. Let $f :]a, b[\to \mathbb{R}$ be a differentiable function. State if the following statements are true or false. If it is true, prove it. If not, give a counter example.

   (a) If f' is bounded, then f is Lipschitz continuous with Lipschitz contstant $k = \sup_{x\in]a,b[} |f'(x)|$.

   (b) If f is Lipschitz continuous, then it is uniformly continuous.

   (c) If f' is bounded then f is uniformly continuous.

8. Study the function $f(x) = \dfrac{x}{x^2-1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

9. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = e^x$. Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

10. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function on $[a,b] \subset D(f)$, $a < b$, and differentiable on $]a,b[$.

    (a) If $f'(x) \geq 0$ for all $x \in]a,b[$, then f is increasing on $[a,b]$.

    (b) If f is increasing on $[a,b]$, then $f'(x) \geq 0$ for all $x \in]a,b[$.

    (c) If f is strictly increasing on $[a,b]$, then $f'(x) > 0$ for all $x \in]a,b[$.

    (d) If $f'(x) > 0$ for all $x \in]a,b[$, then f is strictly increasing on $[a,b]$.

    (e) If $\lim\limits_{x \to a^+} f'(x) = \ell$ exists, then f is differentiable from right at a and the right derivative is $f'_d(a) = \ell$.

11. Using the definition of convex functions, show that the function $f(x) = x^2$ is convex.

12. Find the local extrema and the absolute maximum and minimum of $f(x) = (x-1)^2 - 2\,|2-x|$ in $]2,3[$

13. Show that the following functions are Lipschitz continuous on the given domain.

    (a) $f(x) = |x|$, $f : \mathbb{R} \to \mathbb{R}$.

    (b) $f(x) = \sqrt{x}$, $f : [a, \infty[\to \mathbb{R}$, $a > 0$.

    (c) $f(x) = x^n$, $f : [a,b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

14. Study the function $f(x) = \dfrac{3x^2 - x}{2x-1}$ and sketch its graph (domain, range, symmetries, roots, continuity, differentiability, stationary points, extrema, convexity, inflection points, asymptotes).

15. (a) Show that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

    (b) Let $f : [0,1] \to \mathbb{R}$ be defined by

    $$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ \frac{1}{2}, & x \notin \mathbb{Q}. \end{cases}$$

    Compute the upper and lower Darboux sums for the regular partitions σ_{2n}. Is f integrable?

16. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathbb{R}$ with $g'(x) \neq 0$ for all $x \in \mathbb{R}$.

    (a) If $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$, then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$.

    (b) If $\lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

17. Using the definition of convex functions, show that the function $f(x) = \frac{1}{x}$, $\quad x \in]0, +\infty[$ is convex.

18. Let the function $f(x) : [-4, 4]\backslash\{2\}$ be defined by

$$f(x) = \frac{x^2}{x+2}$$

then

(a) f attains its maximum at $x = -4$ and its minimum at $x = 0$.

(b) f attains its maximum at $x = -4$ and has a local minimum at $x = 0$.

(c) f has a local maximum at $x = -4$ and attains its minimum at $x = 0$.

(d) f does not have a maximum or a minimum on $[-4, 4]\backslash\{2\}$.

19. Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$. Let $f :]a, b[\to \mathbb{R}$ be a differentiable function. State if the following statements are true or false. If it is true, prove it. If not, give a counter example.

(a) If f is Lipschitz continuous, then f' is bounded.

(b) The function $f(x) = \sqrt{x}$ defined on $]0, +\infty[$ is Lipschitz continuous.

(c) If f is uniformly continuous, then f' is bounded.

(d) If f is uniformly continuous, then it is Lipschitz continuous.

20. (a) Show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 2x^2 + 3x - 1$ Compute the upper and lower Darboux sums for the regular partitions σ_n. Is f integrable?

21. State if the following statements are true or false. If it is true, prove it. If not, give a counter example. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) If $f(x) = x + e^x$, then $(f^{-1})'(1) = 1 + \frac{1}{e}$.

(b) If f is differentiable on the interval $I \subset \mathbb{R}$, then f' is continuous on I.