Analysis I (English)
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EPFL

Analysis 1 - Exercise Set 11

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. (a) Let $a, b \in \mathbb{Z}$, $b > 0$. Show that $\sqrt[b]{x^a} = e^{\frac{a}{b}\log(x)}$ for all real numbers $x > 0$.
   (b) Compute the derivative of the following functions:
      (i) $f(x) = x^a : \mathbb{R}_+^* \to \mathbb{R}$, $a \in \mathbb{R}$; show that f is strictly increasing when $a > 0$ and strictly decreasing when $a < 0$;
      (ii) $f(x) = a^x : \mathbb{R} \to \mathbb{R}_+^*$, $a \in \mathbb{R}_+^*$; show that f is strictly increasing when $a > 1$ and strictly decreasing when $a < 1$;
      (iii) $f(x) = \log_a(x) : \mathbb{R}_+^* \to \mathbb{R}$, $a \in \mathbb{R}_+^*$; show that f is strictly increasing when $a > 1$ and strictly decreasing when $a < 1$;

2. State if the following are true or false.
   (a) If $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable and has two roots, that is, there exist $x \neq y \in \mathbb{R}$, $f(x) = 0 = f(y)$, then f' has at least one root.
   (b) The function $f(x) = \frac{\sin(x^2-2)}{e^{3x+1}+\sqrt{2x}}$ has a critical value in $]\sqrt{2}, \sqrt{2+\pi/2}[$.

3. Calculate the following limits:
   (a) $\lim\limits_{x\to 0} \big(1+\sin(x)\big)^{1/x}$
   (*Hint: Write* $\big(1+\sin(x)\big)^{1/x} = e^{\left(\frac{1}{x}\log\left(1+\sin(x)\right)\right)}$ *and first calculate the limit of the exponent.*)
   (b) $\lim_{x\to\sqrt{3}} \frac{x^x - \sqrt{3^{\sqrt{3}}}}{x-\sqrt{3}}$. Remember that $x^x = e^{x\log(x)}$.

4. Find the Taylor expansion of order 5 at $x = 0$ of the following functions.
   (a) $f(x) = \sin(x)$
   (b) $f(x) = \log(1+x)$
   (c) $f(x) = \tan(x)$
   (d) $f(x) = \arccos(x)$
   (e) $f(x) = \sinh(x)$
   (f) $f(x) = \log(\cos(x))$

5. Use Taylor expansion to find the following limits.
   (a) $\lim_{x\to 0} \dfrac{x - \frac{x^3}{6} - \sin(x)}{x^5}$
   (b) $\lim_{x\to 0} \dfrac{e^x + \sin(x) - \cos(x) - 2x}{x - \log(1+x)}$

(c) $\lim_{x\to 0} \dfrac{x\sin(\sin(x)) - \sin(x)^2}{x^6}$

(d) $\lim_{x\to 0} \frac{\sqrt[3]{1-x}-1}{\sqrt[4]{1-x}-1}$

6. For a complex number of the form $e^{ix},\ x \in \mathbb{R}$, we defined

$$\cosh(ix) := \frac{e^{ix}+e^{-ix}}{2}, \quad \sinh(ix) := \frac{e^{ix}-e^{-ix}}{2}.$$

   (a) Compute the complex numbers $\cosh(ix)$, $\sinh(ix)$;

   (b) For each of the functions $\cosh(x), \sinh(x), \tanh(x), \coth(x)$ compute the derivative and the domain of the derivative.
   Which of these functions are invertible on the domain $\mathbb{R}$? which on $\mathbb{R}_+^*$? Recall that

$$\cosh(x) = \frac{e^x+e^{-x}}{2},\ \sinh(x) = \frac{e^x-e^{-x}}{2},\ \tanh(x) = \frac{\sinh(x)}{\cosh(x)},\ \coth(x) = \frac{1}{\tanh(x)}.$$

   (c) Compute the derivatives of the inverses of the functions in (b) and their domains.

7. For the following functions, find the stationary points and discuss whether they are points at which the function attains a local maximum or minimum.

   (a) $f(x) = x\log^2(x)$ in $]0,+\infty[$

   (b) $f(x) = 2\sin(x) + \cos(2x)$ in $[-\frac{1}{10}, \frac{1}{15}]$

8. Calculate the following limits:

   (a) $\lim\limits_{x\to+\infty} x\big(\tanh(x)-1\big)$

   (b) $\lim\limits_{x\to+\infty} \frac{e^x}{x^n}$ where $n\in\mathbb{N}$. First find the limit using $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ and then using l'Hopital's rule.

   (c) $\lim\limits_{x\to 0^+} x^n\log(x)$

   (d) $\lim\limits_{x\to\infty} \frac{x^n}{\log(x)}$

9. Calculate the derivative f' of the function $f(x) = \log_3\big(\cosh(x)\big)$ and give the domain of f and f'.

10. State if the following are true or false.

   (a) The function $f : [0,+\infty[\to [1,+\infty[$ defined by $f(x) = x^3 - x + e^x$ is invertible.

   (b) Let $a, b \in \mathbb{R}$, $a < b$. Given a continuous function $f : (a,b) \to \mathbb{R}$ which is not monotone, there exists a point $x_0 \in (a,b)$ at which f admits a local minimum.

11. Calculate the following limit $\lim_{x\to 2} \dfrac{\log(x-1)}{x-2}$

12. Let the function $f :]-\pi/2, \pi/2[\to \mathbb{R}$ be defined by $f(x) = \log(1+\sin(x))$. What is the Taylor expansion of order 3 of f at $x = 0$.

   (a) $x - \frac{x^3}{6} + \cdots$

   (b) $x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$

   (c) $x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

(d) $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

Revision Exercises

13. Consider the bijective function $f :]1, \infty[\to]-\infty, -2[$ defined as $f(x) = \log(x) - 2x$. Then the derivative of the inverse function $f^{-1}(y)$ at $y = -2$ is

    (a) $(f^{-1})'(-2) = -1$
    (b) $(f^{-1})'(-2) = 1$
    (c) $(f^{-1})'(-2) = -\frac{2}{5}$
    (d) $(f^{-1})'(-2) = \frac{2}{5}$

14. For which of the following item you can prove, using the intermediate value theorem, that there exists a c in I such that $f(c) = k$.

    (a) $f(x) = \frac{x^2+8}{x}, \quad k = 5, \quad I = [1, 3]$
    (b) $f(x) = x^2 + x + 1, \quad k = 2, \quad I = [-2, 3]$
    (c) $f(x) = \frac{1}{2x-1}, \quad k = 0, \quad I = [0, 1]$
    (d) $f(x) = \frac{10}{x^2+1}, \quad k = 8, \quad I = [0, 1]$

15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)$$
then.

    (a) $f'(x) = \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{1+x^2}$
    (b) $f'(x) = \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{(1+x^2)^2}$
    (c) $f'(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{(1+x^2)^2}$
    (d) $f'(x) = \sin\left(\frac{1-x^2}{1+x^2}\right)\frac{-4x}{1+x^2}$

16. Let the function $f : \mathbb{R}\backslash\{1/2\} \to \mathbb{R}$ be defined by
$$f(x) = \frac{2x^3 + x^2 + 2x + 1}{2x - 1}$$
then

    (a) f has at least one root in $[-1, 0]$
    (b) f has at least one root in $[0, 1]$
    (c) f has at least two roots in $[-1, 1]$
    (d) f has no roots

17. Let the function $f :]-1, 1[\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = \frac{x\log(1+x)}{\cos(x)-1}$. Let $g :]-1, 1[\to \mathbb{R}$ be an extension of f that is continuous at 0. Then

    (a) g exist and $g(0) = -2$.
    (b) g exist and $g(0) = 2$.
    (c) g exist and $g(0) = 0$.
    (d) f does not have a continuous extension at 0.

(*Hint: Note that* $\log(1) = 0$ *so* $\log(1+x) = \log(1+x) - \log(1)$. *Then use the definition of derivative.*)

18. Check if the following series are convergent

    (a) $\sum_{n=0}^{+\infty} \frac{n}{(n+1)!}$

    (b) $\sum_{n=0}^{+\infty} \frac{3n+1}{n^2(n+1)^2}$

19. For what values of $t > 0$ the following series converges? If convergent, what is the limit?

$$\sum_{n=0}^{+\infty} \left(\frac{t}{t+1}\right)^{2n}$$