Analysis I (English)
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Analysis 1 - Exercise Set 10

Remember to check the correctness of your solutions whenever possible.

To solve the exercises you can use only the material you learned in the course.

1. Calculate the derivative f' of the function f and give the domain of f and f'.
   (a) $f(x) = \dfrac{5x+2}{3x^2-1}$
   (b) $f(x) = \tan(x)$
   (c) $f(x) = x\sin(x) + \frac{\cos(x)^2}{x^2+2}$

2. Let I be some open interval and $f : I \to \mathbb{R}$ be a function that is continuous at $x_0 \in I$. Prove that if $f(x_0) > 0$ then $f(x) > 0$ on some open interval containing x_0.

3. Prove the quotient rule for derivatives:
   if $f : I \to \mathbb{R}$, $f(x) = \frac{g(x)}{h(x)}$, $x_0 \in I$ and both g and h are differentiable at x_0, with $h(x_0) \neq 0$, then, $f'(x_0) = \frac{g'(x_0)h(x_0) - g(x_0)h'(x_0)}{h(x_0)^2}$.

4. For each of the following functions, find the inverse function and the derivative of the inverse function.
   (a) $f(x) = \cos x$, $x \in \,]0, \pi[$.
   (b) $f(x) = \tan x$, $x \in \,]-\frac{\pi}{2}, \frac{\pi}{2}[$.

5. Calculate the derivative f' of the function f and give the domain of f and f'.
   (a) $f(x) = \dfrac{x^2}{\sqrt{1-x^2}}$
   (b) $f(x) = \sin(x)^2 \cdot \cos(x^2)$
   (c) $f(x) = \sqrt{\sin\left(\sqrt{\sin(x)}\right)}$
   (d) $f(x) = \sin(x)\log(\sin(x))e^{\cos(x)}$

6. For $x \in \mathbb{R}$, e^x has been defined as $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Hence, this definition gives rise to a function $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. Prove the following properties of the exponential e^x:
   (a) $e^0 = 1$;
   (b) $e^x \cdot e^y = e^{x+y}$; [For this part of the exercise you can assume the following result:
   Let $(a_n), (b_n)$ be sequences. Assume that both $\sum_{i=0}^{\infty} a_i, \sum_{i=0}^{\infty} b_i$ converge to a finite limit, and, moreover, that at least one of $\sum_{i=0}^{\infty} a_i, \sum_{i=0}^{\infty} b_i$ converges absolutely. Then the sequence (z_n), $z_n := \sum_{l=0}^{n} a_l b_{n-l}$ satisfies
   $$\sum_{i=0}^{\infty} a_i \cdot \sum_{i=0}^{\infty} b_i = \sum_{i=0}^{\infty} z_i.$$

(c) $e^{-x} = \frac{1}{e^x}$

(d) e^x is a strictly increasing function of x; e^{-x} is a strictly decreasing function of x;

(e) Use the definition of $\log(x)$ as inverse of the function e^x to show that

(i) $\log(ab) = \log(a) + \log(b)$ for all $a, b > 0$.

(ii) $\log(a^b) = b\log(a)$ for all $a > 0$ and all $b \in \mathbb{R}$.

(iii) $\log(x)$ is a strictly increasing function of x.

7. For each function, calculate $f^{(n)}$, the n-th order derivative of f.

(a) $f(x) = x^m \quad (m \in \mathbb{Z})$

(b) $f(x) = \sin(2x) + 2\cos(x)$

(c) $f(x) = \log(x)$

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. State if the following are true or false.

(a) f even $\quad \Rightarrow \quad f'$ odd,

(b) f odd $\quad \Rightarrow \quad f'$ even,

(c) f' even $\quad \Rightarrow \quad f$ odd,

(d) f periodic $\quad \Rightarrow \quad f'$ periodic.

9. Find maximum and minimum of the following functions

(a) $f(x) = x$ in $[-\pi, \pi]$

(b) $f(x) = \sin(x) + \cos(x)$ in $[0, \frac{2\pi}{3}]$

10. Calculate $(g \circ f)'(0)$ for the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined by

(a) $f(x) = 2x + 3 + (e^x - 1)\sin(x)^7\cos(x)^4$ and $g(x) = \log(x)^3$.

(b) $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) + 2x\,, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$ and $g(x) = (x-1)^4$.

11. Calculate the derivative f' of the function f and give the domain of f and f'.

(a) $f(x) = \sqrt[5]{\left(2x^4 + e^{-(4x+3)}\right)^3}$

(b) $f(x) = e^{\sqrt[3]{\log(4x)^2}}$

(c) $f(x) = \log\left(4^{\sin(x)}\right)e^{\cos(4x)}$

12. State if the following are true or false.

(a) If f is differentiable at $a \in \mathbb{R}$, Then there is $\delta > 0$ such that f is continuous on $]a-\delta, a+\delta[$.

(b) If f is differentiable from left and right at $a \in \mathbb{R}$, then f is differentiable at a.

(c) If f is differentiable on $\mathbb{R}$, then $g(x) = \sqrt{f^2(x)}$ is differentiable on $\mathbb{R}$.

13. For each of the following functions, find the inverse function. Find the derivative of the inverse function once by direct calculation and once by the inverse function derivative.

(a) $f(x) = \sqrt{x^2 + 9}$, $x \geq 0$.

(b) $f(x) = \frac{1}{1+x}$, $x \neq -1$.

14. Show that the derivative of the function

$$f(x) = \begin{cases} x^2 \cos \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at $x = 0$ is zero and then find $f'(x)$. Is f' continuous?

15. Find maximum and minimum of the following functions

(a) $f(x) = x^2 - 5$ in $[-\pi, \pi]$

(b) $f(x) = \sqrt[3]{(x-1)(x-2)^2}$ in $[1 + \frac{1}{10}, 2 - \frac{1}{10}]$

16. Calculate f'

(a) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{\cos x}{2+\sin(\log x)}$

(b) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \log(a|x|)$, $a > 0$

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x^2 \sin x}$

17. State if the following are true or false.

(a) If $f : E \to F$ is strictly increasing and bijective, then the inverse function is strictly increasing.

(b) If $f(x) = x^2 - 2x$, then $(f \circ f)'(1) = 0$.

(c) If a car traveled 210 km in 3 hours, then the speedometer must have read 70 km/h at least once.

18. Find the inverse of the following functions if they exist. Give the domain of both functions.

(a) $f(x) = \left(\frac{1}{8}\right)^{1-x}$

(b) $f(x) = \log x - \log 2x + \log 3x$

19. Compute

$$\lim_{x \to +\infty} \log(x).$$

20. The limit

$$\lim_{n \to \infty} \frac{\cos(n)}{\log(n)}$$

is

(a) 0

(b) -1

(c) $+1$

(d) $+\infty$