Numerous Proofs of $\zeta(2) = \frac{\pi^2}{6}$

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Abstract

In this talk, we will investigate how the late, great Leonhard Euler originally proved the identity $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ way back in 1735. This will briefly lead us astray into the bewildering forest of complex analysis where we will point to some important theorems and lemmas whose proofs are, alas, too far off the beaten path. On our journey out of said forest, we will visit the temple of the Riemann zeta function and marvel at its significance in number theory and its relation to the problem at hand, and we will bow to the uber-famously-unsolved Riemann hypothesis. From there, we will travel far and wide through the kingdom of analysis, whizzing through a number N of proofs of the same original fact in this talk's title, where N is not to exceed 5 but is no less than 3. Nothing beyond a familiarity with standard calculus and the notion of imaginary numbers will be presumed.

Note: These were notes I typed up for myself to give this seminar talk. I only got through a portion of the material written down here in the actual presentation, so I figured I'd just share my notes and let you read through them. Many of these proofs were discovered in a survey article by Robin Chapman (linked below). I chose particular ones to work through based on the intended audience; I also added a section about justifying the $\sin(x)$ "factoring" as an infinite product (a fact upon which two of Euler's proofs depend) and one about the Riemann Zeta function and its use in number theory. (Admittedly, I still don't understand it, but I tried to share whatever information I could glean!)

http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf

The Basel Problem was first posed by the Italian mathematician Pietro Mengoli in 1644. His question was simple:

$$\text{What is the value of the infinite sum} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad ?$$

Of course, the connection to the Riemann zeta function came later. We'll use the notation for now and discuss where it came from, and its significance in number theory, later. Presumably, Mengoli was interested in infinite sums, since he had proven already not only that the harmonic series is divergent, but also

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$$

and that Wallis' product

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \frac{\pi}{2}.$$

is correct. Let's tackle the problem from the perspective of Euler, who first "solved" the problem in 1735; at least, that's when he first announced his result to the mathematical community. A rigorous proof followed a few years later in 1741 after Euler made some headway in complex analysis. First, let's discuss his original "proof" and then fill in some of the gaps with some rigorous analysis afterwards.

Theorem 1. $\zeta(2) = \frac{\pi^2}{6}$

Proof #1, Euler (1735). Consider the Maclaurin series for $\sin(\pi x)$

$$\sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^{2n+1} x^{2n+1}}{(2n+1)!} =: p(x)$$

We know that the roots of $\sin(\pi x)$ are the integers $\mathbb{Z}$. For *finite* polynomials $q(x)$, we know that we can write the function as a product of linear factors of the form $(1 - \frac{x}{a})$, where $q(a) = 0$. Euler conjectured that the same trick would work here for $\sin(\pi x)$. Assuming, for the moment, that this is correct, we have

$$\begin{aligned}\hat{p}(x) :&= \pi x \left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{1}\right)\left(1 - \frac{x}{2}\right)\left(1 + \frac{x}{2}\right)\left(1 - \frac{x}{3}\right)\left(1 + \frac{x}{3}\right)\cdots \\ &= \pi x \left(1 - \frac{x^2}{1}\right)\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{x^2}{9}\right)\cdots = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)\end{aligned}$$

Notice that we have included the leading x factor to account for the root at 0, and the π factor to make things work when $x = 1$. Now, let's examine the coefficient of x^3 in this formula. By choosing the leading πx term, and then $-\frac{x^2}{n^2}$ from one of the factors and 1 from all of the other factors, we see that

$$\hat{p}(x)[x^3] = -\pi \left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots\right) = -\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Comparing this to the coefficient from the Maclaurin series, $p(x)[x^3] = -\frac{\pi^3}{6}$, we obtain the desired result!

$$-\frac{\pi^3}{6} = -\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

So why is it that we can "factor" the function $\sin(\pi x)$ by using what we know about its roots? We can appeal to the powerful Weierstrass factorization theorem which states that we can perform this root factorization process for any entire function over $\mathbb{C}$.

Definition 2. *A function $f : D \to \mathbb{C}$ is said to be* holomorphic *on a domain $D \subseteq \mathbb{C}$ provided $\forall z \in D\ \exists \delta$ such that the derivative*

$$f'(z_0) = \lim_{y \to z_0} \frac{f(y) - f(z_0)}{y - z_0}$$

exists $\forall z_0 \in B(z, \delta)$. A function f is said to be entire *if it is holomorphic over the domain $D = \mathbb{C}$.*

There are two forms of the theorem, and they are essentially converses of each other. Basically, an entire function can be decomposed into factors that represent its roots (and their respective multiplicities) and a nonzero entire function. Conversely, given a sequence of complex numbers and a corresponding sequence of integers satisfying a specific property, we can construct an entire function having exactly those roots.

Theorem 3 (Weierstrass factorization theorem)**.** *Let f be an entire function and let $\{a_n\}$ be the nonzero zeros of f repeated according to multiplicity. Suppose f has a zero at $z = 0$ of order $m \geq 0$ (where order 0 means $f(0) \neq 0$). Then $\exists g$ an entire function and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

where

$$E_n(y) = \begin{cases} (1-y) & \text{if } n = 0, \\ (1-y)\exp\left(\frac{y^1}{1} + \frac{y^2}{2} + \cdots + \frac{y^n}{n}\right) & \text{if } n = 1, 2, \ldots \end{cases}$$

This is a direct generalization of the Fundamental Theorem of Algebra. It turns out that for $\sin(\pi x)$, the sequence $p_n = 1$ and the function $g(z) = \log(\pi)$ works. Here, we attempt to briefly explain why this works. We start by using the functional representation

$$\sin(\pi z) = \frac{1}{2i}\left(\mathrm{e}^{i\pi z} - \mathrm{e}^{-i\pi z}\right)$$

and recognizing that the zeros are precisely the integers $n \in \mathbb{Z}$. One of the Lemmas that provides the bulk of the proof of the Factorization Theorem requires that the sum

$$\sum_{n=-\infty}^{\hat{+\infty}} \left(\frac{r}{|a_n|}\right)^{1+p_n} < +\infty$$

be finite for all $r > 0$, where the hat $\hat{}$ indicates the $n = 0$ term is removed. Since $|a_n| = n$, we see that $p_n = 1 \; \forall n$ suffices, and so

$$\sin(\pi x) = z \exp(g(z)) \hat{\prod_{n=-\infty}^{\infty}} \left(1 - \frac{z}{n}\right) \exp(z/n)$$

and we can cancel the terms $\exp(z/n) \cdot \exp(-z/n)$ and combine the factors $(1 \pm z/n)$ to say

$$f(z) := \sin(\pi x) = z \exp(g(z)) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) =: \exp(g(z)) z h(z)$$

for some entire function $g(z)$. Now, a useful Lemma states that for analytic functions f_n and a function $f = \prod_n f_n$, we have

$$\sum_{k=1}^{\infty} \left[f_k'(z) \prod_{n \neq k} f_n(z) \right]$$

which immediately implies

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f(z)}$$

This allows us to write

$$\begin{aligned} \pi \cot(\pi z) = \frac{f'(z)}{f(z)} &= \frac{g'(z) \exp(g(z))}{\exp(g(z))} + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - z^2/n^2} \\ &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \end{aligned}$$

and according to previous analysis, we know

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

which is based on integrating $\int_\gamma \pi(z^2 - a^2)^{-1} \cot(\pi z) \, dz$ for a non-integral and where γ is an appropriately chosen rectangle. As we enlarge γ, the integral goes to 0, and we get what we want. This means $g(z) = c$ for some c. Putting this back into the formula above, we have

$$\frac{\sin(\pi z)}{\pi z} = \frac{\mathrm{e}^c}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

for all $0 < |z| < 1$. Taking $z \to 0$ tells us $\mathrm{e}^a = \pi$, and we're done!

Remark 4. *Notice that plugging in $z = \frac{1}{2}$ and rearranging yields the aforementioned Wallis' product for π*

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}$$

Now, let's define the Riemann zeta function and discuss some of the interesting number theoretical applications thereof.

Definition 5. *The Riemann zeta function is defined as the analytic continuation of the function defined by the sum of the series*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \Re(s) > 1$$

This function is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. A remarkable elementary result in this field is the following.

Theorem 6 (Euler product formula)**.** *For all $s > 1$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

Sketch. Start with the sum definition for $\zeta(s)$ and subtract off the sum $\frac{1}{2^s}\zeta(s)$:

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \\ \frac{1}{2^s}\zeta(s) &= \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\end{aligned}$$

We see that this removes all terms $\frac{1}{n^s}$ where $2 \mid n$. We repeat this process by taking the difference between

$$\begin{aligned}\left(1 - \frac{1}{2^s}\right)\zeta(s) &= 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots \\ \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) &= \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \cdots\end{aligned}$$

and we see that this removes all of the terms $\frac{1}{n^s}$ where $2 \mid n$ or $3 \mid n$ or both. Continuing ad infinitum, we have

$$\cdots\left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1$$

and dividing by the factors on the left yields the desired result. □

Remark 7. *To prove a neat consequence of this formula, let's consider the case $s = 2$. We have*

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.6079271016$$

Let's think about what the product on the left hand side represents. Given two random integers m, n, the probability that $2 \mid m$ is $\frac{1}{2}$ (and so is $2 \mid n$) since "roughly half" of the integers are divisible by two. Likewise, the probability that $3 \mid m$ is $\frac{1}{3}$ (and same for $3 \mid n$). Thus, each term in the product is just the probability that a prime p does not *divide both m and n. Multiplying over all primes p gives us the probability that m, n have no common factors, i.e. that m and n are* relatively prime*, or $\gcd(m, n) = 1$.*