Rappel : Techniques d'intégration

Proposition. Formule de changement de variable.

$f : [a,b] \to \mathbb{R}$ continue, $\varphi : [\alpha,\beta] \to [a,b]$ continûment dérivable sur $I \supset [\alpha,\beta]$.

Alors
$$\boxed{\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)\,dt}$$
où $x = \varphi(t)$

Dém : Soit $G(t) = \int_a^{\overset{\varphi(t)\in[a,b]}{\varphi(t)}} f(x)\,dx \Rightarrow G'(t) = f(\varphi(t))\cdot\varphi'(t) \overset{\text{def}}{=} g(t)$

$\Rightarrow G(t)$ est une primitive de $g(t) \Rightarrow \int_\alpha^\beta g(t)\,dt = G(\beta) - G(\alpha)$

$$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt = \int_a^{\varphi(\beta)} f(x)\,dx - \int_a^{\varphi(\alpha)} f(x)\,dx$$

$$\int_a^{\varphi(\beta)} f(x)\,dx + \int_{\varphi(\alpha)}^{a} f(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

$$\Rightarrow \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

$\blacksquare$

Ex 1.

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{\overbrace{-2x\,dx}^{du}}{\underbrace{\sqrt{1-x^2}}_{\sqrt{u}}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}\cdot\frac{1}{\frac{1}{2}}u^{1/2}+C = -u^{\frac{1}{2}}+C =$$

$$= -\sqrt{1-x^2}+C.$$

integrale indéfinie = la plus générale primitive

$u = 1-x^2 \Rightarrow du = -2x\,dx$

Ex 2.

$$\int \frac{dx}{\sin x} = \int \frac{\sin x\,dx}{\sin^2 x} = -\int \frac{\overbrace{-\sin x\,dx}^{du}}{\underbrace{1-\cos^2 x}_{1-u^2}} = -\int \frac{du}{1-u^2} = -\int \frac{du}{1-u^2} = -\frac{1}{2}\int\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du =$$

$u = \cos x \Rightarrow du = -\sin x\,dx$

$$= \frac{1}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$$

$$\begin{cases} 1-\cos x = 2\sin^2\frac{x}{2} \\ 1+\cos x = 2\cos^2\frac{x}{2}\end{cases}$$

$$\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| = \frac{1}{2}\ln\left|\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right| = \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right|$$

$$\int \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \frac{1}{2}\int\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx + \frac{1}{2}\int\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx =$$

$$= \int\frac{\sin\frac{x}{2}\,d\left(\frac{x}{2}\right)}{\cos\frac{x}{2}} + \int\frac{\cos\frac{x}{2}\,d\left(\frac{x}{2}\right)}{\sin\frac{x}{2}} = -\int\frac{d\left(\cos\frac{x}{2}\right)}{\cos\frac{x}{2}} + \int\frac{d\left(\sin\frac{x}{2}\right)}{\sin\frac{x}{2}} = -\ln\left|\cos\frac{x}{2}\right| + \ln\left|\sin\frac{x}{2}\right| + C$$

Remarque. Parfois deux expressions différentes sont égales à une constante près

$$\ln 5x + C_1 = \ln x + C_2 \quad : \quad \ln 5x = \ln 5 + \ln x$$

$$\frac{1}{\cos^2 x} = \tan^2 x + C \quad : \quad \tan^2 x = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

Calcul des dérivée: =

Calcul des intégrales =

Proposition. Formule d'intégration par parties.

$g, f : I \to \mathbb{R}$ continûment dérivable ; $[a,b] \subset I$. Alors

$$\boxed{\int_a^b f(x)\, g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b g(x) f'(x)\, dx.}$$

Dém. $(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \Rightarrow f(x)\cdot g'(x) = (f\cdot g)'(x) - f'(x)g(x)$

$$\Rightarrow \int_a^b f(x)g'(x)\,dx = \int_a^b (f\cdot g)'(x)\,dx - \int_a^b f'(x)\cdot g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

primitive de $(f\cdot g)'$

$\blacksquare$

$$\boxed{\int_a^b f\,dg = fg\Big|_a^b - \int_a^b g\,df}$$

Notations utiles:

$$d f(x) = f'(x)\,dx$$

$$f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a).$$

Ex 3. $\displaystyle\int_1^2 \underbrace{x^2}_{g'(x)} \underbrace{\ln x}_{f(x)}\, dx = \frac{1}{3}\int_1^2 \ln x \cdot x^2 dx = \frac{1}{3}x^3 \ln x \Big|_1^2 - \frac{1}{3}\int_1^2 x^3 \frac{1}{x} dx = \frac{1}{3}x^3 \ln x \Big|_1^2 -$

$g(x) = \frac{1}{3}x^3$

$\displaystyle - \frac{1}{9}x^3 \Big|_1^2 = \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{7}{9} > 0$

Marche bien pour $\int e^{ax}(\text{poly})$, $\int (\ln x)^k(\text{poly})$, $\int (\sin x, \cos x)(\text{poly})$, $\int (e^{ax})(\sin x \cos x)$.

Ex 4. $J = \displaystyle\int \sin x\, e^x dx$

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$g'(x) = e^x \Rightarrow g(x) = e^x$

$J = e^x \sin x - \displaystyle\int e^x \cos x\, dx$

$f(x) = \cos x \Rightarrow f'(x) = -\sin x$

$g(x) = e^x \Rightarrow g(x) = e^x$

$J = e^x \sin x - \left(e^x \cos x + \displaystyle\int \sin x\, e^x dx\right) =$

$\Rightarrow J = e^x \sin x - e^x \cos x - J$

$\Rightarrow 2J = e^x(\sin x - \cos x)$

$\Rightarrow J = \frac{1}{2}e^x(\sin x - \cos x) + C$

Ex 5.

$$\int \underbrace{\ln\left(1+\sqrt{1+x}\right)}_{f(x)} \underbrace{dx}_{g(x)} = x\ln\left(1+\sqrt{1+x}\right) - \int \frac{x}{2\sqrt{1+x}\left(1+\sqrt{1+x}\right)}\,dx$$

$u = \sqrt{1+x} \Leftrightarrow u^2-1 = x$

$du = \frac{1}{2\sqrt{1+x}}\,dx$

$$= x\ln\left(1+\sqrt{1+x}\right) - \int \frac{(u^2-1)\,du}{u+1}$$

$$= x\ln\left(1+\sqrt{1+x}\right) - \int \frac{(u+1)(u-1)}{(1+u)}\,du$$

$$= x\ln\left(1+\sqrt{1+x}\right) - \int (u-1)\,du$$

$$= x\ln\left(1+\sqrt{1+x}\right) - \frac{1}{2}u^2 + u + C$$

$$= x\ln\left(1+\sqrt{1+x}\right) - \frac{1}{2}(x+1) + \sqrt{x+1} + C$$

$$= x\ln\left(1+\sqrt{1+x}\right) - \frac{1}{2}x \cancel{-\frac{1}{2}} + \sqrt{x+1} + C$$

$$= x\ln\left(1+\sqrt{1+x}\right) - \frac{1}{2}x + \sqrt{x+1} + C$$

Intégration des fonctions rationnelles.

$\int \frac{P(x)}{Q(x)}\,dx$ s'exprime toujours en termes des fonctions élémentaires.

(Par contre, $\int \sin x \cdot \ln x \, dx$ ne s'exprime pas en termes des fonctions élémentaires).

poly

$$\frac{f(x)}{g(x)} = p(x) + \sum_i \frac{f_i(x)}{g_i(x)} \quad \text{où} \quad g_i(x) = \begin{cases} (ax+b)^k \Rightarrow f_i(x) = A_i \\ (ax^2+bx+c)^m \Rightarrow f_i(x) = A_i x + B_i \\ \text{si } b^2-4ac<0 \end{cases}$$

poly   poly

(1) $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$

$a \neq 0$

$u = ax+b \Rightarrow du = a\,dx$

(2) $\displaystyle\int \frac{(cx+d)\,dx}{(x-a)(x-b)} = \int\left(\frac{A}{x-a} + \frac{B}{x-b}\right)dx = A\ln|x-a| + B\ln|x-b| + C$

$a \neq b$

Coefficients indéterminés $\quad \dfrac{A}{x-a} + \dfrac{B}{x-b} = \dfrac{cx+d}{(x-a)(x-b)} = \dfrac{A(x-b)+B(x-a)}{(x-a)(x-b)}$

$$\begin{array}{l} 1: \\ x: \end{array}\begin{cases} -Ab-Ba = d \\ A+B = c \end{cases} \Rightarrow \begin{array}{l} B = c-A \\ -Ab-ac+Aa = d \end{array} \Rightarrow \begin{array}{l} A = \dfrac{ac+d}{a-b} \\ B = c-A \end{array} \quad a \neq b$$

(3) $\displaystyle\int \frac{dx}{(ax+b)^k} = \frac{1}{a}\int\frac{du}{u^k} = \frac{1}{a}\int u^{-k}du = \frac{1}{a}\,\frac{1}{-k+1}\,u^{-k+1} + C = \frac{1}{a(1-k)}(ax+b)^{-k+1} + C$

$k \geq 2$ $\quad u = ax+b$, $du = a\,dx$

(4) $\displaystyle\int \frac{dx}{x^2+c^2} = \frac{1}{c^2}\int\frac{dx}{\left(\frac{x}{c}\right)^2+1} = \frac{1}{c^2}\int\frac{c\,dt}{t^2+1} = \frac{1}{c}\arctan t + C = \frac{1}{c}\arctan\frac{x}{c} + C$

$t = \frac{x}{c} \Rightarrow dt = \frac{1}{c}dx$

$\left(\text{Aussi } \displaystyle\int\frac{dx}{x^2+px+q}\right) = \displaystyle\int\frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)} = \int\frac{du}{u^2+c^2} \qquad u = \left(x+\frac{p}{2}\right)$, $du = dx$

$p^2-4q<0$

(5) $\displaystyle\int \frac{x\,dx}{x^2+c^2} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+c^2| + C$

$u = x^2+c^2 \Rightarrow du = 2x\,dx$

Ex. 6 $\displaystyle\int_0^1 \frac{dx}{(x-2)(x+3)} = \frac{1}{5}\int_0^1 \left(\frac{1}{x-2} - \frac{1}{x+3}\right)dx = \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3|\Big|_0^1 =$

$\displaystyle\frac{1}{(x-2)(x+3)} = \left(\frac{1}{x-2} - \frac{1}{x+3}\right)\frac{1}{5}$

$\displaystyle = \frac{1}{5}(\ln 1 - \ln 2) - \frac{1}{5}\ln(\underbrace{\ln 4}_{=2\ln 2} - \ln 3) =$

$\displaystyle = \frac{1}{5}(-3\ln 2 + \ln 3) = \frac{1}{5}\ln\frac{3}{8}$

Ex. 7. $\displaystyle\int_{-2}^0 \frac{dx}{x^2+4x+8} = \int_{-2}^0 \frac{dx}{(x^2+4x+4)+4} = \int_{-2}^0 \frac{dx}{(x+2)^2+4} = \int_0^2 \frac{du}{u^2+4} =$

$u = x+2$ $\quad x=-2 \Rightarrow u=0$

$du = dx$ $\quad x=0 \Rightarrow u=2$

$\displaystyle = \frac{1}{4}\int_0^2 \frac{du}{\left(\frac{u}{2}\right)^2+1} = \frac{1}{2}\int_0^1 \frac{dt}{t^2+1} = \frac{1}{2}\arctan t\Big|_0^1 = \frac{1}{2}\cdot\frac{\pi}{4} - 0 = \frac{\pi}{8}.$

$t = \frac{u}{2}$, $dt = \frac{1}{2}du$ $\quad du = 2dt$

$u=0 \quad t=0$

$u=2 \quad t=1$

Question 25

L'intégrale $\int_{-1}^{1} \frac{dx}{e^x+1}$ vaut

A. 1

B. $\frac{1}{e+1} - \frac{1}{e^{-1}+1}$

C. $2-\ln(e+1)+\ln(e^{-1}+1)$

D. $1-\ln(e+1)$

E. 2

F. e^2-e^{-2}

$e^x = t$

$x = \ln t$

$dx = \frac{1}{t}dt$

$x=-1,\ t=e^{-1}$

$x=1 \quad t=e$

$$= \int_{e^{-1}}^{e} \frac{\frac{1}{t}dt}{t+1} = \int_{e^{-1}}^{e} \frac{dt}{t(t+1)} =$$

$$= \int_{e^{-1}}^{e} \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \ln t - \ln(t+1)\Big|_{e^{-1}}^{e}$$

$$= \ln e - \ln(e^{-1}) - \ln(1+e) + \ln(1+e^{-1}) =$$

$$= 2-\ln(e+1)+\ln(e^{-1}+1) \Rightarrow C$$

$$2 + \ln \frac{e^{-1}+1}{e+1} = 2 + \ln\left(\frac{e^{-1/2}\cancel{(e^{-1/2}+e^{1/2})}}{e^{1/2}\cancel{(e^{1/2}+e^{-1/2})}}\right)$$

$$= 2 + \ln e^{-1} = 1. \Rightarrow A$$

Deux réponses justes: A et C.