

among the mathematicians, and Bernard Cohen, Kenneth May, Thomas Hawkins, Morris Kline, Albert Lewis, and Carl Boyer among the historians. On this occasion, she gave a talk titled "The Mathematician, the Historian and the History of Mathematics." With this distinguished audience of mathematicians and historians, she considered the different questions they bring to the table when they consider mathematics. In particular, she noted that mathematicians are often interested in "the past as part of the present," while historians of mathematics view the present as laden with "many and diverse relics from the past and as the end of long, complex processes" [16, p. 440].³ She also underscored the value of the history of mathematics for its contributions to human culture and its role in the teaching and understanding of mathematics. This inextricable link between research in the history of mathematics and its helpful presence in the classroom remained a constant in Grabiner's work over the course of her career.

This observation leads naturally to four themes that emerge when considering the whole of Grabiner's work. (1) First and foremost, there is her deep erudition, founded on her training as a mathematician and as a historian of science, firmly embedded in a liberal arts tradition that also encompasses the history of art, literature, and philosophy. For example, her celebrated paper asking "Why did Lagrange 'prove' the parallel postulate?" features the work of artists Piero della Francesca, Leonardo da Vinci, and Raphael in her discussion of perspective. (2) Although her name is most often associated with Cauchy, it is her work on Lagrange, and the eighteenth century in particular, that forms the bedrock of her scholarship. With Lagrange as a focal point, she moved forward to Cauchy and backwards to Colin Maclaurin and Leonhard Euler, giving her a deep understanding of the creation of the calculus over more than 150 years, which she shares in many publications and leverages in the classroom. (3) The concept of change is everywhere present in her work as a scholar. There is the obvious language of change in her studies of the calculus, but we mean something much more than

the mathematics of calculus. It is her understanding of how viewpoints, including her own, change over time. She regularly encourages her readers to put themselves in the mindset of the scholars she discusses. When considering the progress of calculus throughout the eighteenth century, for example, she reminds her readers that the guiding question was to produce mathematical results, not to build a foundation for calculus. Consequently, it is "unfair" to criticize Euler and others for what more modern mathematicians would call a "lack of rigor" [17, p. 187]. (4) Finally, she is a brilliant expositor. Her numerous Halmos–Ford and Allendoerfer awards, combined with her Beckenbach Book Prize, testify to her expository excellence. But these external validations are no substitute for sitting down with one of her articles and reveling in the compelling questions she motivates and poses at the beginning, her plan for addressing them, the strategy she follows to accomplish her goal, and the conclusion she reaches that inevitably ties up the work with the equivalent of a beautiful bow. She is a master of her craft.

By way of an example, which, incidentally, is how Grabiner often guides her readers, let us consider her discussion of an eighteenth-century proof in her award-winning "Who gave you the epsilon? Cauchy and the origins of rigorous calculus." To illustrate how mathematicians made important discoveries in their efforts to find results rather than to explore rigorous foundations, she offers Euler's 1734 solution to the famous " Basel problem" of finding the precise value of the sum of the reciprocals of the squares of the natural numbers:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \cdots$$

Grabiner writes:

It clearly has a finite sum since it is bounded above by the series

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k-1) \cdot k} + \cdots$$

whose sum was known to be 2; Johann Bernoulli had found this sum by treating $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$ as the difference between the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ and the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, and observing that this difference telescopes.

Euler's summation of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ makes use of a lemma from the theory of equations: given a polynomial equation whose constant term is one, the coefficient of the linear term is the sum⁴ of the reciprocals of the roots with the signs changed. This result was both discovered and demonstrated by

³For a complete list of workshop participants and an overview of the workshop, including who participated in the discussions following talks, see [7, pp. 16–25]. In particular, following introductory remarks by the organizers, Grabiner gave the first talk of the workshop, and Jean Dieudonné, J. P. Kahane, Alberto Dou, Hilary Putnam, and George Mackey participated in the discussion. Grabiner's published version of the talk, "The Mathematician, the Historian and the History of Mathematics," in *Historia Mathematica* 2 (1975), 439–447, includes details of the discussion, particularly Dieudonné's comments and Grabiner's responses to them. The discussion of her paper also mentions that George Mackey "expressed interest in history, but felt that because of the pressures of his discipline, he could not be interested in too detailed a history" [p. 447]. For more on Mackey, including some mini-histories of mathematics he offered in his correspondence, see Della Dumbaugh, "Extensive cooperation with rugged individualism: George Mackey's guide for practitioners of mathematics," *Notices Amer. Math. Soc.* 66 (June/July 2019), 883–891, <https://www.ams.org/journals/notices/201906/rnoti-p883.pdf>.

⁴The original article says "product" here [17, p. 187]. This typo is corrected in [6, p. 6].

considering the equation $(x - a)(x - b) = 0$, having roots a and b . Multiplying and then dividing out ab , we obtain

$$\left(\frac{1}{ab}\right)x^2 - \left(\frac{1}{a} + \frac{1}{b}\right)x + 1 = 0;$$

the result is now obvious, as is the extension to equations of higher degree.

Euler's solution then considers the equation $\sin x = 0$.

Expanding this as an infinite series, Euler obtained

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = 0.$$

Dividing by x yields

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots = 0.$$

Finally, substituting $x^2 = u$ produces

$$1 - \frac{u}{3!} + \frac{u^2}{5!} - \dots = 0.$$

But Euler thought that power series could be manipulated just like polynomials. Thus, we now have a polynomial equation in u , whose constant term is one. Applying the lemma to it, the coefficient of the linear term with the sign changed is $\frac{1}{3!} = \frac{1}{6}$. The roots of the equation in u are the roots of $\sin x = 0$ with the substitution $u = x^2$, namely $\pi^2, 4\pi^2, 9\pi^2, \dots$. Thus the lemma implies

$$\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots$$

Multiplying by π^2 yields the sum of the reciprocal series

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} + \dots = \frac{\pi^2}{6}$$

[17, p. 187].

Following this presentation of Euler's proof, Grabiner seems to know the mind of her more modern reader. She encourages a broader view of Euler's result, one that takes into account the priorities of the eighteenth century while acknowledging current mathematical conventions. In particular, since for Euler and his contemporaries the emphasis was on obtaining mathematical results rather than answering questions about foundations, she argues that mathematicians were free to make these types of "important new discoveries" without the procedural strictures of today's more rigorous analysis.

As Grabiner noted, Cauchy would later prove the same result by "calculating the difference between the n th partial sum and $\frac{\pi^2}{6}$ and showing that it was arbitrarily small" [17, p. 193]. Let the present authors expand on Grabiner's brief remark so that the reader can see, by contrasting Euler's proof with the detail provided by Cauchy in his *Cours*

d'analyse (1821), how much more attention was given to matters of convergence some eighty-seven years later. Starting with an expression

$$\frac{(m+2)(m-2)}{3!} = \frac{1}{\sin^2 \frac{2\pi}{2m}} + \frac{1}{\sin^2 \frac{4\pi}{2m}} + \dots + \frac{1}{\sin^2 \frac{(m-2)\pi}{2m}}$$

derived from standard trigonometric identities, where m is a positive even integer, Cauchy multiplied through by $\left(\frac{\pi}{m}\right)^2$ to obtain

$$\begin{aligned} \frac{\pi^2}{6} \left(1 - \frac{4}{m^2}\right) &= \frac{\left(\frac{\pi}{m}\right)^2}{\sin^2 \frac{\pi}{m}} + \frac{1}{4} \frac{\left(\frac{2\pi}{m}\right)^2}{\sin^2 \frac{2\pi}{m}} + \frac{1}{9} \frac{\left(\frac{3\pi}{m}\right)^2}{\sin^2 \frac{3\pi}{m}} \\ &+ \dots + \frac{1}{\left(\frac{m}{2} - 1\right)^2} \frac{\left(\frac{(m-2)\pi}{2m}\right)^2}{\sin^2 \frac{(m-2)\pi}{2m}} \end{aligned}$$

or

$$\frac{\pi^2}{6} \left(1 - \frac{4}{m^2}\right) = \sum_{k=1}^n \frac{1}{k^2} \frac{\left(\frac{k\pi}{m}\right)^2}{\sin^2 \frac{k\pi}{m}} + \sum_{k=n+1}^{\frac{m}{2}-1} \frac{1}{k^2} \frac{\left(\frac{k\pi}{m}\right)^2}{\sin^2 \frac{k\pi}{m}}, \quad (1)$$

where n is a positive integer less than $\frac{m}{2}$. Cauchy claimed that for any two collections of numbers, $a, b, c, \dots, n$ and $a', b', c', \dots, n'$, there always exists a constant, denoted $M(a, b, c, \dots, n)$,⁵ such that

$$\begin{aligned} a'a + b'b + c'c + \dots + n'n \\ = (a' + b' + c' + \dots + n') \cdot M(a, b, c, \dots, n), \end{aligned}$$

which, since

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x},$$

allowed him to express the first sum on the right-hand side of (1) as

$$\left(1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}\right) \cdot M\left(1, \frac{1}{\cos^2 \frac{n\pi}{m}}\right).$$

Similarly, since for positive $x < \frac{\pi}{2}$,

$$\frac{x}{\sin x} < \frac{\frac{1}{2}x}{\sin \frac{1}{2}x} \frac{1}{\cos \frac{1}{2}x} < \frac{1}{\cos^2 \frac{1}{2}x} < \frac{1}{\cos^2 \frac{\pi}{4}} = 2,$$

every term in the second sum of (1) is strictly less than $\frac{4}{n^2}$, leading to that sum being bounded by 0 and $\frac{2m}{n^2}$. Cauchy

⁵Cauchy referred to this constant as "moyenne" even though, in general, it need not be the arithmetic mean. [8, p. 14] translates "moyenne" as "average." See also [9, pp. 28–29].

could thus rewrite equation (1) as

$$\frac{\pi^2}{6} \left(1 - \frac{4}{m^2}\right) = \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}\right) \cdot M\left(1, \frac{1}{\cos^2 \frac{n\pi}{m}}\right) + \frac{2m}{n^2} \cdot M(0, 1)$$

or, rearranging, as

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} = \frac{\pi^2}{6} \left(1 - \frac{4}{m^2}\right) \cdot M\left(1, \cos^2 \frac{n\pi}{m}\right) - \frac{2m}{n^2} \cdot M(0, 1). \quad (2)$$

Remembering that $\frac{1}{2}m > n$, pushing n to infinity produced the required infinite series on the left-hand side while giving the limit of zero for $4/m^2$ on the right. One final condition was needed to obtain limits for $n\pi/m$ and $2m/n^2$, although Cauchy's explanation is less than satisfactory by modern standards: "It is easy to see that if we always take for $\frac{1}{2}m$ the smallest of the integers greater than n^a (where a denotes a number contained between 1 and 2), the ratios $\frac{n}{m}$ and $\frac{m}{n^2}$ converge together, for increasing values of n , towards the limit zero" [8, p. 383]. Thus, Cauchy concluded, the right-hand side of (2) converges to the limit of $\pi^2/6$.⁶

Grabiner was well aware of what happened in the intervening decades between these proofs of Euler and Cauchy. She captured many of these ideas in her *The Origins of Cauchy's Rigorous Calculus*. Here, however, as elsewhere in her work, we are reminded to take into consideration the long journey of the calculus from Newton and Leibniz to Cauchy, especially as it relates to the students in our calculus classrooms. As she put it in "The changing concept of change: The derivative from Fermat to Weierstrass":

The real historical development of mathematics—the order of discovery—reveals the creative mathematician at work, and it is creation that makes doing mathematics so exciting. The order of exposition, on the other hand, is what gives mathematics its characteristic logical structure and its incomparable deductive certainty. Unfortunately, once the classic exposition has been given, the order of discovery is often forgotten. The task of the historian is to recapture the order of discovery: not as we think it might have been, not as we think it should have been, but as it really was. [15, p. 206]

This is to say that students benefit from understanding the process of the development of mathematics and, in this case, the calculus. Grabiner's celebration of Euler's proof (and Lagrange's approach in viewing calculus from

an algebraic perspective, for that matter) encourages students to realize that they *can* understand the epsilon-delta foundations of calculus simply by recognizing that they don't have to start there. They can start with Euler and be in very good company.

We cannot leave this discussion of "Who gave you the epsilon?" without calling attention to its brilliant conclusion. By the time readers reach the final pages of this work, they may have lost track of the title. But not Grabiner. To underscore her enduring point of considering the rigorous form of calculus as a "completed whole" of the work of many mathematicians over more than 150 years, she offers the epsilon (literally!) as a tangible reminder. "In Cauchy's work," she asserts, "one trace indeed was left of the origin of rigorous calculus in approximation—the letter epsilon. The ϵ corresponds to the initial letter in the word 'erreur' (or 'error'), and Cauchy in fact used ϵ for 'error' in some of his work on probability. It is both amusing and historically appropriate that the ' ϵ ,' once used to designate the 'error' in approximations, has become transformed into the characteristic symbol of precision and rigor in the calculus. As Cauchy transformed the algebra of inequalities from a tool of approximation to a tool of rigor, so he transformed the calculus from a powerful method of generating results to the rigorous subject we know today" [17, p. 193].

But Grabiner does not just encourage a "whole" view of mathematics for the sake of the discipline: she also encourages a view of mathematics as part of the whole history of ideas. In her 1986 International Congress address in Berkeley, California, Grabiner chose the topic "The Centrality of Mathematics in the History of Western Thought." She began with a student and a teaching experience, another reflection of the strong link between the history of mathematics and teaching. In particular, she aimed to "recapture" the critical and beautiful moment when a student read Euclid's *Elements of Geometry* and exclaimed, "I never realized mathematics was like this. Why, it's like philosophy!" [14, p. 220]. She discussed "major developments in the history of ideas" where mathematics played a key role. Among them, she highlighted the role of Euclidean thinking in the Declaration of Independence of the United States. This foundational document

... is one more example of an argument whose authors tried to inspire faith in its certainty by using the Euclidean form. "We hold these truths to be self-evident..." not that all right angles are equal, but "that all men are created equal." These self-evident truths include that if any government does not obey these postulates, "it is the right of the people to alter or abolish it." The central section begins by saying that they will "prove" King George's government does not obey them. The conclusion is, "We, therefore... declare, that these

⁶These details are taken from [9, pp. 455–457, 556–559]. For an English translation, see [8, pp. 381–383].

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