

Asset Pricing Theory

Problem Set 7: Dynamic Portfolio Choice

1. Portfolio choice: Dynamic Programming

Consider a discrete-time economy where the dynamics of the risky asset and the risk-free asset is given by:

$$\begin{aligned} S_{t+1} &= e^{\mu + \sigma \tilde{w}_{t+1}} \cdot S_t \\ S_{t+1}^0 &= e^{r_f} \cdot S_t^0 \end{aligned}$$

where $\tilde{w}_t \forall t = 0, 1, 2, 3 \dots$ are i.i.d. binomial random variables with 0.5 probability equal to +1 and 0.5 probability equal to -1.

Suppose an agent maximizes her expected utility of intertemporal consumption $E[\sum_{t=0}^T e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$ by choosing in every period the optimal consumption $\{C_t\}_{t=0,1,2,\dots}$ and optimal fraction of remaining wealth to invest in the risky asset $\{\pi_t\}_{t=0,1,2,\dots}$ (the fraction $1 - \pi_t$ is invested in the risk-free asset) given an initial wealth level W_0 .

1. Write the dynamics of wealth for the investor.
2. Setup the HJB equation and derive the first-order conditions for consumption and investment policy.
3. Guess that the optimal value function is of the form $J(t, W) = e^{-\rho t} A_t \frac{W^{1-\gamma}}{1-\gamma}$ and derive the deterministic function A_t . Find the optimal consumption and investment policy.
4. Use the solution you derived to compute the solution to the infinite horizon problem $E[\sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$. Find the optimal consumption and investment policy in that case. Give the conditions on the parameters (the so-called ‘transversality condition’) under which the infinite horizon problem has a finite value.
5. How does an increase in risk-aversion γ , in the expected return μ , and in the volatility σ affect the optimal investment in the risky security and the optimal consumption? (If you cannot do this analytically, then do it numerically for some choice of parameters).

Proof

The solution can be done in two ways. The first one is to use the Bellman equation. This has been done in the lecture:

- *Let’s use dynamic programming to solve the optimal portfolio consumption problem of an agent with CRRA utility $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ when returns are i.i.d.*
- *The value function is $J(t, W_t) = \max_{c_n, \pi_n} E[\sum_{n=t}^T \delta^n u(C_n)]$.*
- *Subject to $W_{t+1} = (W_t - c_t)R_p(t+1)$.*

- We also assume that there exists a risk-free rate and solve for π_t , the vector of fractions of wealth invested in risky assets. (so we have substituted the constraint $\pi' \mathbf{1} = 1$). Thus we define $R_p(t+1) = (R_f + \pi'_t(R_{t+1} - R_f \mathbf{1}))$.
- The HJB equation is $J(t, W_t) = \max_{\pi, C} \{u(C)\delta^t + E_t[J(t+1, W_{t+1})]\}$
- The FOC is

$$\begin{aligned} 0 &= \delta^t u'(C) - E_t[J_W(t+1, W_{t+1})R_p(t+1)] \\ 0 &= E_t[J_W(t+1, W_{t+1})(R_{t+1}^j - R_f)] \quad \forall j = 1, \dots, d \end{aligned}$$

- Using the “envelope condition” the FOC become:

$$\begin{aligned} \delta^t u'(C_t) &= J_W(t, W_t) \\ 0 &= E_t\left[\frac{\delta u'(C_{t+1})}{u'(C_t)}(R_{t+1}^j - R_f)\right] \quad \forall j = 1, \dots, d \end{aligned}$$

We recognize the Euler equation derived previously.

- For a CRRA investor we guess that $J(t, W) = \delta^t A_t^{-\gamma} \frac{W^{1-\gamma}}{1-\gamma}$
- Substituting into the FOC conditions, we get:

$$\begin{aligned} C_t &= A_t W_t \\ 0 &= E_t[(R_p(t+1))^{-\gamma}(R_{t+1}^j - R_f)] \quad \forall j = 1, \dots, d \end{aligned}$$

Note that the optimal solution for the portfolio strategy is a constant vector π^* since the returns are assumed to be i.i.d. Further, that solution is identical to that obtained in the one-period case. Since returns are i.i.d., agents act myopically concerning their portfolio choice. Not so, however, for their consumption decision.

Here, because we only have one risky asset, we can actually solve the portfolio problem. We have $R_f = e^{r_f}$ and $S_{t+1} = e^{\mu+\sigma\tilde{w}_{t+1}} \cdot S_t$ implies $R_{t+1} = e^{\mu+\sigma\tilde{w}_{t+1}}$, so that

$$E_t[(R_f + \pi_t(e^{\mu+\sigma\tilde{w}_{t+1}} - R_f))^{-\gamma}(e^{\mu+\sigma\tilde{w}_{t+1}} - R_f)] = 0$$

can be rewritten as

$$0.5(R_f + \pi_t(e^{\mu+\sigma} - R_f))^{-\gamma}(e^{\mu+\sigma} - R_f) + 0.5(R_f + \pi_t(e^{\mu-\sigma} - R_f))^{-\gamma}(e^{\mu-\sigma} - R_f) = 0.$$

which is equivalent to

$$\frac{(R_f + \pi_t(e^{\mu+\sigma} - R_f))^{-\gamma}}{(R_f + \pi_t(e^{\mu-\sigma} - R_f))^{-\gamma}} = \frac{R_f - e^{\mu-\sigma}}{e^{\mu+\sigma} - R_f}$$

which is equivalent to

$$\frac{(R_f + \pi_t(e^{\mu+\sigma} - R_f))}{(R_f + \pi_t(e^{\mu-\sigma} - R_f))} = \left(\frac{R_f - e^{\mu-\sigma}}{e^{\mu+\sigma} - R_f}\right)^{-1/\gamma}$$

which is equivalent to

$$(R_f + \pi_t(e^{\mu+\sigma} - R_f)) = \left(\frac{R_f - e^{\mu-\sigma}}{e^{\mu+\sigma} - R_f}\right)^{-1/\gamma} (R_f + \pi_t(e^{\mu-\sigma} - R_f))$$

which implies

$$\pi_t = \frac{\left(\frac{R_f - e^{\mu-\sigma}}{e^{\mu+\sigma} - R_f}\right)^{-1/\gamma} R_f - R_f}{(e^{\mu+\sigma} - R_f) - \left(\frac{R_f - e^{\mu-\sigma}}{e^{\mu+\sigma} - R_f}\right)^{-1/\gamma} (e^{\mu-\sigma} - R_f)}$$

- Now, for the consumption, substituting the guess into the HJB equation, we obtain $C_t^* = A_t W_t$ where A_t solves a recursive equation:

$$A_t = \frac{A_{t+1}}{A_{t+1} + (\delta B)^{1/\gamma}}$$

subject to $A_T = 1$ and where $B = E_t[(R_p^*(t+1))^{1-\gamma}]$ a constant since the optimal portfolio π^* is constant and returns are iid.

- The solution is easily derived $A_{T-t} = \frac{1}{\sum_{n=0}^t (\delta B)^{n/\gamma}}$.
- Note that when $T \rightarrow \infty$ we obtain a simple solution $C_t^* = A W_t$ with $A = 1 - (\delta B)^{1/\gamma}$ if it is positive (what happens else?).
- In that case the value function is simply $J(t, W) = \delta^t A^{-\gamma} \frac{W^{1-\gamma}}{1-\gamma}$
- One can also attack the problem directly by looking for a stationary solution in the HJB equation subject to a transversality condition $\lim_{T \rightarrow \infty} E[J(T, W_T)] = 0$

2. Portfolio choice: Complete Markets

Consider the same economy as in Problem 1 above and consider the infinite horizon problem $\max E[\sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$.

1. Show that in this economy there exists a unique pricing kernel M_t with the property that $\frac{M_{t+1}}{M_t} = e^{m_{t+1}}$, where m_{t+1} is an i.i.d. binomial process you can characterize uniquely (not necessarily in closed-form) in terms of μ, σ, r_f . Conclude that markets are complete.
2. Show that any feasible consumption plan must satisfy the intertemporal budget constraint:

$$E[\sum_{t=0}^{\infty} M_t C_t] = M_0 W_0$$

3. Consider then the problem of maximizing $E[\sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$, subject to the intertemporal budget constraint. Write the first-order condition for consumption, and show that the optimal consumption of the investor must satisfy:

$$\frac{C_{t+1}}{C_t} = e^{\frac{\rho}{\gamma} - \frac{1}{\gamma} m_{t+1}} \quad (\star)$$

4. Assume the agent consumes a constant fraction of wealth $C_t = a W_t$ in every period and invests a constant fraction π of the remaining wealth into the risky asset. Show that one can find two constants a^*, π^* so that $C_t = a^* W_t$ is budget feasible and satisfies the dynamics of optimal consumption identified in $(\star)$ above. Conclude that this is the optimal consumption and portfolio choice.

Proof.

We now present a second solution based on the complete market approach. First, we build the state price density process

$$M_t = \prod_{\tau=0}^{t-1} M_{\tau, \tau+1},$$

where $M_{\tau, \tau+1}$ satisfies

$$E_{\tau}[M_{\tau, \tau+1}] = 1/R_f, \quad E_{\tau}[M_{\tau, \tau+1} e^{\mu + \sigma \tilde{w}_{\tau+1}}] = 1.$$

Clearly, there are only two states of the world from the point of view of time τ , and, since data is i.i.d., $M_{\tau, \tau+1}$ will only depend on the realization of $\tilde{w}_{\tau+1}$, and take only two values, $M_{\tau, \tau+1}(1)$ and $M_{\tau, \tau+1}(-1)$, for the respective values of the SDF. This gives a system of two equations and two unknowns:

$$\begin{aligned} 0.5(M_{\tau, \tau+1}(-1) + M_{\tau, \tau+1}(1)) &= R_f^{-1} \\ 0.5(M_{\tau, \tau+1}(-1)e^{\mu - \sigma} + M_{\tau, \tau+1}(1)e^{\mu + \sigma}) &= 1 \end{aligned} \quad (1)$$

Solving this system gives $M_{\tau, \tau+1}(\tilde{w}_{\tau+1})$. Now, having built M_t , we can solve the utility optimization problem:

$$e^{-\rho t} C_t^{-\gamma} = M_t \lambda, \quad (2)$$

where λ is the Lagrange multiplier, determined through the budget constraint

$$E[\sum_t C_t M_t] = W_0 \quad (3)$$

Thus,

$$C_t = \lambda^{-1/\gamma} e^{-\rho t/\gamma} M_t^{-1/\gamma} \quad (4)$$

and the budget constraint gives

$$W_0 = E\left[\sum_t C_t M_t\right] = \lambda^{-1/\gamma} E\left[\sum_t e^{-\rho t/\gamma} M_t^{1-1/\gamma}\right]$$

so that

$$\lambda^{-1/\gamma} = \frac{W_0}{E\left[\sum_t e^{-\rho t/\gamma} M_t^{1-1/\gamma}\right]}$$

Furthermore, M_t is a geometric random walk process (a cumulative product of independent, identically distributed random variables $M_{t-1,t}$). Hence, we can compute the expectation using this independence:

$$\begin{aligned} E\left[\sum_t e^{-\rho t/\gamma} M_t^{1-1/\gamma}\right] &= E\left[\sum_t e^{-\rho t/\gamma} \prod_{\tau=1}^t (M_{\tau-1,\tau})^{1-1/\gamma}\right] \\ &\stackrel{\text{independence}}{=} \sum_t e^{-\rho t/\gamma} \prod_{\tau=1}^t E[(M_{\tau-1,\tau})^{1-1/\gamma}] \\ &= \sum_t e^{-\rho t/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}])^t = \frac{e^{-\rho(T+1)/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}])^{T+1} - 1}{e^{-\rho/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}]) - 1} \end{aligned} \quad (5)$$

Now, using the inter-temporal budget constraint (absent endowment, wealth today has to be entirely spent on future consumption) and the following identities

$$\begin{aligned} C_\tau &= \lambda^{-1/\gamma} e^{-\rho \tau/\gamma} M_\tau^{-1/\gamma} = \underbrace{\lambda^{-1/\gamma} e^{-\rho t/\gamma} M_t^{-1/\gamma}}_{=C_t} e^{-\rho(\tau-t)/\gamma} (M_\tau/M_t)^{-1/\gamma} \\ &= C_t e^{-\rho(\tau-t)/\gamma} (M_\tau/M_t)^{-1/\gamma}, \end{aligned} \quad (6)$$

we get

$$\begin{aligned} W_t &= E_t \left[\sum_{\tau=t}^T C_\tau \frac{M_\tau}{M_t} \right] = E_t \left[\sum_{\tau=t}^T \lambda^{-1/\gamma} e^{-\rho \tau/\gamma} M_\tau^{-1/\gamma} \frac{M_\tau}{M_t} \right] \\ &= C_t E_t \left[\sum_{\tau=t}^T e^{-\rho(\tau-t)/\gamma} \left(\frac{M_\tau}{M_t} \right)^{1-1/\gamma} \right] \\ &\stackrel{\text{under } M_\tau/M_t \text{ is independent of the past}}{=} C_t \sum_{\tau=t}^{T-t} e^{-\rho(\tau-t)/\gamma} E \left[\left(\frac{M_\tau}{M_t} \right)^{1-1/\gamma} \right] \\ &= C_t \sum_{\tau=t}^T e^{-\rho(\tau-t)/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}])^{\tau-t} \\ &= C_t \frac{e^{-\rho(T-t+1)/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}])^{T-t+1} - 1}{e^{-\rho/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}]) - 1} \\ &= C_t / A_t \end{aligned} \quad (7)$$

where

$$A_t = \frac{e^{-\rho/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}]) - 1}{e^{-\rho(T-t+1)/\gamma} (E[(M_{\tau-1,\tau})^{1-1/\gamma}])^{T-t+1} - 1}$$

is the marginal propensity to consume. Now we can solve for the optimal portfolio. We have

$$W_{t+1}(\pm) = (W_t - C_t)(R_f + \pi_t(e^{\mu \pm \sigma} - R_f)) \quad (8)$$

where the two states, $\pm$, correspond to the two realizations of $\tilde{w}_{t+1}$. One might be worried that we have two equations (for $+1$ and -1), and only one number (π_t) to solve for. However,

$$0.5(M_{t,t+1}(1)W_{t+1}(1) + M_{t,t+1}(-1)W_{t+1}(-1)) = E_t[M_{t,t+1}W_{t+1}] = (W_t - C_t) \quad (9)$$

implies that there is a link between W_{t+1} in the two states; hence, it is enough to pin down just one of them. The other is then determined automatically. Thus, we can solve

$$W_{t+1}(1) = (W_t - C_t)(R_f + \pi_t(e^{\mu + \sigma} - R_f)) \quad (10)$$

and get

$$\pi_t = \frac{\frac{W_{t+1}(1)}{(W_t - C_t)} - R_f}{e^{\mu + \sigma} - R_f}$$

A very long and very tedious calculation implies that the π_t derived this way coincides with the π_t derived the other way.