EPFL & SFI

**Asset Pricing Theory**

# Problem Set 6: Dynamic Arbitrage Pricing

## 1. First Theorem of Asset Pricing

Consider a discrete-time economy, as considered in the lecture notes, with two risky assets, $B_t$ and $S_t$. Assume that both are strictly positive at all times and that ownership of $S_t$ at $t$ entitles you to a dividend $D_{t+1}$ paid at $t+1$.

1. Consider the value of a self-financing portfolio $V_t = \Delta_t^0 B_t + \Delta_t^1 S_t$. Write down its dynamics.
2. Consider the value $V_t^* = \frac{V_t}{B_t}$ of the self-financing portfolio expressed in units of asset $B_t$, the numeraire asset. Write down its dynamics in terms of the discounted risky asset $S_t^* = S_t/B_t$ and the discounted dividend $D_t^* = D_t/B_t$.
3. Formulate and then prove the first fundamental theorem of asset pricing for this economy. In particular, show that if there is no arbitrage, then there exists a measure $Q$ equivalent to $P$ under which the following holds:
   - $S_t^* = E_t^Q[S_{t+1}^* + D_{t+1}^*]$, and equivalently,
   - for $T > t$: $S_t^* = E_t^Q[\sum_{n=t+1}^T D_n^* + S_T^*]$
4. Show that if there is no arbitrage, then there exists a stochastic discount factor (or pricing kernel) process $M_t$ that is strictly positive and such that :
   - $M_t S_t = E_t[M_{t+1}(S_{t+1} + D_{t+1})]$, and equivalently,
   - $1 = E_t[\frac{M_{t+1}}{M_t} R_{t+1}]$ for the gross return $R_{t+1} = \frac{S_{t+1}+D_{t+1}}{S_t}$, as well as:
   - for $T > t$: $M_t S_t = E_t[\sum_{n=t+1}^T M_n D_n + M_T S_T]$

## 2. Multiple Periods, Complete Markets: Cox, Ross Rubinstein

The purpose of this exercise is to take you through Cox, Ross, and Rubinstein's derivation of the Black and Scholes formula as a limit of the discrete-time multiperiod binomial model.

Suppose now that period $T$ is subdivided into $n$ periods of $\Delta = T/n$. At each time $t_i = i\Delta \ \forall i = 0, \dots, n-1$, the dynamics of the risky asset and the risk-free asset are given by:

$$\begin{aligned} S_{i+1} &= \tilde{w}_{i+1}^n \cdot S_i \\ S_{i+1}^0 &= R_n \cdot S_i^0 \end{aligned}$$

where $\tilde{w}_i^n \ \forall i = 1, \dots, n$ are i.i.d. random variables, which can take each of two values $u_n$ with probability $p$ and $d_n$ with probability $1-p$.

1. Show that the market $(S^0, S)$ is dynamically complete, in the sense that any time-T random variable $h(S_T)$ can be perfectly replicated by a self-financing dynamic trading strategy in the risk-free and risky asset. Importantly, we impose the no-arbitrage restrictions
$$u_n > R_n > d_n.$$
Here, $R_n$ and $\tilde{w}_i^n$ indicates the dependence on $n$ in the sense that $n$ is the time frequency that controls the approximation to the *continuous time limit.*

*Hint: use a recursive argument.*

**Proof.** *Please see item 6 at the end of this problem set: Replication in binomial models: You find the optimal portfolio recursively.*

*The equivalent martingale measure assigns probabilities* $1 - q = (u_n - R_n)/(u_n - d_n)$ *and* $q = (R_n - d_n)/(u_n - d_n)$ *to children nodes of the tree. Then, we can multiply these probabilities to get the full probability measure*

$$d\mathcal{Q} \;=\; \xi d\mathbb{P} \text{ with } \xi \;=\; \prod_{i=1}^{n}(q\mathbf{1}_{\tilde{w}_i^n=u_n} + (1-q)\mathbf{1}_{\tilde{w}_i^n=d_n}) \tag{1}$$

*Now, we know from item 6 below that markets are complete and*

$$\tilde{X}_n \;=\; X_n/S_n^0 \;=\; E^{\mathcal{Q}}[X_n] \;+\; \sum_i \pi_i(\tilde{S}_{i+1} - \tilde{S}_i),\; \tilde{S}_i \;=\; S_i/S_i^0$$

*and*

$$\tilde{X}_{n-1} = E_{n-1}^{\mathcal{Q}}[X_n] \;=\; qX_n(u) + (1-q)X_n(d)$$

*whereas*

$$\tilde{X}_n \;=\; \tilde{X}_{n-1} \;+\; \pi_{n-1}(\tilde{S}_{i+1} - \tilde{S}_i)$$

*implying that*

$$\pi_{n-1} \;=\; \frac{\tilde{X}_n(u) \;-\tilde{X}_{n-1}}{(\tilde{S}_{i+1}(u) - \tilde{S}_i)} \;=\; \frac{\tilde{X}_n(d) \;-\tilde{X}_{n-1}}{(\tilde{S}_{i+1}(d) - \tilde{S}_i)}$$

*where we use* $X_n(u)$ *to denote the value of* $X_n$ *is the node of the binomial tree corresponding to an up-move in S. Now,*

$$\tilde{X}_n(u) - \tilde{X}_{n-1} \;=\; (1-q)(\tilde{X}_n(u) - \tilde{X}_n(d))$$

*and*

$$\tilde{S}_{i+1}(u) - \tilde{S}_i \;=\; \tilde{S}_i(u/R_n - 1)$$

*so that*

$$\tilde{S}_{n-1}\pi_{n-1} \;=\; \frac{(\tilde{X}_n(u) - \tilde{X}_n(d))(u_n - R_n)}{(u/R_n - 1)(u_n - d_n)} \;=\; R_n^{-1}\frac{\tilde{X}_n(u) - \tilde{X}_n(d)}{u_n - d_n}$$

*Clearly, the same works for any other* $i$*. This gives a constructive formula for the replicating portfolio: compute* $\tilde{X}_{i+1} = E_{i+1}[\tilde{X}_n]$ *and then*

$$\tilde{S}_i\pi_i \;=\; R_n^{-1}\frac{\tilde{X}_{i+1}(u) - \tilde{X}_{i+1}(d)}{u_n - d_n}$$

*The same argument works for any Markov chain with two states.*

2. Prove that there is a unique equivalent martingale measure. Find the distribution of $\tilde{w}_1^n$ under the equivalent measure and define the associated probabilities $\pi_n^Q$ and $1 - \pi_n^Q$.

3. We now want to find the price of a contingent claim with payoff:
$h(S_T) = \max(S_T - K, 0)$, i.e. a European call option. And in particular, we would like to compute the limit of that formula as $n \to \infty$ for a given $T$, i.e. $\Delta \to 0$.
Let us define $Y_n = \sum_{i=1}^{n} \log(\tilde{w_i}^n)$. Clearly $S_T = S_0 \cdot \exp(Y_n)$. We have to pick $R_n, u_n, d_n$ so that the distribution of the stock price converges to that of a geometric brownian motion, for which we have : $S_T = S_0 \exp\left((\mu - \frac{\sigma^2}{2})T + \sigma Z(T)\right)$ where $Z(T)$ is a normally distributed random variable with zero mean and variance $T$ (i.e., $Z(T) \sim N(0,T)$).
Show that if we pick $p = 0.5$, $u_n = \exp(\alpha_n + \sigma_n)$, $d_n = \exp(\alpha_n - \sigma_n)$ for appropriate $\alpha_n, \sigma_n$ we get the desired convergence. Show that the natural choice of $R_n$ is $\exp(r\Delta)$ where $r$ is the continuously compounded risk-free rate.
*Hint: prove it by showing the convergence of the characteristic function of $Y_n$ to that of a standard normal distribution (recall that $X$ is a standard normal if and only if its characteristic function $\phi(t) = E[\exp(itX)] = \exp(\frac{-t^2}{2})$ - for the general theorem on characteristic functions see the book by Williams chap. 16 for example).*
4. We now turn to the convergence of the option price. Show that the call option price can be written as follows:
$$C(S_0, 0) = S_0\, B(n, \eta, \pi_n^R) - \frac{K}{R_n^n} B(n, \eta, \pi_n^Q)$$
where $B(n, \eta, \pi) = 1 - P(\frac{Y_n - E^\pi[Y_n]}{\sqrt{V^\pi[Y_n]}} < \frac{\eta - E^\pi[Y_n]}{\sqrt{V^\pi[Y_n]}})$ Find $\eta$ and $\pi^Q$, $\pi^R$ the probability of an up realization for $\tilde{w}_1^n$ under two different measures.
*Hint: Since markets are complete, the price of any payoff must equal the value of a self-financing trading strategy. The price of a call option $C(S_0, 0)$ can then be expressed as an expectation of its discounted final payoff under the risk-neutral measure (why?). Then notice that the distribution of* $\exp(Y_n)$ *under the historial measure is $P(\exp(Y_n) = u_n^k d_n^{n-k}) = C_n^k p^k (1-p)^{n-k}\ \forall k = 0, \ldots, n$, where we use the standard notation $C_n^k = \frac{n!}{k!(n-k)!}$.*
Notice that the formula could easily be implemented on a computer to find the values of a European call option. Here we are interested in the continuous-time limit of that formula.
5. Show that $X_n = \frac{Y_n - E_n^\pi[Y_n]}{\sqrt{V_n^\pi[Y_n]}}$ converges in distribution towards a centered gaussian random variable.
*Hint: Prove the convergence of the characteristic function of $X_n$. Also remember that $Y_n$ is the sum of i.i.d. random variables.*
6. Compute and find the limit as $n \to \infty$ of $E^{\pi_n^Q}[Y_n]$, $E^{\pi_n^R}[Y_n]$, $V^{\pi_n^Q}[Y_n]$, $V^{\pi_n^R}[Y_n]$ to derive the Black and Scholes formula as the limit of the discrete time price of the European call option.
*Hint: Remember $Y_n$ is the sum of i.i.d random variables.*
7. Write the resulting limiting Option Pricing Formula.

## 3. Kolmogorov Equations I

Let $X_t$ be a Markov chain with values $x_i$ and transition probabilities $\pi(x_i, x_j)$. Prove Kolmogorov equations:

1.
$$E[g(X_T)\,|\,\mathcal{F}_t] \;=\; E[g(X_T)\,|\,X_t] \;=\; G(t,X_t)$$

satisfies
$$G(t,x_i) \;=\; \sum_j p(x_i,x_j)\,G(t+1,x_j)$$

We have by the law of iterated expectations that

$$\begin{aligned} E[g(X_T)\,|\,X_t] \quad & \underbrace{=}_{iterated\ expectations} \quad = \; E[E_{t+1}[g(X_T)]|\,X_t] \\ \underbrace{=}_{Markov\ property} \quad & E[E[g(X_T)|X_{t+1}]|\,X_t] \\ = \; E[G(t+1,X_{t+1})|X_t] \; & = \; \sum_j p(x_i,x_j)\,G(t+1,x_j) \end{aligned} \tag{2}$$

2. define transition matrix $\Pi$ with $\Pi \;=\; (p(x_i,x_j))$. Then, prove
$$G(t,x) \;=\; \Pi^{T-t}\,g(x)$$

This follows by induction: The above calculation implies

$$\begin{aligned} G(t,x) \;&=\; \Pi\, G(t+1,x) \;=\; \Pi^2 G(t+2,x) \\ =\; \cdots \;&=\; \Pi^{T-t}G(T,x) = \Pi^{T-t}E[g(X_T)|X_T=x] = \Pi^{T-t}\,g(x)\,. \end{aligned} \tag{3}$$

3. Let
$$V(x) \;=\; \sum_{t=0}^{\infty} e^{-rt}\,E[X_t\,|\,X_0=x]$$

Prove that
$$V(x) \;=\; x \,+\, e^{-r}\,\Pi\,V(x) \;\Leftrightarrow\; V(x) \;=\; (\mathrm{Id}-e^{-r}\,\Pi)^{-1}\,x$$

There are two ways. First,

$$\begin{aligned} V(x) \;&=\; \sum_{t=0}^{\infty} e^{-rt}\,E[X_t\,|\,X_0=x] \;=\; E[\sum_{t=0}^{\infty} e^{-rt}\,X_t\,|\,X_0=x] \\ &=\; E[E_1[\sum_{t=0}^{\infty} e^{-rt}\,X_t\,]|\,X_0=x] \\ &=\; E[E[\sum_{t=0}^{\infty} e^{-rt}\,X_t\,|X_1]|\,X_0=x] \\ &=\; E[X_0 \;+\; E[\sum_{t=1}^{\infty} e^{-rt}\,X_t\,|X_1]|\,X_0=x] \\ &=\; x \;+\; e^{-r}E[E[\sum_{t=0}^{\infty} e^{-rt}\,X_{t+1}\,|X_1]|\,X_0=x] \\ &=\; x \;+\; e^{-r}E[V(X_1)|X_0=x] \;=\; x \,+\, e^{-r}\,\Pi\,V(x) \end{aligned} \tag{4}$$

Alternatively,
$$E[X_t|X_0=x] \;=\; \Pi^t x$$

by the above and therefore

$$V(x) \;=\; \sum_{t=0}^{\infty} e^{-rt}\Pi^t x \;=\; (\mathrm{Id}-e^{-r}\,\Pi)^{-1}\,x\,.$$

# 4. Kolmogorov Equations II

1. Suppose $X_t = Y_1 \cdots Y_t$ so that

$$X_{t+1} = X_t Y_{t+1} \tag{5}$$

2. $Y_t$ is a Markov process with transition probabilities $\Pi$
3. $X_t$ is **not a Markov process**
4. $(X_t, Y_t)$ is a two-dimensional Markov process
5. $X_t$ takes "too many values"
6. Define

$$V(X_t, Y_t) = E_t\left[\sum_{s=0}^{\infty} e^{-rs} X_{t+s}\right] = E\left[\sum_{s=0}^{\infty} e^{-rs} X_{t+s} | X_t, Y_t\right] \tag{6}$$

7. Derive Kolmogorov equation

$$V(X_t, Y_t) = X_t + e^{-r} E_t[V(X_{t+1}, Y_{t+1})]$$

Indeed,

$$\begin{aligned} V(X_t, Y_t) &= E\left[\sum_{s=0}^{\infty} e^{-rs} X_{t+s} | X_t, Y_t\right] \\ &= X_t + E\left[\sum_{s=1}^{\infty} e^{-rs} X_{t+s} | X_t, Y_t\right] \\ &= X_t + e^{-r} E\left[\sum_{s=1}^{\infty} e^{-r(s-1)} X_{t+s} | X_t, Y_t\right] \\ &\underbrace{=}_{iterated\ expectations} X_t + e^{-r} E\left[E_{t+1}[\sum_{s=1}^{\infty} e^{-r(s-1)} X_{t+s}] | X_t, Y_t\right] \\ &\underbrace{=}_{Markov\ property} X_t + e^{-r} E\left[E[\sum_{s=1}^{\infty} e^{-r(s-1)} X_{t+s} | X_{t+1}, Y_{t+1}] | X_t, Y_t\right] \\ &= X_t + e^{-r} E\left[\underbrace{E[\sum_{s=0}^{\infty} e^{-rs} X_{t+s+1} | X_{t+1}, Y_{t+1}]}_{=V(X_{t+1}, Y_{t+1})} | X_t, Y_t\right] \\ &= X_t + e^{-r} E_t[V(X_{t+1}, Y_{t+1})] \end{aligned} \tag{7}$$

8. Make an Ansatz

$$V(x, y) = x v(y)$$

to get that

$$
\begin{aligned}
&\underbrace{V(X_t, Y_t)}_{=X_t v(Y_t)} \\
&= X_t \ + \ e^{-r} E_t[\underbrace{V(X_{t+1}, Y_{t+1})}_{=X_{t+1} v(Y_{t+1})}] \\
&= X_t \ + \ e^{-r} E_t[X_{t+1} v(Y_{t+1})] \\
&\underbrace{=}_{(5)} \ X_t \ + \ e^{-r} E_t[X_t Y_{t+1} v(Y_{t+1})] \\
&= X_t \ + \ e^{-r} E_t[X_t Y_{t+1} v(Y_{t+1})]
\end{aligned} \tag{8}
$$

and therefore

$$
v(Y_t) \ = \ 1 \ + \ e^{-r} E_t[Y_{t+1} v(Y_{t+1})] \ = \ 1 \ + \ e^{-r} \sum_j p(Y_t, y_j) y_j v(y_j),
$$

which in vector form becomes

$$
v(y) \ = \ \mathbf{1} \ + \ e^{-r} \Pi diag(y) v(y)
$$

where $\mathbf{1}$ is the vector of ones.

9. Prove

$$
v(y) \ = \ (\mathrm{Id} - e^{-r} \Pi \, diag(y))^{-1} \mathbf{1} \ = \ \sum_{\tau=0}^{\infty} e^{-r\tau} (\Pi \, diag(y))^{\tau} \, \mathbf{1}
$$

**Theorem** This series converges if and only if the *spectral radius*

$$
\rho(\Pi \, diag(y)) \ = \ \max(|eig(\Pi \, diag(y))|)
$$

satisfies

$$
\rho(\Pi \, diag(y)) \ < \ e^r \tag{9}
$$

# 5. Kolmogorov Equations III

1. Suppose $X_t \ = \ Y_1 \cdots Y_t$
2. $Z_t$ is a Markov process with transition probabilities $\Pi$, $Y_t = Y(Z_t)$
3. $r_t = r(Z_t)$ (monetary policy)
4. $X_t$ is **not a Markov process**
5. $(X_t, Y_t)$ is a two-dimensional Markov process
6. $X_t$ takes "too many values"
7. Define

$$
V(X_t, Z_t) \ = \ E_t \left[ \sum_{s=0}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} \right] \ = \ E \left[ \sum_{s=0}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} | X_t, Z_t \right] \tag{10}
$$

8. Derive Kolmogorov equation

$$V(X_t, Z_t) \;=\; X_t \;+\; e^{-r_t} E_t[V(X_{t+1}, Z_{t+1})]$$

Indeed,

$$\begin{aligned}
V(X_t, Z_t) \;&=\; E\left[\sum_{s=0}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} | X_t, Z_t\right] \\
&=\; X_t \;+\; E\left[\sum_{s=1}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} | X_t, Y_t\right] \\
&=\; X_t \;+\; e^{-r} E\left[\sum_{s=1}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} | X_t, Y_t\right] \\
&\underbrace{=}_{iterated\ expectations} \; X_t \;+\; e^{-r_t} E\left[E_{t+1}[\sum_{s=1}^{\infty} e^{-\sum_{\tau=1}^{s-1} r_{t+\tau}} X_{t+s}] | X_t, Y_t\right] \\
&\underbrace{=}_{Markov\ property} \; X_t \;+\; e^{-r_t} E\left[E[\sum_{s=1}^{\infty} e^{-\sum_{\tau=1}^{s-1} r_{t+\tau}} X_{t+s} | X_{t+1}, Y_{t+1}] | X_t, Y_t\right] \\
&=\; X_t \;+\; e^{-r_t} E\left[\underbrace{E[\sum_{s=0}^{\infty} e^{-\sum_{\tau=0}^{s} r_{t+\tau}} X_{t+s+1} | X_{t+1}, Y_{t+1}]}_{=V(X_{t+1}, Y_{t+1})} | X_t, Y_t\right] \\
&=\; X_t \;+\; e^{-r_t} E_t[V(X_{t+1}, Y_{t+1})]
\end{aligned} \tag{11}$$

9. We now make the Ansatz

$$V(X_t, Y_t) \;=\; X_t v(Y_t)$$

10. Substituting that $X_{t+1} = X_t Y_{t+1}$ and $V(X_t, Y_t) \;=\; X_t v(Y_t)$ and $V(X_{t+1}, Y_{t+1}) \;=\; X_{t+1} v(Y_{t+1}) = X_{t+1} = X_t Y_{t+1} v(Y_{t+1})$, we get

$$X_t v(Y_t) \;=\; X_t \;+\; e^{-r(Z_t)} E_t[X_t Y(Z_{t+1}) v(Z_{t+1})] \;\Leftrightarrow$$

$$v(Z_t) \;=\; 1 \;+\; e^{-r(Z_t)} E_t[Y(Z_{t+1}) v(Z_{t+1})] \;=\; 1 \;+\; e^{-r(Z_t)} \sum_j p(Z_t, z_j) Y(z_j) v(z_j),$$

which in vector form becomes

$$v(z) \;=\; \mathbf{1} \;+\; diag(e^{-r(z)}) \Pi diag(Y(z)) v(z)$$

11. Prove

$$v(z) \;=\; (\mathrm{Id} - diag(e^{-r(z)}) \Pi diag(Y(z)))^{-1} \mathbf{1} \;=\; \sum_{\tau=0}^{\infty} (diag(e^{-r(z)}) \Pi \, diag(Y(z)))^{\tau} \, \mathbf{1}$$

**Theorem** This series converges if and only if the *spectral radius*

$$\rho(\Pi \, diag(y)) \;=\; \max(|eig(diag(e^{-r(z)}) \Pi \, diag(Y(z)))|) \;<\; 1$$

# 6. Replication and Binomial Trees

**Stochastic Integrals**

**Definition** Given a martingale $M_t$, the process

$$X_t = \sum_{s=0}^{t-1} \pi_s \left(M_{s+1} - M_s\right) = \pi \cdot M = \int_0^t \pi_s \, dM_s$$

is called the stochastic integral of $\pi$ with w.r.t. $M$

**Lemma** $X_t$ is a Martingale

**Proof** $X_{t+1} = X_t + \pi_t \left(M_{t+1} - M_t\right)$ implies

$$E_t[X_{t+1}] = X_t + \pi_t \, E_t[M_{t+1} - M_t] = X_t$$

**Self-Financing Portfolio Gains Processes**

- there are two investment opportunities: risk-less with zero interest rate (bank account) and stock with price process $M_t$
- $\pi_t$ is the number of shares of the stock purchased at time $t$
- the total gains process change is

$$X_{t+1} - X_t = \pi_t \left(M_{t+1} - M_t\right)$$

- that is

$$X_t = X_0 + \pi \cdot M$$

**Replication and Martingale Representation for the Binomial Model**

- Let

$$M_t = Y_1 \cdots Y_t$$

  where $Y_t$ are i.i.d., $Y_t = u$ or $d$ with prob. $p$ such that $pu + (1-p)d = 1$. Then, $M$ is a martingale.
- $\mathcal{F}_t$ is the natural filtration of $Y_t$
- $X_T$ is a $\mathcal{F}_T$-measurable random variable
- then,

$$X_{T-1} = E_{T-1}[X_T]$$

  and solving

$$\begin{cases} X_{T-1} + \pi_{T-1} \, (u-1) \, M_{T-1} = X_T(u, X_{T-1}) \\ X_{T-1} + \pi_{T-1} \, (d-1) \, M_{T-1} = X_T(d, X_{T-1}) \end{cases}$$

  gives the replicating portfolio $\pi_{T-1}$

**Why is there a solution: first derivation**

- $$pu + (1-p)d = 1 \Leftrightarrow p = \frac{1-d}{u-d}$$
- the system has a solution if and only if

$$\frac{X_T(u, X_{T-1}) - X_{T-1}}{(u-1) \, M_{T-1}} = \frac{X_T(d, X_{T-1}) - X_{T-1}}{(d-1) \, M_{T-1}}$$

- by direct algebraic calculation, this is equivalent to

$$\frac{1-d}{u-d} X_T(u, X_{T-1}) + \frac{u-1}{u-d} X_T(d, X_{T-1}) = X_{T-1} \;\Leftrightarrow$$

$$pX_T(u, X_{T-1}) + (1-p) X_T(d, X_{T-1}) = X_{T-1}$$

  that is

$$X_{T-1} \;=\; E_{T-1}[X_T].$$

**Second Derivation**

- for any $\mathcal{F}_{T-1}$-measurable random variable, we have

$$E_{T-1}[X_{T-1} + \pi_{T-1}\,(M_T - M_{T-1})] \;=\; X_{T-1} \tag{12}$$

- if $E_{T-1}[X_T] = X_{T-1}$ then

$$X_T(d, X_{T-1}) = (X_{T-1} - p\, X_T(u, X_{T-1}))/(1-p)$$

- thus, if we have two variables $Z_1$ and $Z_2$ that have the same value at the state $u$ and both satisfy $E_{T-1}[Z_1] = E_{T-1}[Z_2]$, they have the same value in the state $d$
- thus, if $X_{T-1} + \pi_{T-1}\,(u-1)M_{T-1} = X_T(u, X_{T-1})$ then we get

$$X_{T-1} + \pi_{T-1}\,(d-1)M_{T-1} \;=\; X_T(d, X_{T-1})$$