Asset pricing
Homework 4 Solution

Reminder If we have S states and complete markets with a payoff matrix X then

- $p \;=\; Xq$ defines states prices $q = X^{-1}p$ which are prices of Arrow Securities e_s
- a portfolio θ maps into a portfolio $\tilde{\theta} = X'\theta$ of Arrow securities in the sense that they have the same payoff $(\theta' X)(s) = \tilde{\theta}_s$. So, once we have found a portfolio $\tilde{\theta}$, we can immediately recover $\theta = (X')^{-1}\tilde{\theta}$
- budget constrains with Arrow securities

$$c_0^i \;=\; \omega_0^i - \sum_s q_s \theta_s^i,\; c_1^i(s) \;=\; \omega_1^i(s) \;+\; \theta^i(s) \tag{1}$$

imply a unique inter-temporal budget constraint

$$\underbrace{c_0^i + \sum_s q_s c_1^i(s)}_{market\ value\ of\ consumption} \qquad = \qquad \underbrace{\omega_0^i + \sum_s q_s \omega_1^i(s)}_{market\ value\ of\ endowment} \tag{2}$$

(assuming endowment is fully pledgeable). As a result, the first order condition for

$$u_i(c_0^i) + \delta_i E[u_i(c_1^i)] \;=\; u_i(c_0^i) + \delta_i \sum_s \pi_s u_i(c_1^i(s)) \tag{3}$$

is

$$u_i'(c_0^i) = \lambda_i,\; \delta_i \pi_s u_i'(c_1^i(s)) \;=\; q_s \lambda_i \tag{4}$$

where λ_i is the Lagrange multiplier for the budget constraint. We thus get

$$c_0^i = (u_i')^{-1}(\lambda_i),\; c_1^i(s) \;=\; (u_i')^{-1}(\delta_i^{-1} M_s \lambda_i) \tag{5}$$

where we have defined the state price density

$$M_s \;=\; q_s/\pi_s\,, \tag{6}$$

and the Lagrange multiplier λ_i is determined by the binding budget constraint

$$(u_i')^{-1}(\lambda_i) + E[M (u_i')^{-1}(\delta_i^{-1} M \lambda_i)] \;=\; \omega_0^i + E[M\omega_1^i]\,. \tag{7}$$

Then, equilibrium state prices M are determined by consumption market clearing:

$$\sum_i (u_i')^{-1}(\delta_i^{-1} M_s \lambda_i) \;=\; \Omega(s) \tag{8}$$

for each state s implying that

$$M_s \;=\; H(\Omega(s)), \tag{9}$$

where H is the inverse function of

$$h(x) \;=\; \sum_i (u_i')^{-1}(\delta_i^{-1} x \lambda_i) \tag{10}$$

so that $h(H(x)) = x$.

- we immediately recognize the link with the social planner: in complete markets, equilibrium coincides with a Pareto efficient social planner allocation maximizing

$$\sum_i \mu_i (u_i(c_0^i) + \delta_i \sum_s \pi_s u_i(c_1^i(s))) \tag{11}$$

under the feasibility constraint $\sum_i c_i = \Omega$, but with a specific equilibrium choice of social utility weights

$$\mu_i \;=\; \lambda_i^{-1} \tag{12}$$

We can see from the above that λ_i is monotone decreasing in total inter-temporal wealth $\omega_0^i + E[M\omega_1^i]$. Thus, the invisible market hand (Adam Smith) allocates resources like a social planner who puts a larger weight on richer individuals.

- **Representative agent.** Imagine the market is populated by just one person who trades with himself :-) And he/she has some weird utility U. The analysis above still applies, and state prices are proportional to his marginal utility $U'(\Omega) = \lambda M$. Thus, we can think of the function H above as the marginal utility of an artificial agent. It is then straightforward to show that, in fact, his utility is the social planner utility at equilibrium weights:

$$U(x) \;=\; \max\{\sum_i \mu_i \sum_s \pi_s \delta_i u_i(c_1^i) : \; \sum_i c_1^i \;=\; x\} \tag{13}$$

- Sometimes, we have *effectively complete markets.* Suppose that markets are incomplete. Yet let us first, solve for equilibrium with complete markets. Suppose then we find that in this complete market equilibrium, $c^i - \omega^i$ belongs to the span of X. This means that even in this artificial complete market equilibrium, agents end up trading only securities from X. Thus, incompleteness has no bite in equilibrium, and agents can achieve the same consumption allocation by trading the original securities. Thus, the artificial complete market equilibrium we have constructed is, in fact, also an equilibrium in the original, incomplete market model.

1. Equilibrium with Linear Risk Tolerance

Consider an economy with two dates and three equiprobable states with two agents that have the following expected utility preference:

$$\frac{C_0^{1-\gamma}}{1-\gamma} + \delta \mathbb{E}[\frac{C_1^{1-\gamma}}{1-\gamma}]$$

with $\delta = 0.95$ and $\gamma = 2$. Suppose both agents are endowed with $\omega_0^a = 1$ and $\omega_0^b = 2$ unit of consumption good at date 0 and with $\omega_1^a = [1; 2; 3]$ and $\omega_1^b = [3; 2; 1]$ at time 1 respectively.

- Suppose that three state-contingent claims are traded with price q_i that each pays off 1 in state i for $i = 1, 2, 3$. Determine the equilibrium consumption of each agent, the equilibrium values of the state prices, and the corresponding risk-free rate.
- Determine the utility function of the representative agent. Explain why such an agent exists. Derive the Pareto Optimal Sharing Rule ($C_1(\Omega)$ and $C_2(\Omega)$ where Ω is the aggregate endowment).
- Suppose that instead of the three state contingent claims, there are three securities with price P_1, P_2, P_3 and payoff $[1; 1; 0]$, $[0; 1; 2]$ and $[2; 0; 1]$ respectively that are traded. Determine the equilibrium consumption allocation, the trading strategies of the agents, and prices of these securities.
- Now assume that the only security that is traded is a risk-free bond with price P_0 and payoff $[1, 1, 1]$. Show that any Pareto optimal consumption allocation in this economy lies in the span of P_0. Can agents achieve the same Pareto optimal allocation as in the previous section by trading only in that bond?

Solutions to all items above

$$\delta_i C_{1,i}^{-\gamma_i} \;=\; \lambda_i\, M,\; C_{0,i}^{-\gamma} \;=\; \lambda_i \tag{14}$$

so that

$$C_{1,i} \;=\; (\delta_i \lambda_i^{-1} M^{-1})^{1/\gamma_i} \tag{15}$$

and

$$C_{0,i} + E[C_{1,i} M] \;=\; W_i \;=\; \omega_{0,i} \;+\; E[\omega_{1,i} M]\,, \tag{16}$$

and substituting, we get

$$\lambda_i^{-1/\gamma_i} + E[(\delta_i \lambda_i^{-1} M^{-1})^{1/\gamma_i} M] \;=\; W_i \;=\; \omega_{0,i} \;+\; E[\omega_{1,i} M] \tag{17}$$

so that

$$\lambda_i^{-1/\gamma_i} \;=\; \frac{W_i}{1 + E[(\delta_i M^{-1})^{1/\gamma_i} M]}\,, \tag{18}$$

and market clearing for consumption gives

$$\sum_i (\delta_i \lambda_i^{-1} M^{-1})^{1/\gamma_i} \;=\; \Omega_1 \tag{19}$$

In all γ_i are the same, we get

$$M \;=\; \Omega_1^{-\gamma} \left(\sum_i (\delta_i \lambda_i^{-1})^{1/\gamma} \right)^{\gamma} \;=\; H(\Omega_1) \tag{20}$$

and hence any agent with $U_0,\ U_1$ maximizing $E[U_0(C_0) + U_1(C_1)]$ satisfying

$$H(\Omega_1) \;=\; U_1'(\Omega_1)/U_0'(\Omega_0) \tag{21}$$

For example,

$$U_1(x) \;=\; \left(\sum_i (\delta_i \lambda_i^{-1})^{1/\gamma} \right)^{\gamma} x^{1-\gamma}/(1-\gamma) \tag{22}$$

and

$$U_0(\Omega_0) \;=\; \Omega_0 \tag{23}$$

does the job.

Let us define

$$\psi_i \;=\; \lambda_i^{-1/\gamma} \tag{24}$$

Then, the budget constraint implies

$$\psi_i \;=\; \frac{\omega_{0,i} \;+\; E[\omega_{1,i}M]}{1+\delta_i^{1/\gamma}E[(\Omega_1^{-\gamma}\;\left(\sum_i(\delta_i\lambda_i^{-1})^{1/\gamma}\right)^{\gamma})^{1-1/\gamma}]} \tag{25}$$

that is

$$\psi_i \;=\; \frac{\omega_{0,i} \;+\; E[\omega_{1,i}\Omega_1^{-\gamma}\;\left(\sum_i(\delta_i)^{1/\gamma}\psi_i\right)^{\gamma}]}{1+\delta_i^{1/\gamma}E[(\Omega_1^{-\gamma}\;\left(\sum_i(\delta_i)^{1/\gamma}\psi_i\right)^{\gamma})^{1-1/\gamma}]} \tag{26}$$

Let

$$\Psi \;=\; \sum_i(\delta_i)^{1/\gamma}\psi_i \tag{27}$$

Then, summing up gives

$$\Psi \;=\; \frac{\sum_i\delta_i^{1/\gamma}(\omega_{0,i} \;+\; E[\omega_{1,i}\Omega_1^{-\gamma}]\Psi^{\gamma})}{1+\delta_i^{1/\gamma}E[(\Omega_1^{-\gamma}\,\Psi^{\gamma})^{1-1/\gamma}]} \tag{28}$$

When $\delta_i=\delta$ for all i then

$$M \;=\; \delta\,(\Omega_1/\Omega_0)^{-\gamma} \tag{29}$$

Markets are effectively complete if

$$C_{1,i}-\omega_{1,i} \;\in\; span \tag{30}$$

Equilibrium sharing rules are

$$C_{1,i}/\Omega_1 \;=\; \xi_i \tag{31}$$

where

$$\xi_i \;=\; \frac{(\delta_i\lambda_i^{-1})^{1/\gamma}}{\sum_j(\delta_j\lambda_j^{-1})^{1/\gamma}} \tag{32}$$

Span: If you have several securities with payoff vectors $(X_k(s))_{s=1}^S$ and you buy π_k units of security k then

$$C_{1,i}(s) \;=\; \omega_{1,i}(s) \;+\; \sum_k \pi_k X_k(s) \tag{33}$$

Thus, to find the portfolio π_k, we need to solve the system

$$\sum_k \pi_k X_k(s) \;=\; C_{1,i}(s) \;-\; \omega_{1,i}(s) \;=\; \theta \tag{34}$$

This system has a solution $(\pi_k)_{k=1}^K$ if and only if the vector $(C_{1,i}(s)\;-\;\omega_{1,i}(s))_{s=1}^S$ belongs to the span of $(X_k(s))_{s=1}^S$. For example, in our setting, $(\Omega_1(s))=\sum_i\omega_{1,i}=(4,4,4)$ and hence

$$C_{1,a}(s) \;-\; \omega_{1,a}(s) \;=\; (4,4,4)\xi_a-(1,2,3) \tag{35}$$

For example, if there is only one security, $X_1(s) = 1$ (the bond), then, clearly, $C_{1,a}(s) - \omega_{1,a}(s)$ is not in the span. By contrast, if you have also $X_2(s) = (1, 2, 3)$ then

$$C_{1,a}(s) \; - \; \omega_{1,a}(s) \; = \; (4,4,4)\xi_a - (1,2,3) \; = \; 4\xi_a X_1 \; - \; X_2 \tag{36}$$

End of Solution

- Determine the optimal asset allocation, and hence the equilibrium consumption allocation, and the price of the only traded security (the risk-free bond) in this market (with the same endowment structure as above). Can you construct a representative agent in this economy? Explain why.

Solution:

$$C_{0,i} \; = \; \omega_{0,i} \; - \; x\,p, \; C_{1,i} \; = \; \omega_{1,i} \; + \; x\,, \tag{37}$$

and the first-order conditions are

$$\delta_i E[(\omega_{1,i} \; + \; x)^{-\gamma}] \; = \; p(\omega_{0,i} \; - \; x\,p)^{-\gamma}\,, \tag{38}$$

and this implicitly defines

$$x \; = \; X_i(p) \tag{39}$$

and so p is determined by

$$\sum_i X_i(p) \; = \; 0 \tag{40}$$

If

$$\delta_i E[(\omega_{1,i})^{-\gamma}]\omega_{0,i}^{\gamma} \tag{41}$$

is independent of i, then autarky is an equilibrium with

$$p \; = \; \delta_i E[(\omega_{1,i})^{-\gamma}]\omega_{0,i}^{\gamma} \tag{42}$$

End of Solution

2. Representative Agent Economy with non-linear sharing rules

Consider an economy with two dates and two equiprobable states with two agents (a,b) who have the following expected utility preference:

$$\frac{C_0^{1-\gamma_i}}{1-\gamma_i} + \delta_i \mathbb{E}[\frac{C_1^{1-\gamma_i}}{1-\gamma_i}]$$

with $\delta_a = 0.95$, $\delta_b = 0.9$ and $\gamma_a = 2$ and $\gamma_b = 2\gamma_a = 4$.

Suppose that agent a is endowed with $\omega_0^a = 2$ units of the consumption good at date 0 and that agent b owns 1 share of a company that will pay $X = [2; 3]$ tomorrow depending on the state. Assume both agents can trade today's shares in that company at a price S_0 and borrow and lend from each other at a risk-free rate of R_f.

- Show that markets are complete when agents trade in S_0 and can borrow or lend at the risk-free rate from each other.
- Determine the utility function of the representative agent. Derive the non-linear risk-sharing rule $C_i(\Omega_s)$ $i = 1, b$ for that agent, where Ω_s is the aggregate amount of consumption good in state s.
- Use the marginal utility of any agent or the representative agent to find the equilibrium Arrow Debreu prices in this economy.
- Derive the equilibrium risk-free rate R_f and stock price S_0.

Solution Markets are complete because there are only two states of the world. Equilibrium is determined by (19) (with the Lagrange multipliers determined by (18)). In our case, (19) takes the form

$$(\delta_a \lambda_a^{-1} M^{-1})^{1/2} \ + \ (\delta_b \lambda_b^{-1} M^{-1})^{1/4} \ = \ \Omega_1 \tag{43}$$

where $\Omega_1 = (2, 3)$ (the payoff of the company share. Note that the shares pay in real goods that are then consumed!)

Define

$$x \ = \ M^{-1/4}. \tag{44}$$

Then, (43) takes the form

$$\alpha_a x^2 + \alpha_b x = \Omega_1, \ \alpha_i \ = \ (\delta_i \lambda_i^{-1})^{1/\gamma_i}, \ i = a, b\,. \tag{45}$$

Solving this quadratic equation gives

$$x \ = \ (-\alpha_b \ + \ \sqrt{\alpha_b^2 + 4\alpha_a \Omega_1})/(2\alpha_a) \tag{46}$$

and therefore

$$M \ = \ H(\Omega_1) \ = \ ((-\alpha_b \ + \ \sqrt{\alpha_b^2 + 4\alpha_a \Omega_1})/(2\alpha_a))^{-4} \tag{47}$$

and the representative agent can be defined using (21); and then consumption-sharing rules are

$$C_i \ = \ (\delta_i \lambda_i^{-1} H(\Omega_1)^{-1})^{1/\gamma_i}\,. \tag{48}$$

Note that in this expression, it is not at all obvious whether $C_a + C_b = \Omega_1$.

The risk free rate is

$$R_f \ = \ 1/E[M] \tag{49}$$

and

$$S_0 \ = \ E[XM] \ = \ E[\Omega_1 M] \tag{50}$$

End of Solution

3. Consumption Sharing Rules with Linear Risk Tolerance

- Show that if agents have linear risk-tolerances of the form $-\frac{U_i'(c)}{U_i''(c)} = \alpha_i + \beta c \ \ \forall i = 1, \ldots, n$, then the consumption of each individual agent in a Pareto-efficient equilibrium with n agents is linear in aggregate consumption.
- Conversely, show that if in a Pareto-efficient equilibrium, consumption sharing rules are linear, then agents have linear risk-tolerances.

Solution We know from the above that

$$\sum_i (u_i')^{-1}(\delta_i^{-1} M_s \lambda_i) \ = \ \Omega(s) \tag{51}$$

and

$$M_s \ = \ H(\Omega(s)), \tag{52}$$

where H is the inverse function of

$$h(x) \ = \ \sum_i (u_i')^{-1}(\delta_i^{-1} x \lambda_i) \tag{53}$$

so that $h(H(x)) = x$. We thus have

$$C_i(\Omega) \ = \ (u_i')^{-1}(\delta_i^{-1} \lambda_i H(\Omega)) \tag{54}$$

and

$$u_i'((u_i')^{-1}(x)) = x \tag{55}$$

so that

$$u_i'(C_i) \ = \ \delta_i^{-1} \lambda_i H(\Omega)$$

implies

$$u_i''((u_i')^{-1}(x))((u_i')^{-1}(x))' \ = \ 1$$

Similarly,

$$h'(H(x))H'(x) = 1$$

gives

$$\begin{aligned}
H'(x) \;&=\; \frac{1}{h'(H(x))} \;=\; \frac{1}{\sum_i((u_i')^{-1}(\delta_i^{-1}H(x)\lambda_i))'\delta_i^{-1}\lambda_i} \\
&=\; \frac{1}{\sum_i \delta_i^{-1}\lambda_i/u_i''((u_i')^{-1}(\delta_i^{-1}H(x)\lambda_i))} \\
&=\; \frac{1}{\sum_i \delta_i^{-1}\lambda_i/u_i''(C_i)} \\
&=\; \frac{1}{\sum_i \delta_i^{-1}\lambda_i H(x)/(H(x)u_i''(C_i))} \\
&=\; \frac{1}{\sum_i u_i'(C_i)/(H(x)u_i''(C_i))}
\end{aligned} \tag{56}$$

so that

$$\begin{aligned}
C_i'(\Omega) \;&=\; ((u_i')^{-1}(\delta_i^{-1}\lambda_i H(\Omega)))' \;=\; ((u_i')^{-1})'(\delta_i^{-1}\lambda_i H(\Omega))\delta_i^{-1}\lambda_i H'(\Omega) \\
&=\; (1/u_i''(C_i))\delta_i^{-1}\lambda_i H'(\Omega) \\
&=\; (1/u_i''(C_i))\delta_i^{-1}\lambda_i H(\Omega)H'(\Omega)/H(\Omega) \\
&=\; (1/u_i''(C_i))u'(C_i)H'(\Omega)/H(\Omega) \\
&=\; (1/u_i''(C_i))u'(C_i)\frac{1}{\sum_i u_i'(C_i)/u_i''(C_i))} \;=\; \frac{T_i(C_i)}{\sum_j T_j(C_j)}\,.
\end{aligned} \tag{57}$$

where

$$T_i(c) = -u'(c)/u''(c) \tag{58}$$

is the risk tolerance function. This is very intuitive: Exposure of consumption to shocks is proportional to the relative risk tolerance of the agent. The claim now follows: To get linearity, we need the slopes C_i' to be constant, and the only way to achieve it is to have a linear risk tolerance.

End of Solution