# Asset pricing
# Homework 3 Solution

## Exercise 1

1. Let $w$ be the agent's wealth. The agent invests a fraction x of her wealth in the risky asset R and the rest in the risk-free asset, which makes the dollar amount invested in the risky asset equal $xw$. Let the risky asset return be $R$ and the risk-free asset return be $R_f$. The terminal wealth is
$$W_1 = wR_f + x(w)w\,(R - R_f) \tag{1}$$

2. Let $u(\cdots)$ be the utility function. From the lecture notes, we use payoff $X$, pricing kernel $M$ and asset price to demonstrate that $E\left[M\left(R - R_f\right)\right] = 0$, based on which we get the first order condition
$$E\left[u'\left(wR_f + xw\left(R - R_f\right)\right)\left(R - R_f\right)\right] = 0, \tag{2}$$
while the second order condition is satisfied because $u$ is strictly concave and hence $u'' < 0$.[1]

3. Using the same lemma as in the previous problem set, we get: If $x < 0$, then $f(R - R_f) = u'\left(wR_f + xw\left(R - R_f\right)\right)$ is monotone increasing in $R - R_f$ and therefore
$$0 \;=\; E\left[u'\left(wR_f + xw\left(R - R_f\right)\right)\left(R - R_f\right)\right] \;\geq\; E\left[u'\left(wR_f + xw\left(R - R_f\right)\right)\right]E[R - R_f] \;>\; 0,$$
which is a contradiction. Thus $x \geq 0$.

4. Differentiating the first-order condition in equation (2) with respect to $w$, we get
$$E\left[u''\left(wR_f + x(w)w\left(R - R_f\right)\right)\left(R - R_f\right)\left(R_f + \left(R - R_f\right)\partial_w\left(x(w)w\right)\right)\right] = 0\,, \tag{3}$$
and therefore
$$wx'(w) \;=\; -\frac{E\left[u''\left(wR_f + x(w)w\left(R - R_f\right)\right)w^{-1}(R - R_f)\left(R_f w + (R - R_f)xw\right)\right]}{E\left[u''\left(wR_f + xw\left(R - R_f\right)\right)\left(R - R_f\right)^2\right]} \tag{4}$$
and our goal is to show that, under the technical conditions stated in the homework, $x'(w) > 0$. Since $u'' \leq 0$, we just need to show that
$$E\left[u''\left(wR_f + x(w)w\left(R - R_f\right)\right)w^{-1}(R - R_f)\left(R_f w + (R - R_f)xw\right)\right] \;>\; 0 \tag{5}$$

[1] Second order conditions guarantee that the extremum is not a saddle point but a genuine local maximum. Proving global maximality requires some global properties, such as concavity.

Now, let

$$\mathcal{RA}(R-R_f) \;=\; -\frac{u''\left(wR_f+x(w)w\left(R-R_f\right)\right)\left(wR_f+x(w)w\left(R-R_f\right)\right)}{u'\left(wR_f+x(w)w\left(R-R_f\right)\right)} \tag{6}$$

be the relative risk aversion evaluated at the optimal wealth. By assumption, relative risk aversion is monotone decreasing, meaning that $\mathcal{RA}(R-R_f)$ is a monotone decreasing function in $R-R_f$. Furthermore, by assumption

$$-u''(x)x/u'(x) \;\le\; 1\,, \tag{7}$$

which implies

$$\begin{aligned}
&-u''\left(wR_f+x(w)w\left(R-R_f\right)\right)xw(R-R_f)\\
&\le\; -u''\left(wR_f+x(w)w\left(R-R_f\right)\right)\left(wR_f+xw(R-R_f)\right)\\
&\le\; u'\left(wR_f+x(w)w\left(R-R_f\right)\right)
\end{aligned} \tag{8}$$

By direct calculation, this implies that

$$g(R-R_f) \;=\; u'\left(wR_f+x(w)w\left(R-R_f\right)\right)\left(R-R_f\right) \tag{9}$$

is monotone increasing in $R-R_f$, By the same Lemma as in the previous problem set,

$$\begin{aligned}
&E\left[u''\left(wR_f+x(w)w\left(R-R_f\right)\right)\left(R-R_f\right)\left(R_fw+(R-R_f)xw\right)\right]\\
&=-E\left[\mathcal{RA}(R-R_f)\,u'\left(wR_f+x(w)w\left(R-R_f\right)\right)\left(R-R_f\right)\right]\\
&=\;E[-\mathcal{RA}(R-R_f)g(R-R_f)] \;>\; E[-\mathcal{RA}(R-R_f)]E[g(R-R_f)] \;=\; 0
\end{aligned} \tag{10}$$

where we have used that, by the first order condition (2), we have $E[g(R-R_f)] \;=\; 0.$

5. if preferences are constant relative risk aversion, we know from the calculation in the class that $x(w)$ is constant even in the multi-asset case.

## Exercise 2

1. •

$$\lim_{\gamma\to 1}\frac{(\alpha+\gamma w)^{\left(1-\frac{1}{\gamma}\right)}-1}{\gamma-1}$$

Applying l'Hôpital's rule:

$$\lim_{\gamma\to 1}\frac{(\alpha+w\gamma)^{-\frac{1}{\gamma}}\left((\alpha+w\gamma)\log(\alpha+w\gamma)+w(\gamma-1)\gamma\right)}{\gamma^2}=\log(\alpha+w)$$

□

•

$$\lim_{\gamma\to 0}\frac{(1+\frac{\gamma}{\alpha}w)^{\left(1-\frac{1}{\gamma}\right)}}{\gamma-1}$$

$$\frac{\lim_{\gamma\to 0}(1+\frac{\gamma}{\alpha}w)^{\left(1-\frac{1}{\gamma}\right)}}{\lim_{\gamma\to 0}\gamma-1}$$

$$-\lim_{\gamma\to 0}(1+\frac{\gamma}{\alpha}w)^{\left(1-\frac{1}{\gamma}\right)}$$

Let $u = \frac{\gamma w}{\alpha}$. Then

$$\begin{aligned}
-\lim_{\gamma\to 0}(1+\frac{\gamma}{\alpha}w)^{\left(1-\frac{1}{\gamma}\right)} &= -\lim_{u\to 0}(1+u)^{\left(1-\frac{w}{u\alpha}\right)} \\
&= -\frac{\lim_{u\to 0}(1+u)}{\lim_{u\to 0}(1+u)^{\left(\frac{1}{u}\right)^{\frac{w}{\alpha}}}} \\
&= -\frac{1}{e^{\frac{w}{\alpha}}} \\
&= -e^{-\frac{w}{\alpha}}
\end{aligned}$$

□

2.

$$\begin{aligned}
A(w) &= -\frac{u''(w)}{u'(w)} \\
T(w) &= -\frac{u'(w)}{u''(w)} \\
&= -\frac{(\alpha+\gamma w)^{-\frac{1}{\gamma}}}{-(\alpha+\gamma w)^{-\frac{1}{\gamma}-1}} \\
&= \alpha+\gamma w
\end{aligned}$$

□

3. Take the *ansatz* that $a_i = (\alpha+\gamma w R_f)b_i$ where $b_i$ is independent of $\alpha$ and $w$. We know that the solution for the optimal portfolio choice problem should obey:

$$\mathbb{E}\left(u'\left(wR_f+\sum_{i=1}^{n}a_i(R_i-R_f)\right)(R_i-R_f)\right)=0$$
$$\mathbb{E}\left(\left(\alpha+\gamma\left(wR_f+\sum_{i=1}^{n}a_i(R_i-R_f)\right)\right)^{-\frac{1}{\gamma}}(R_i-R_f)\right)=0$$

Substituting $a_i$ we have that

$$\mathbb{E}\left(\left(\alpha+\gamma\left(wR_f+\sum_{i=1}^{n}((\alpha+\gamma wR_f)b_i)(R_i-R_f)\right)\right)^{-\frac{1}{\gamma}}(R_i-R_f)\right)=0$$

$$\mathbb{E}\left((\alpha+\gamma wR_f)^{-\frac{1}{\gamma}}\left(1+\sum_{i=1}^{n}\gamma b_i(R_i-R_f)\right)^{-\frac{1}{\gamma}}(R_i-R_f)\right)=0$$

$$\mathbb{E}\left(\left(1+\sum_{i=1}^{n}\gamma b_i(R_i-R_f)\right)^{-\frac{1}{\gamma}}(R_i-R_f)\right)=0$$

Now notice that the system of equations above for every $i \in \{1,2,...,n\}$ does not involve $w$ or $\alpha$, and therefore neither does the solution vector $b$.

4. The system of equations that lead to the optimal allocation vector of risky assets $b$ does not involve $w$ or $\alpha$ and is therefore independent of both. It does, however, involve $\gamma$ and is therefore not independent of the coefficient of relative risk aversion.

## Exercise 3

Let $R$ be elliptically distributed with probability density function $f(x) = kg((x-\mu)^T\Sigma^{-1}(x-\mu))$. To show that $\mathbb{E}[e^{\alpha^T R}] = e^{\alpha^T\mu}\psi(\alpha^T\Sigma\alpha)$, we will proceed as following: From known properties of elliptical distributions,

$$R \;=\; \mu \;+\; \Sigma^{1/2}Y \tag{11}$$

where $Y$ is a spherically symmetric random vector, so that $\alpha^\top Y$ has the same distribution as $\|\alpha\|Y_1$. Thus,

$$E[e^{\beta Y}] \;=\; E[e^{\|\beta\|Y_1}] \;=\; \psi(\|\beta\|^2)$$

and

$$E[-e^{-\alpha^\top R}] \;=\; E[-e^{-\alpha^\top\mu\; -\alpha^\top\Sigma^{1/2}Y}] \;=\; -e^{-\alpha^\top\mu}\psi(\|\Sigma^{1/2}\alpha\|^2) \;=\; -e^{-\alpha^\top\mu}\psi(\alpha^\top\Sigma\alpha) \tag{12}$$

The portfolio choice problem is the maximization task for the expected future utility. Assume there is an economy with $n$ risky and no risk-free assets. We are given an exponential utility function; then the task is as follows:

$$\max_{\alpha}\mathbb{E}[-e^{-\alpha^T R}] = \min_{\alpha} e^{-\alpha^T\mu}\psi(\alpha^T\Sigma\alpha)$$

Denote $\alpha^T\Sigma\alpha = \xi$. Differentiating the function with respect to $\alpha$, one gets

$$\frac{\partial}{\partial\alpha^T}e^{-\alpha^T\mu}\psi(\alpha^T\Sigma\alpha) = 0$$

or, equivalently,

$$2\Sigma\alpha\psi'(\xi) = \mu\psi(\xi)$$

Isolating $\alpha$,

$$\alpha = \frac{\psi(\xi)}{2\psi'(\xi)}\Sigma^{-1}\mu$$

## Exercise 4

We are maximizing

$$E[-e^{-\gamma(wR_f+D(X-R_fp))}|\theta] \;=\; -e^{-\gamma wR_f+\gamma DR_fp}E[e^{-\gamma DX}|\theta] \tag{13}$$

Now,

$$E[e^{-\gamma DX}|\theta] \;=\; \int e^{a(\theta)+\theta X+b(X)}e^{-\gamma DX}dX \;=\; \int e^{a(\theta)+(\theta-\gamma D)X+b(X)}dX \tag{14}$$

Instead of maximizing this objective, we can equivalently assume that the agent is minimizing

$$\log E[e^{-\gamma(wR_f+D(X-R_fp))}|\theta] \;=\; -\gamma wR_f+\gamma DR_fp+\psi(\theta-\gamma D)$$

where we have defined the cumulant generating function (CGF)

$$\psi(y) \;=\; \log\int e^{a(\theta)+yX+b(X)}dX \tag{15}$$

The first-order condition takes the form

$$R_fp \;=\; \psi'(\theta-\gamma D) \tag{16}$$

By the well-know properties of CGFs, they are convex and thus $\psi'$ is monotone increasing and has a well-defined inverse $(\psi')^{-1}(x)$ that we denote by $g(x)$. In this case, we get

$$g(p) \;=\; \theta-\gamma D \;\Leftrightarrow D(p) \;=\; \frac{\theta-g(p)}{\gamma}\,. \tag{17}$$

In the case of a Gaussian distribution, we have $g(p) \;=\; p/\sigma^2$ and $\theta=\mu/\sigma^2$, and we get the standard mean-variance efficient portfolio. The general formula for $D(p)$ shows the key properties are preserved:

- Demand is *downward-sloping*: $D$ is decreasing in $p$
- Demand sensitivity to prices depends inversely on $\gamma$: The higher $\gamma$, the more inelastic the demand.

However, in general, demand is non-linear in $p$