Asset pricing
Homework 2 Solutions

October 23, 2023

Exercise 1

- Fix $L^a = \{(x_1, p_1^a), \ldots, (x_S, p_S^a)\}$, $L^b = \left\{\left(x_1, p_1^b\right), \ldots, \left(x_S, p_S^b\right)\right\}$ and $L^c = \{(x_1, p_1^c), \ldots, (x_S, p_S^c)\}$. Let $L^i \succcurlyeq L^j := \mathbb{E}\left[u\left(L^i\right)\right] \geq \mathbb{E}\left[u\left(L^j\right)\right]$.

  Note that all lotteries have the same payoff, so taking combinations of lotteries only affects probabilities:

  $$\mathbb{E}[u(L^a)] \;=\; \sum_s u(x_s)p_s^a \tag{1}$$

  and for a convex combination

  $$L^* \;=\; \alpha L^a + (1-\alpha)L^b \;=\; \left\{\left(x_1, \alpha p_1^a + (1-\alpha)p_1^b\right), \ldots, \left(x_S, \alpha p_S^a + (1-\alpha)p_S^b\right)\right\}$$

  and therefore

  $$\mathbb{E}[u(L^*)] \;=\; \sum_s u(x_s)(\alpha p_s^a + (1-\alpha)p_s^b) \;=\; \alpha\mathbb{E}[u(L^a)] + (1-\alpha)\mathbb{E}[u(L^b)]$$

  1. *Completeness*

     Since $u(\cdot) \in \mathbb{R}$, then $\mathbb{E}[u(\cdot)] \in \mathbb{R}$. We know that the $\geq$ relation is complete in $\mathbb{R}$. Therefore, $\succcurlyeq$ is complete in the space of lotteries.

  2. *Reflexivity*

     $\mathbb{E}\left[u\left(L^i\right)\right] = \mathbb{E}\left[u\left(L^i\right)\right] \therefore \mathbb{E}\left[u\left(L^i\right)\right] \geq \mathbb{E}\left[u\left(L^i\right)\right] \implies L^i \succcurlyeq L^i$.

  3. *Transitivity*

     Assume $L^a \succcurlyeq L^b$ and $L^b \succcurlyeq L^c$. Thus $\mathbb{E}\left[u\left(L^a\right)\right] \geq \mathbb{E}\left[u\left(L^b\right)\right] \geq \mathbb{E}\left[u\left(L^c\right)\right] \therefore \mathbb{E}\left[u\left(L^a\right)\right] \geq \mathbb{E}\left[u\left(L^c\right)\right] \implies L^a \succcurlyeq L^c$.

  4. *Continuity*

     Assume $L^a \succcurlyeq L^b$ and $L^b \succcurlyeq L^c$. Remember $\mathbb{E}\left[u\left(L^i\right)\right] \in \mathbb{R}$. The real line is continuous in the sense that $\mathbb{E}\left[u\left(L^a\right)\right] \geq \mathbb{E}\left[u\left(L^b\right)\right] \geq \mathbb{E}\left[u\left(L^c\right)\right] \implies \exists\, \alpha \in [0,1] : \mathbb{E}\left[u\left(L^b\right)\right] = \alpha\mathbb{E}\left[u\left(L^a\right)\right] + (1-\alpha)\mathbb{E}\left[u\left(L^c\right)\right] \implies L^b \sim \alpha L^a + (1-\alpha)L^c$.

  5. *Independence*

     Assume $L^a \succcurlyeq L^b$. Fix $\alpha \in [0,1]$. We shall prove that

$$\alpha L^a + (1-\alpha)L^c \succcurlyeq \alpha L^b + (1-\alpha)L^c$$

Indeed,

$$\begin{aligned}
\mathbb{E}\left[u\left(L^a\right)\right] &\geq \mathbb{E}\left[u\left(L^b\right)\right] \\
\alpha\mathbb{E}\left[u\left(L^a\right)\right] &\geq \alpha\mathbb{E}\left[u\left(L^b\right)\right] \\
\alpha\mathbb{E}\left[u\left(L^a\right)\right] + (1-\alpha)\mathbb{E}\left[u\left(L^c\right)\right] &\geq \alpha\mathbb{E}\left[u\left(L^b\right)\right] + (1-\alpha)\mathbb{E}\left[u\left(L^c\right)\right] \\
\alpha L^a + (1-\alpha)L^c &\succcurlyeq \alpha L^b + (1-\alpha)L^c
\end{aligned}$$

$\square$

- $(\Leftarrow)$

  Let $a, b \in \mathbb{R}$, $a > 0$ and $v(\cdot) = au(\cdot) + b$. Fix L^a, L^b and L^c and assume $L^a \succ L^b \succ L^c$. Then

$$\begin{aligned}
\mathbb{E}\left[u\left(L^a\right)\right] &> \mathbb{E}\left[u\left(L^b\right)\right] \\
\mathbb{E}\left[u\left(L^a\right)\right] + b &> \mathbb{E}\left[u\left(L^b\right)\right] + b \\
a\mathbb{E}\left[u\left(L^a\right)\right] + b &> a\mathbb{E}\left[u\left(L^b\right)\right] + b \\
\mathbb{E}\left[au\left(L^a\right) + b\right] &> \mathbb{E}\left[au\left(L^b\right) + b\right] \\
\mathbb{E}\left[v\left(L^a\right)\right] &> \mathbb{E}\left[v\left(L^b\right)\right]
\end{aligned}$$

And thus v represents the same ordering as u.

$\square$

$(\Rightarrow)$
Now assume v represents the same ordering as u. We have that

$$\mathbb{E}\left[u\left(L^a\right)\right] > \mathbb{E}\left[u\left(L^b\right)\right] > \mathbb{E}\left[u\left(L^c\right)\right]$$

For simplicity of notation, let us denote $\mathbb{E}\left[u\left(L^a\right)\right]$ as $u(L^a)$ and $\mathbb{E}\left[v\left(L^a\right)\right]$ as $v(L^a)$. We know there must exist $\alpha \in [0,1]$ such that

$$u(L^b) = \alpha u(L^a) + (1-\alpha)u(L^c)$$

In fact,

$$\alpha = \frac{u(L^b) - u(L^c)}{u(L^a) - u(L^c)}$$

Now, since $L^b \sim \alpha L^a + (1-\alpha)L^c$ we must have

$$v(L^b) = \alpha v(L^a) + (1-\alpha)v(L^c)$$

$$v(L^b) = \frac{u(L^b) - u(L^c)}{u(L^a) - u(L^c)} v(L^a) + \frac{u(L^a) - u(L^b)}{u(L^a) - u(L^c)} v(L^c)$$

$$v(L^b) = u(L^b)\frac{v(L^a)}{u(L^a) - u(L^c)} - \frac{v(L^a)u(L^c)}{u(L^a) - u(L^c)} + \frac{u(L^a)v(L^c)}{u(L^a) - u(L^c)} - u(L^b)\frac{v(L^c)}{u(L^a) - u(L^c)}$$

Now let $a = \frac{v(L^a)-v(L^c)}{u(L^a)-u(L^c)}$ and $b = \frac{u(L^a)v(L^c)}{u(L^a)-u(L^c)} - \frac{v(L^a)u(L^c)}{u(L^a)-u(L^c)}$ and it is clear that $v(x) = au(x) + b$. Also, since $L^a \succ L^c$, it is clear that $a > 0$.

$\square$

Exercise 2

- Let $A \succ B$. Then

$$\begin{aligned}
\underbrace{0.9u(6) + 0.1u(0)}_{A} &> \underbrace{0.45u(12) + 0.55u(0)}_{B} \\
0.9u(6) &> 0.45u(12) + 0.45u(0) \\
2u(6) &> u(12) + u(0) \\
0.002u(6) &> 0.001u(12) + 0.001u(0) \\
0.002u(6) &> 0.001u(12) + 0.999u(0) - 0.998u(0) \\
\underbrace{0.002u(6) + 0.998u(0)}_{D} &> \underbrace{0.001u(12) + 0.999u(0)}_{C}
\end{aligned}$$

  And therefore $D \succ C$.

- Assume $A \succ B$ and $C \succ D$. By the axiom of independence, we must have

$$\alpha C + (1-\alpha)B \succsim \alpha D + (1-\alpha)B \quad \forall\, \alpha \in [0,1]$$

  In the previous problem, we proved that $A \succ B \iff D \succ C$. Fix $\alpha = 1$. Then $C \succ D$, but we assumed $A \succ B$. Thus, we have a contradiction.

Exercise 3

3.1

We are given the utility function and the gamble of the form

$$u(x) = -e^{-\gamma x}, \quad W_1 = W_0 + \varepsilon, \quad \varepsilon \sim \mathcal{N}(\mu, \sigma^2), \quad W_0 = const \tag{2}$$

The expectation of the future utility is

$$\mathbb{E}[u(W_1)] = -e^{-\gamma W_0}\mathbb{E}[e^{-\gamma\varepsilon}] \tag{3}$$

Making use of the moment-generating function of normal distribution, we get

$$\mathbb{E}[u(W_1)] = -e^{-\gamma W_0}e^{-\gamma\mu+\frac{1}{2}\gamma^2\sigma^2} \tag{4}$$

The utility identity takes on the following form

$$-e^{-\gamma W_0}e^{-\gamma\mu+\frac{1}{2}\gamma^2\sigma^2} = e^{-\gamma(W_0-\pi)} \tag{5}$$

Applying logarithm to both sides of the identity, one simplifies (5) to

$$-\gamma W_0 - \gamma\mu + \frac{1}{2}\gamma^2\sigma^2 = -\gamma W_0 + \gamma\pi \tag{6}$$

Therefore, the **certainty equivalent** is

$$\pi = -\mu + \frac{1}{2}\gamma\sigma^2 \tag{7}$$

3.2

Here are the main observations:

1. π does not depend on W_0
2. π is proportional to $-\mu$, i.e., the more we expect to win in the gamble, the less we need to insure against it
3. π is proportional to $\gamma\sigma^2$, i.e. proportional to **variance** of ε with factor $\frac{\gamma}{2}$

3.3

Now, let $W_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$ and $Corr(W_0, \varepsilon) = \rho$. The expected present utility takes on the form

$$\mathbb{E}[u(W_0 - \pi)] = -e^{\gamma\pi}\mathbb{E}[e^{-\gamma W_0}] = -e^{\gamma\pi}e^{-\gamma\mu_0+\frac{1}{2}\gamma^2\sigma_0^2} \tag{8}$$

By properties of Gaussian distribution, $W_0 + \varepsilon$ is again normal with $\widehat{\mu} = \mu_0 + \mu$ and $\widehat{\sigma}^2 = \sigma^2 + \sigma_0^2 + 2\rho\sigma\sigma_0$, therefore

$$\mathbb{E}[u(W_1)] = \mathbb{E}[-e^{-\gamma(W_0+\varepsilon)}] = -e^{-\gamma(\mu_0+\mu)+\frac{1}{2}\gamma^2(\sigma^2+\sigma_0^2+2\rho\sigma\sigma_0)} \tag{9}$$

Equating present and future utilities and applying logarithm to both sides of the equation, one obtains

$$\gamma\pi - \gamma\mu_0 + \frac{1}{2}\gamma^2\sigma_0^2 = -\gamma(\mu_0 + \mu) + \frac{1}{2}\gamma^2(\sigma^2 + \sigma_0^2 + 2\rho\sigma\sigma_0) \tag{10}$$

Thus, the certainty equivalent is as following

$$\pi = -\mu + \frac{1}{2}\gamma(\sigma^2 + 2\rho\sigma\sigma_0) \tag{11}$$

Note that the certainty equivalent (11) is the one obtained in (7) plus the positive term $\gamma\rho\sigma\sigma_0$. It converges to (7) as $\sigma_0 \to 0$ or $\rho \to 0$. This means that present risk (σ_0^2), if uncorrelated with gamble risk (σ^2), does not add up to the certainty equivalent. Further,

- π does not depend on μ_0
- π linearly depends on σ_0, i.e. proportional to **volatility** of W_0
- ρ controls exposure of π to volatility of W_0 and ε.

Exercise 4

- If agent **b** dislikes a gamble given by the random variable $\tilde{\epsilon}$ with $\mathbb{E}[\tilde{\epsilon}] = 0$ at all levels of wealth, we can express it in the following way

  $$\mathbb{E}\left[U_b(w + \tilde{\epsilon})\right] < U_b(w).$$

  Using the strictly increasing property of $\Phi(x)$, we can transform both sides by $\Phi(x)$ to obtain

  $$\Phi\left(\mathbb{E}\left[U_b(w + \tilde{\epsilon})\right]\right) < \Phi\left(U_b(w)\right).$$

  Jensen's inequality tells us that for a strictly concave function, we have

  $$\mathbb{E}\left[\Phi\left(U_b(w + \tilde{\epsilon})\right)\right] < \Phi\left(\mathbb{E}[U_b(w + \tilde{\epsilon})]\right).$$

  Combining the last two equations and $U_a(w) = \Phi\left(U_b(w)\right)$ gives

  $$\begin{gathered}\mathbb{E}\left[\Phi\left(U_b(w + \tilde{\epsilon})\right)\right] < \Phi\left(U_b(w)\right)\\ \mathbb{E}\left[U_a(w + \tilde{\epsilon})\right] < U_a(w).\end{gathered}$$

  Therefore agent **a** also dislikes the same gamble. This is true for all gambles that agent **b** dislikes.

  If $\Phi(x)$ wasn't strictly increasing and concave, we can find a gamble $\tilde{\epsilon}$ and a level of wealth w^* where

  $$\mathbb{E}\left[U_b(w^* + \tilde{\epsilon})\right] < U_b(w^*)$$

but

$$\Phi\left(\mathbb{E}\left[U_b(w^* + \tilde{\epsilon})\right]\right) \geq \Phi\left(U_b(w^*)\right)$$

and

$$\mathbb{E}\left[\Phi\left(U_b(w^* + \tilde{\epsilon})\right)\right] \geq \Phi\left(\mathbb{E}[U_b(w^* + \tilde{\epsilon})]\right)$$

and therefore

$$\mathbb{E}\left[U_a(w^* + \tilde{\epsilon})\right] \geq U_a(w^*).$$

- $$\begin{aligned}\frac{\partial}{\partial x} U_a(x) &= \frac{\partial}{\partial x} \Phi\left(U_b(x)\right) \\ &= \Phi'\left(U_b(x)\right) \frac{\partial}{\partial x} U_b(x)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} U_a(x) &= \frac{\partial^2}{\partial x^2} \Phi\left(U_b(x)\right) \\ &= \Phi'\left(U_b(x)\right) \frac{\partial^2}{\partial x^2} U_b(x) + \Phi''\left(U_b(x)\right) \left(\frac{\partial}{\partial x} U_b(x)\right)^2\end{aligned}$$

$$\begin{aligned}A_a(x) &= -\frac{U_a''(x)}{U_a'(x)} \\ &= -\frac{\frac{\partial^2}{\partial x^2} \Phi\left(U_b(x)\right)}{\frac{\partial}{\partial x} \Phi\left(U_b(x)\right)} \\ &= -\frac{\Phi'\left(U_b(x)\right) \frac{\partial^2}{\partial x^2} U_b(x) + \Phi''\left(U_b(x)\right) \left(\frac{\partial}{\partial x} U_b(x)\right)^2}{\Phi'\left(U_b(x)\right) \frac{\partial}{\partial x} U_b(x)} \\ &= -\frac{U_b''(x)}{U_b'(x)} - U_b'(x) \frac{\Phi''\left(U_b(x)\right)}{\Phi'\left(U_b(x)\right)} \\ &= A_b(x) - U_b'(x) \frac{\Phi''\left(U_b(w)\right)}{\Phi'\left(U_b(w)\right)}\end{aligned}$$

We know that $U_b'(x) > 0$ and $\Phi'(x) > 0$ and $\Phi''(x) < 0$ because $\Phi(x)$ is strictly increasing and concave.

Therefore

$$U_b'(x) \frac{\Phi''\left(U_b(w)\right)}{\Phi'\left(U_b(w)\right)} < 0$$

and $A_a > A_b$.

Exercise 5

- Let $Y \in [0,1]$ and $X \in [-y, y]$. In such way, $E(X) = E(X \mid Y = y) = 0$, so the two random variables are mean-independent. However, $P\left(X < -\frac{1}{2} \mid Y > \frac{1}{4}\right) = 0$ but $P\left(X < -\frac{1}{2}\right) = \frac{1}{8}$. Therefore, the two random variables are not independent.

- Let $X \sim N(0,1)$ and $Y = X^2$. However, $P(X < 0) = \frac{1}{2}$ while $P(X < 0 \mid Y < 0) = 0$. Thus, the two random variables are not independent.

- The first observation is: if $E[C_a] = E[C_b]$, then with $u(x) = -x^2$ we get that C_a dominating C_b implies $E[C_a^2] < E[C_b^2]$ and, hence $\mathrm{Var}[C_a] \le \mathrm{Var}[C_b]$. Suppose now that C_a, C_b are Gaussian, $C_a \sim N(\mu, \sigma_a^2)$, $C_b \sim N(\mu, \sigma_b^2)$ with $\sigma_a^2 < \sigma_b^2$. Let $\epsilon \sim N(0, \sigma_b^2 - \sigma_a^2)$ be independent of C_a. Then, $C_a + \epsilon$ has the same distribution as C_b, and is obviously a mean-preserving spread. Thus,

$$E[u(C_a)] \;>\; E[u(C_a + \epsilon)] \;=\; E[u(C_b)]$$

  where the latter follows because $E[u(Z)]$ only depends on the distribution of a random variable Z.

- The example goes in the following way:

$$C_a = \left\{0, 2, 4, 6; \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\} \tag{12}$$

$$C_b = \left\{0.1, 3, 5.9; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\} \tag{13}$$

  The means and variances are:

  C_a: Mean $= 3$, Variance $= 5$

  C_b: Mean $= 3$, Variance $= 5.607$

  The utility function could be set as follows:

$$u(W) = \begin{cases} 2W & if W \leqslant 3 \\ 3 + W & if W > 3 \end{cases} \tag{14}$$

  The expected utilities will be:

$$E[U(C_a)] = 5 \tag{15}$$

$$E[U(C_b)] = 5.033 \tag{16}$$

  Therefore, although C_a has a lower variance, the lower variance does not guarantee that the expected utility of C_a is larger than that of C_b, and C_a does not second order stochastically dominate C_b.

- If u_1 is more willing to pay the full insurance, we can say that u_1 is more risk averse than u_2. We define the insurance premium as the risk premium π, and the above judgement is equivalent to $\pi_1 > \pi_2$; to be more specific, $\pi_1 (W_0, \epsilon) > \pi_2 (W_0, \epsilon)$.According to the Arrow-Pratt measure of Absolute Risk-aversion,
$$\pi_i (W_0) = \frac{1}{2}\sigma_\epsilon^2 A (W_0), i = 1, 2. \tag{17}$$
So that $\pi_1 > \pi_2$ indicates a $A_1 (W_0) > A_2 (W_0)$.
Because $u_1 (x) = \phi [u_2 (x)]$, and $Ai (x) = -\frac{u_i''(x)}{u_i'(x)}$, we can have
$$-\frac{u_1'' (x)}{u_1' (x)} = -\frac{u_2 (x)'' \phi' [u_2 (x)] + \phi'' [u_2']^2 [u_2 (x)]}{\phi' [u_2 (x)] u_2 (x)'} = -\frac{u_2'' (x)}{u_2 (x)'} - \frac{\phi'' [u_2 (x)] u_2' (x)}{\phi' [u_2 (x)]} \tag{18}$$
Only when $\phi'' (x) < 0$, the inequality $-\frac{u_1''(x)}{u_1'(x)} > -\frac{u_2''(x)}{u_2(x)'}$ could be correctly established.

- **First, consider an investor who simply dislikes variance.** We now show that $\gamma > \lambda$ implies $Var (C_\gamma) > Var (C_\lambda)$. Because $E (\epsilon \mid \mu) = 0$,the C_i will not be affected by μ, that is to say $E (C_\gamma \mid \mu) = E (C_\lambda \mid \mu) = E (\mu)$. And $Var (C_i \mid \mu) (where i = \gamma, \lambda)$ could be calculated as:
$$Var (C_i) = E \left(C_i^2 \mid \mu\right) - E^2 (C_i \mid \mu) \tag{19}$$
Substitute $C_i = \mu + i \cdot \epsilon$, the function become
$$Var (C_i) = E \left(\mu^2\right) + E \left(i^2\epsilon^2 \mid \mu\right) - E (\mu)^2 = Var (\mu) + i^2 \cdot Var (\epsilon \mid \mu)) \tag{20}$$
Because $Var (\epsilon \mid \mu)) > 0$, we will have $Var (C_\gamma) > (C_\lambda)$.

  Now, consider generic preferences We will need

  Lemma For any increasing $f_1(\epsilon)$ and decreasing $f_2(\epsilon)$, we have
$$E[f_1(\epsilon)f_2(\epsilon)] \;\le\; E[f_1(\epsilon)]\, E[f_2(\epsilon)]\,. \tag{21}$$
If both f_1, f_2 are increasing, then the inequality sign is reversed.

  Let
$$f(\gamma) \;=\; E[u(C_\gamma)] \;=\; E[u(\mu + \gamma\epsilon)]\,. \tag{22}$$
Then, since $\gamma > 0$, we have
$$f'(\gamma) \;=\; E[E[u'(\mu + \gamma\epsilon)\epsilon|\mu]] \;\le\; E[u'(\mu + \gamma\epsilon)|\mu]E[\epsilon\mu]] \;=\; 0 \tag{23}$$
and hence f is decreasing in γ

- **First, consider an investor who simply dislikes variance.** We have
$$Var (C_\gamma) = \gamma^2\sigma_{\epsilon_1}^2 + (1 - \gamma)^2 \sigma_{\epsilon_2}^2 + 2\gamma (1 - \gamma) \underbrace{Cov (\epsilon_1, \epsilon_2)}_{=0} \tag{24}$$
To minimize the equation above, we take the first derivative of the equation over γ, and will get:
$$\sigma_{\epsilon_1}^2\gamma + \sigma_{\epsilon_2}^2\gamma - \sigma_{\epsilon_2}^2 = 0 \tag{25}$$

Thus,

$$\gamma = \frac{\sigma_{\epsilon_2}^2}{\sigma_{\epsilon_1}^2 + \sigma_{\epsilon_2}^2} \tag{26}$$

As mentioned above, both ϵ_1 and ϵ_2 are *i.i.d*, so they have the same variance and thus should be canceled out. The final result for equation (22) will be $\frac{1}{2}$.

Now, consider generic preferences Let

$$f(\gamma) \;=\; E[u(C_\gamma)] \;=\; E[u(\gamma\epsilon_1 + (1-\gamma)\epsilon_2)]\,. \tag{27}$$

Since ϵ_1 and ϵ_2 have the same distribution and are independent, $f(\gamma) = f(1-\gamma)$ (because we can interchange ϵ_1 and ϵ_2.

$$f'(\gamma) \;=\; E[u'(\gamma\epsilon_1 + (1-\gamma)\epsilon_2)(\epsilon_1 - \epsilon_2)],\; f''(\gamma) \;=\; E[u''(\gamma\epsilon_1 + (1-\gamma)\epsilon_2)(\epsilon_1 - \epsilon_2)^2] < 0\,. \tag{28}$$

Thus, f is concave in γ and from $f(\gamma) = f(1-\gamma)$ we get $f'(\gamma) = -f'(1-\gamma)$. For $\gamma = 0.5$, we get $f'(0.5) = -f'(0.5)$, implying $f'(0.5) = 0$. Thus, $\gamma = 0.5$ is the global maximum of the concave function f.

Exercise 6

We are asked to maximize the consumption function for three different cases, specifically.

6.1

The utility function is given by $\frac{c^{1-\gamma}}{1-\gamma} - 1$. Consumption at $t = 0$ and $t = 1$ is as following

$$c_0 = w_0 - x, \quad c_1 = w_1 + rx, \quad \gamma \neq 1 \tag{29}$$

The task is to maximise the function

$$u(c_0) + e^{-\rho}\mathbb{E}[u(c_1)] = \frac{(w_0 - x)^{1-\gamma}}{1-\gamma} - 1 + e^{-\rho}\Big(\frac{(w_1 + rx)^{1-\gamma}}{1-\gamma} - 1\Big) \tag{30}$$

with respect to investment x, in other words

$$\frac{\partial}{\partial x}\Big(\frac{(w_0 - x)^{1-\gamma}}{1-\gamma} + \frac{e^{-\rho}}{1-\gamma}(w_1 + rx)^{1-\gamma}\Big) = 0 \tag{31}$$

Taking derivative, one gets

$$(w_0 - x)^{-\gamma} = e^{-\rho}(w_1 + rx)^{-\gamma} \tag{32}$$

which yields

$$x = \frac{w_0 - w_1(re^{-\rho})^{-\frac{1}{\gamma}}}{1 + r(re^{-\rho})^{-\frac{1}{\gamma}}} \tag{33}$$

6.2

Now, w_1 can take on two values: w_H with probability p and w_L with probability $1-p$. The utility function for the special case $\gamma = 0$ degenerates to $u(c) = \ln c$. We have

$$\mathbb{E}[u(c_1)] = \mathbb{E}[\ln(w_1 + rx)] = p\ln(w_H + rx) + (1-p)\ln(w_L + rx) \tag{34}$$

and

$$u(c_0) = \ln(w_0 - x) \tag{35}$$

The task is to maximize the following function

$$\max_x \left[\ln(w_0 - x) + e^{-\rho} p \ln(w_H + rx) + e^{-\rho}(1-p)\ln(w_L + rx)\right] \tag{36}$$

This is equivalent to maximizing the third-order polynomial under the logarithm. The first order condition is

$$-\frac{1}{w_0 - x} + rpe^{-\rho}\frac{1}{w_H + rx} + r(1-p)e^{-\rho}\frac{1}{w_L + rx} = 0 \tag{37}$$

which is equivalent to

$$-(w_H + rx)(w_L + rx) + rpe^{-\rho}(w_0 - x)(w_L + rx) + r(1-p)e^{-\rho}(w_0 - x)(w_H + rx) = 0\,. \tag{38}$$

Solving this quadratic equation gives the required solution.

6.3

In the case of the exponential utility function, the expectation of the utility of future consumption is

$$\mathbb{E}[u(w_1 + rx)] = pe^{-a(w_H + rx)} - (1-p)e^{-a}(w_L + rx) \tag{39}$$

The first order condition gives

$$e^{-aw_0}e^{ax} = re^{-\rho}pe^{-aw_H}e^{-arx} + r(1-p)e^{-\rho}e^{-aw_L}e^{-arx} \tag{40}$$

Denote

$$\begin{aligned} A &= re^{-\rho}pe^{-aw_H} \\ B &= r(1-p)e^{-\rho}e^{-aw_L} \\ C &= e^{-aw_0} \end{aligned} \tag{41}$$

The identity takes on the following form

$$Ce^{ax} = (A + B)e^{-arx} \tag{42}$$

Isolating x, one concludes that

$$x = \frac{\ln(A + B) - \ln C}{a(r + 1)} \tag{43}$$