Asset pricing
Homework 1 Solutions

Exercise 1

The payoff matrix and the price vector are $X = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & 3 & 1 \end{bmatrix}$ and $p = \begin{pmatrix} 2.5 \\ 15 \\ 4 \end{pmatrix}$. X has $rank(X) = 3$ and $det(X) \neq 0$; therefore, the market is complete and X invertible.
We can then calculate the unique state price vector as

$$q = X^{-1}p = \begin{pmatrix} -0.5 \\ 1 \\ 2 \end{pmatrix}.$$

Because $q_1 < 0$, there is arbitrage in the market.
Consider the portfolio $\theta = (8, 0, -5)^\top$. The price of the portfolio is

$$\theta^\top p = 8 \times 2.5 - 4 \times 5 = 0$$

and the payoff

$$\theta^\top X = (14, 1, 3)$$

which is positive in all possible states. Therefore, θ is an arbitrage portfolio.

Exercise 2

2.1

Consider the two risky assets $X_1 = (1, 0, 1)$ with price $p_1 = 3$ and $X_2 = (0, 1, 1)$ with price $p_2 = 1$. The payoff matrix can then be written as $X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ with price vector $p = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. The state price vector

$$q = \begin{pmatrix} 3 - q_3 \\ 1 - q_3 \\ q_3 \end{pmatrix}$$

is a solution to $p = Xq$. For any $q_3 > 1$ (e.g., $q = (-1, -3, 4)^\top$), the state price vector contains negative state prices.

To have arbitrage in the market $\theta^\top p \leq 0$ and $\theta^\top X \geq 0$ has to be fulfilled with at least one strict inequality:

$$\theta^\top X = \begin{pmatrix} \theta_1 + \theta_2 \\ \theta_2 \\ \theta_1 \end{pmatrix} \geq 0,$$

which is fulfilled for $\theta_1, \theta_2 \geq 0$, and

$$\theta^\top p = 3\theta_1 + \theta_2 \leq 0.$$

The two statements can simultaneously only be fulfilled as equalities (i.e., $\theta_1, \theta_2 = 0$), therefore, there is no arbitrage in the market.

2.2

$$\begin{aligned} q(z) &= \{zq : q = X'(XX')^{-1}p, \forall z \in M\} \\ &= \frac{5}{3}z_1 - \frac{1}{3}z_2 + \frac{4}{3}z_3 \quad , \forall z \in M \end{aligned}$$

Consider the vector $\hat{z} = (1,0,0)$, which is outside the span of existing securities. The upper and lower bounds are calculated as

$$\begin{aligned} q_u(\hat{z}) &= \min_\theta\{\theta' p \ \ s.t. \ \ \theta' X \geq \hat{z}\} \\ &= \min_\theta\{3\theta_1 + \theta_2 \ \ s.t. \ \ \theta_1 + \theta_2 \geq 1, \theta_1 \geq 0, \theta_2 \geq 0\} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} q_l(\hat{z}) &= \max_\theta\{\theta' p \ \ s.t. \ \ \theta' X \leq \hat{z}\} \\ &= \max_\theta\{3\theta_1 + \theta_2 \ \ s.t. \ \ \theta_1 + \theta_2 \leq 1, \theta_1 \leq 0, \theta_2 \leq 0\} \\ &= 0. \end{aligned}$$

One can define a valuation functional $Q(y)$ using the augmented payoff matrix and the augmented price vector $\tilde{X} = (X_1, X_2, \hat{z})$ and $\tilde{p} = (p_1, p_2, q_l(\hat{z}))$.

$$\begin{aligned} Q(y) &= \{y\tilde{q} : \tilde{q} = \tilde{X}^{-1}\tilde{p}, \forall y \in \mathbb{R}\} \\ &= -2y_2 + 3y_3 \quad , \forall y \in \mathbb{R} \end{aligned}$$

This valuation functional is not unique as any $q_l(\hat{z}) \leq p_3 \leq q_u(\hat{z})$ could be used to find a different valuation functional.

2.3

Adding the security with $X_3 = (1,0,0)$ with $p_3 = 1$ to the market makes it complete as the augmented payoff matrix is now of rank three and invertible. The state price vector in this market is $q = X^{-1}p = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Because the market is complete, negative state prices mean that there is arbitrage in the market.

The payoff pricing functional and the valuation functional are now identical

$$q(z) = Q(z) = z_1 - z_2 + 2z_3 \quad \forall z \in \mathbb{R},$$

which is not a strictly positive function.

Exercise 3

3.1

The market can be expressed with the payoff matrix

$$X = \begin{bmatrix} uS & dS \\ RS^\circ & RS^\circ \end{bmatrix}$$

and the price vector

$$p = \begin{pmatrix} S \\ S^\circ \end{pmatrix}.$$

There exists a unique state price vector if X is invertible, which is the case if $\det(X) \neq 0$:

$$\begin{aligned} \det(X) &= uSRS^\circ - RS^\circ dS \\ &= SRS^\circ(u-d) \\ &\neq 0 \end{aligned}$$

which is fulfilled since $u > d$ and $S, S^\circ, R \neq 0$.

With the inverse

$$X^{-1} = \frac{1}{SRS^\circ(u-d)} \begin{bmatrix} RS^\circ & -dS \\ -RS^\circ & uS \end{bmatrix}$$

we can calculate

$$q = X^{-1}p = \frac{1}{R(u-d)} \begin{pmatrix} R-d \\ u-R \end{pmatrix}.$$

No arbitrage requires only positive state prices. Since $R > 0$ and $u > d$, $R - d > 0$ and therefore $R > d$. $u - R > 0$ gives $u > R$ and together $u > R > d$.

3.2

$$\begin{aligned} S &= \mathbb{E}^{Q_\circ}[\tilde{\omega}\frac{S}{R}] \\ &= \pi^{Q_\circ}\frac{uS}{R} + (1-\pi^{Q_\circ})\frac{dS}{R} \end{aligned}$$

solving for $\pi^{Q_\circ}$ gives

$$\pi^{Q_\circ} = \frac{R-d}{u-d}$$

and

$$(1-\pi^{Q_\circ}) = \frac{u-R}{u-d}.$$

3.3

We can combine the results for q and $\pi^{Q_\circ}$ from the two previous parts to get

$$q = \frac{1}{R}\begin{pmatrix} \pi^{Q_\circ} \\ 1-\pi^{Q_\circ} \end{pmatrix}.$$

3.4

$$h(\tilde{\omega}S) = \theta X$$

$$\theta = hX'^{-1} = (h(uS), h(dS)) \begin{bmatrix} uS & RS^\circ \\ ds & RS^\circ \end{bmatrix}^{-1}.$$

The system has a unique solution since we already showed in part 1 that X is invertible.

3.5

$$\begin{aligned} h(\tilde{\omega}S)q &= (\theta_1, \theta_2) \begin{bmatrix} uS & dS \\ RS^\circ & RS^\circ \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= (\theta_1 uS + \theta RS^\circ, \theta_1 dS + \theta_2 RS^\circ) \frac{1}{R} \begin{pmatrix} \pi^{Q_\circ} \\ 1-\pi^{Q_\circ} \end{pmatrix} \\ &= \frac{1}{R}(\pi^{Q_\circ}\theta_1 uS + \pi^{Q_\circ}\theta_2 RS^\circ + (1-\pi^{Q_\circ})\theta_1 dS + (1-\pi^{Q_\circ})\theta_2 RS^\circ) \\ &= \frac{1}{R}\mathbb{E}[\theta_1\tilde{\omega}S + \theta_2 RS^\circ] \\ &= \frac{1}{R}\mathbb{E}[h(\tilde{\omega})] \end{aligned}$$

Exercise 4

4.1

Consider the trinomial model with three possible outcomes u, m, d with probabilities q, p, $1-q-p$ respectively. The asset takes on either of three values $\tilde{w}S$ at $t=1$. Equating the expectation of the price of the asset under the martingale measure at $t=1$ and the respective price at $t=0$ yields:

$$pu + qm + (1-p-q)d = R$$

There are 2 variables constrained by one equation, meaning that the rank of the system is less than the number of variables. This leads to infinite solutions. The results obviously hold for models with more than 3 outcomes, so, in general, there is a continuum of possible equivalent martingale measures.

4.1

Consider again the trinomial model. Denote Q_3 as one of the martingale measures of a trinomial model and Q_2 as the martingale measure of a binomial model. For the first inequality, we exploit the fact that the function h is increasing:

$$\mathbb{E}^{Q_3}[h(\tilde{w}S)] = \mathbb{E}[ph(uS) + qh(mS) + (1-p-q)h(dS)],$$

where $h(mS) < h(dS)$ since $m < d$, so

$$\mathbb{E}^{Q_3}[h(\tilde{w}S)] \leq \mathbb{E}[ph(uS) + qh(dS) + (1-p-q)h(dS)] = \mathbb{E}[ph(uS) + (1-p)h(dS)] = \mathbb{E}^{Q_2}[h(\tilde{w}S)],$$

for any such Q_3, the inequality also holds for the supremum.

4.2

It suffices to use the Jensen's inequality here:

$$\mathbb{E}^{Q_3}[h(\tilde{w}S)] \geq h(\mathbb{E}^{Q_3}[\tilde{w}S]) = h(RS)$$

which proves the second inequality. Since the inequality holds any such martingale measure Q_3, it holds for the infimum.

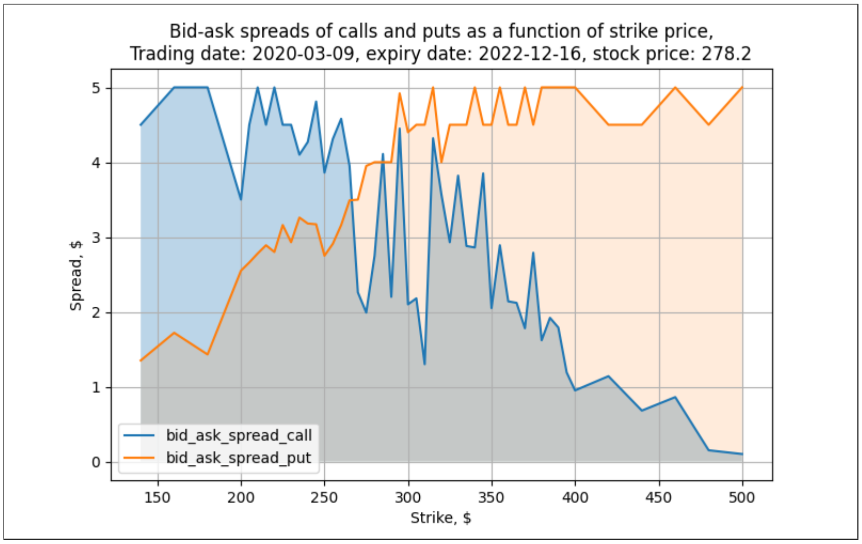
Exercise 5

We examine the given SPY options data set in the following manner:

1. We study bid-ask spreads for puts and calls
2. We examine whether put-call parity holds and whether there are any arbitrage opportunities stemming from put-call parity violation
3. We study quasi-arbitrage opportunities
4. We construct a basis of state prices for each trade date and expiry date
5. We draw general conclusions about the data set

Bid-ask spreads

Bid-ask spreads are positive throughout the data set. Below is the example of call and put bid-ask spreads as a function of a strike price for trading date 2020/03/09 and expiry date 2022/12/16.



Put-call parity and arbitrage opportunities

For each trading date and expiry date, we verify whether inequalities

$$C_A(i) - P_B(i) \geq S_B - K_i(1 - r_-)$$

and

$$C_B(i) - P_A(i) \geq S_A - K_i(1 - r_+)$$

hold. We take the lending rate $r_+ = 0\%$ and the borrowing rate $r_- = 2.5\%$, which is the average borrowing rate for the Bank of America as of 2021. The average SPY bid-ask spread is taken \$ 0.1 based on NASDAQ live SPY spreads.

Out of 55,186 trials, the first inequality holds only for 0.007%. This comes as no surprise: short-selling SPY and non-zero commissions would not allow us to benefit from this seemingly obvious arbitrage opportunity. Second inequality holds on 100% of cases.

Quasi-arbitrage opportunities

Both inequalities

$$P_A(K_i)\frac{K_{i+2} - K_{i+1}}{K_{i+2} - K_i} + P_A(K_{i+2})\frac{K_{i+1} - K_i}{K_{i+2} - K_i} > P_B(K_{i+1})$$

and

$$C_A(K_i)\frac{K_{i+2} - K_{i+1}}{K_{i+2} - K_i} + C_A(K_{i+2})\frac{K_{i+1} - K_i}{K_{i+2} - K_i} > C_B(K_{i+1})$$

hold in 99.9% of cases. Taking into account market frictions, this leaves a place for only occasional and vanishingly small quasi-arbitrage opportunities.

Breeden-Litzenberger approximation to state prices

Below is the example of state prices Breeden-Litzenberger approximation for maturity date 2020/03/09 and expiry date 2022/12/16:

The state prices are not positive in general, but this happens primarily due to imperfection of the numerical approximation procedure used and oscillations in spreads. This instability is observed in for all trading dates and expiry dates (there are more than 400 plots).

In case bid and ask spreads are different, i.e. there is a positive bid-ask spread, we can either average out the spread taking mid-price or compute state prices numerical approximations for bid and ask prices separately. Computing separate approximations for bid and ask prices is perhaps not a good idea because of spread oscillations, so taking mid-price is the preferred option.

Exercise 6

The basic intuition of CIP states that the interest rates implicit in foreign exchange (FX) swap markets coincide with the corresponding interest rates in cash markets. To be more specific, the CIP arbitrage strategy is to borrow currency A, using an FX swap in order to convert the proceeds, and investing in a risk-free asset in currency B – should not yield any profits. The function for CIP goes as follows:

$$(1 + r) = (1 + r^*)\,\frac{F}{S}, \tag{1}$$

so that the currency risk is hedged completely. From the arbitrage perspective, CIP arbitrage is only possible if the cost of hedging the exchange risk is less than the additional return generated by investing in a higher-yielding currency. According to Du et al. (2018) paper, in reality, due to balance sheet constraints, the borrowing and lending ability is limited, which could be a source of CIP deviations.

The CIP deviation leads to a cross-currency basis and could be defined as the difference between the dollar interest rate in the cash market and the implied dollar interest rate in the FX swap market when swapping foreign currency into dollars.

According to the sample in the Du et al. (2018) paper, the deviation is negative from 2007-2017. This suggests that the cash market interest rate is lower than the implied interest rate in the swap market. We can borrow USD from the cash market and lend the USD out in the currency swap market. In this way, we borrow at a lower rate and lend (or earn) at a higher rate, leading to a seemingly profitable arbitrage strategy.