

Asset Pricing I

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Intoduction to Asset Pricing I

- assets $i = 1, \dots, N$ have prices $P_{i,t}$ and excess returns

$$R_{i,t+1} = \frac{P_{i,t+1} + D_{t+1}}{P_{i,t}} - \underbrace{R_{f,t}}_{\text{risk free rate}} \quad (1)$$

- if you invest fraction $\pi_{i,t}$ of your wealth W_t into security i , the rest stays on your bank account and grows at the rate $R_{f,t}$:

$$W_t = \sum_i \underbrace{\pi_{i,t} W_t}_{\text{investment in stock } i} + \underbrace{(W_t - \sum_i \pi_{i,t} W_t)}_{\text{bank account}} \quad (2)$$

Intoduction to Asset Pricing II

and then you sell your investments at time t and collect dividends so that

$$\begin{aligned} W_{t+1} &= \sum_i W_t \pi_{i,t} \frac{P_{i,t+1} + D_{t+1}}{P_{i,t}} + (W_t - \sum_i \pi_{i,t} W_t) R_{f,t} \\ &= W_t R_{f,t} + W_t \sum_i \pi_{i,t} R_{i,t+1} \end{aligned} \quad (3)$$

- Thus, the excess return on your wealth is

$$\frac{W_{t+1}}{W_t} - R_{f,t} = \sum_i \pi_{i,t} R_{i,t+1} = \pi'_t R_{t+1} \quad (4)$$

- Thus, we want π_t that gives good returns. But what is the criterion?

Intoduction to Asset Pricing III

- Recall that, given the information set S_t at time t (S_t is the vector of signals encompassing all useful information available to investors at that time, e.g., all prices of all securities; all macroeconomic variables; etc.), we can define the **conditional expectation** of any random variable Z_{t+1} at time $t + 1$:

$$E_t[Z_{t+1}] = f_*(S_t) \quad (5)$$

where $f(S_t)$ is the best prediction in the sense that

$$f_*(S_t) = \arg \min_{f(S_t)} E[(Z_{t+1} - f(S_t))^2] \quad (6)$$

That is, **we minimize over all possible linear and non-linear functions!**

- Standard models** assume that agents are somehow able to solve this (tremendously complex) minimization problem and find $E_t[Z_{t+1}]$.

Intoduction to Asset Pricing IV

- mean-variance optimization:

$$\pi_t = \arg \max_{\pi_t} \left(E_t[\pi_t' R_{t+1}] - 0.5 \underbrace{\gamma}_{\text{risk aversion}} \text{Var}_t[\pi_t' R_{t+1}] \right) \quad (7)$$

- We have

$$E_t[\pi_t' R_{t+1}] = \underbrace{\pi_t'}_{\in \mathbb{R}^N} \underbrace{E_t[R_{t+1}]}_{\in \mathbb{R}^N}, \quad (8)$$

and

$$\begin{aligned} \text{Var}_t[\pi_t' R_{t+1}] &= \text{Cov}_t(\pi_t' R_{t+1}, \pi_t' R_{t+1}) \\ &= \sum_{i,j} \pi_{i,t} \pi_{j,t} \text{Cov}_t(R_{i,t+1}, R_{j,t+1}) \\ &= \pi_t' \text{Var}_t[R_{t+1}] \pi_t, \end{aligned} \quad (9)$$

Intoduction to Asset Pricing V

and the first-order conditions are

$$\begin{aligned} 0 &= \frac{\partial}{\partial \pi_t} (E_t[\pi_t' R_{t+1}] - 0.5\gamma \text{Var}_t[\pi_t' R_{t+1}]) \\ &= E_t[R_{t+1}] - \gamma \text{Var}_t[R_{t+1}] \pi_t, \end{aligned} \quad (10)$$

and hence the **Mean-Variance Efficient (MVE) portfolio** is

$$\underbrace{\pi_t}_{\text{tangency portfolio}} = \gamma^{-1} \underbrace{(\text{Var}_t[R_{t+1}])^{-1}}_{N \times N \text{ covariance matrix}} \underbrace{E_t[R_{t+1}]}_{N \times 1 \text{ expected returns}} \quad (11)$$

Intoduction to Asset Pricing VI

- Similarly,

$$\tilde{\pi}_t = \underbrace{\gamma^{-1}(E_t[R_{t+1}R'_{t+1}])^{-1}}_{\text{Sherman-Morrison formula}} E_t[R_{t+1}] \frac{1}{1 + E_t[R_{t+1}]' \text{Var}_t[R_{t+1}]^{-1} E_t[R_{t+1}]} \pi_t \quad (12)$$

where

$$E_t[R_{t+1}R'_{t+1}] = \text{Var}_t[R_{t+1}] + E_t[R_{t+1}]E_t[R_{t+1}]' \quad (13)$$

- The magic behind is the

Lemma (Sherman-Morrison formula)

Introduction to Asset Pricing VII

$$(A + xx')^{-1} = A^{-1} - \frac{A^{-1}xx'A^{-1}}{1 + x'A^{-1}x} \quad (14)$$

and

$$(A + xx')^{-1}x = \frac{A^{-1}x}{1 + x'A^{-1}x} \quad (15)$$

Proof of the Sherman-Morrison formula Recall that

$$xx' = (x_i x_j)_{i,j=1}^N$$

Intoduction to Asset Pricing VIII

is a symmetric, positive, semi-definite, $rank - 1$ matrix (all columns are proportional to x). Then,

$$\begin{aligned} & (A + xx')(A^{-1} - \frac{A^{-1}xx'A^{-1}}{1 + x'A^{-1}x}) \\ &= I - \frac{xx'A^{-1}}{1 + x'A^{-1}x} + xx'A^{-1} - xx'\frac{A^{-1}xx'A^{-1}}{1 + x'A^{-1}x} \\ &= I - \frac{xx'A^{-1}}{1 + x'A^{-1}x} + xx'A^{-1} - xx'A^{-1}\frac{x'A^{-1}x}{1 + x'A^{-1}x} = I \end{aligned} \quad (16)$$

and

$$(A + xx')^{-1}x = (A^{-1} - \frac{A^{-1}xx'A^{-1}}{1 + x'A^{-1}x})x = \frac{A^{-1}x}{1 + x'A^{-1}x} \quad (17)$$

- OK, this is the optimal portfolio, but what does this have to do with asset pricing?

Intoduction to Asset Pricing IX

- Intuitively, we expect that

$$P_{i,t} = \underbrace{(R_{f,t})^{-1} E_t[P_{i,t+1} + D_{i,t+1}]}_{\text{Definitely wrong in the data}} \quad (18)$$

because the **discount factor** $(R_{f,t})^{-1}$ is too naive

- We need a smart discount factor (SDF):**

$$P_{i,t} = E_t\left[\underbrace{M_{t,t+1}}_{\text{stochastic discount factor}} (P_{i,t+1} + D_{i,t+1}) \right] \quad (19)$$

- with a bit of algebra, this is equivalent to

$$E_t[R_{i,t+1} M_{t,t+1}] = 0 \quad (20)$$

Intoduction to Asset Pricing X

- By direct calculation,

$$M_{t,t+1} = 1 - \tilde{\pi}'_t R_{t+1} \quad (21)$$

does the job:

$$\begin{aligned} E_t[R_{t+1} M_{t,t+1}] &= E_t[R_{t+1} (1 - \tilde{\pi}'_t R_{t+1})] \\ &= E_t[R_{t+1}] - E_t[R_{t+1} R'_{t+1}] \tilde{\pi}_t = 0 \end{aligned} \quad (22)$$

implies

$$\tilde{\pi}_t = E_t[R_{t+1} R'_{t+1}]^{-1} E_t[R_{t+1}] \quad (23)$$

- Now comes the big question: **How do we measure the conditional** expectations, $E_t[R_{t+1}]$ and $E_t[R_{t+1} R'_{t+1}]$?
- Once can start with a simple prediction problem: measure $E_t[R_{t+1}]$ by running a regression on **observables (economic variables)** S_t using **past data** (time series prediction)

Overview

Readings:

- Leroy & Werner Chap. 1,2,3,5,6
- Back Chap. 4
- Cochrane Chap. 1 & 4

Topics:

- Arbitrage
- Stochastic Discount Factor
- State Prices
- Fundamental Theorems of Asset Pricing
- Complete Markets
- Pricing Kernel
- State Price density
- Risk-Neutral Measure

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Market Structure I

- Risky Asset prices $p = [p_1, \dots, p_N]'$ with **payoff matrix** $X = \{X_{i,s}\}$ for $i = 1, \dots, N$ and $s = 1, \dots, S$.
- A **portfolio** is a $(N, 1)$ vector θ of stock holdings with payoff $\theta'X$.
- The **asset span** is $\mathcal{M} = \{z \in \mathbb{R}^S : z = \theta'X\}$
- Markets are complete if any date 1 consumption plan (in $\mathbb{R}^N$) can be attained as a portfolio payoff:

Theorem

Markets are complete if and only if X has rank S .

Proof.

Market Structure II

Recall that, by standard linear algebra, *rank* is the dimension of the span of columns of X . Given a column $X_i = (X_{i,s})_{s=1}^S \in \mathbb{R}^S$ (the payoff vector of security i), we can write the payoff of a portfolio as

$$\underbrace{z}_{\text{portfolio payoff}} = \theta' X = \sum_i \underbrace{\theta_i}_{\text{portfolio weight}} \underbrace{X_i}_{\text{security } i \text{ payoff}}$$

Thus, $\mathcal{M}$ = span of columns of X and, hence, the same dimension as the rank of X . Since $\mathcal{M} \subset \mathbb{R}^S$, market is complete if and only if $\text{rank } X = S$. □

Arbitrage and State prices

- A **state price** vector is an S -dimensional vector q such that $p = Xq$
- That is,

$$p_i = \sum_s X_{i,s} q_s = \sum_s \underbrace{X_{i,s}}_{\text{payoff in state } s} \underbrace{q_s}_{\text{price of state } s} \quad (24)$$

- An **arbitrage** is a trading strategy θ such that $\theta'p \leq 0$ and $\theta'X \geq 0$ with at least one strict inequality.
- **The first fundamental theorem of asset pricing** for general (possibly incomplete) markets:

Theorem

There are no arbitrage opportunities iff there exists a vector of strictly positive state prices.

- Let us note here that if strictly positive state prices $q \gg 0$ exist, then it is straightforward to show that there is no arbitrage. Indeed, for any θ such that $\theta'X \geq 0$, we have $\theta'Xq \geq 0$, which implies that $\theta'p \geq 0$ (and if $\theta'X > 0$ then $\theta'p > 0$).
- Further, note that when markets are complete, then it is clear that a state price vector exists and that it is unique:

$$p = Xq \Leftrightarrow X'p = X'Xq \Leftarrow q = (X'X)^{-1}X'p \quad (25)$$

When X is a square matrix ($N = S$), we get

$$q = (X'X)^{-1}X'p = q = (X^{-1}(X')^{-1})X'p = X^{-1}p \quad (26)$$

Law of one price

- The LOP says that any two portfolios with the same payoff must have the same price:
if $\theta'_1 X = \theta'_2 X$ then $\theta'_1 p = \theta'_2 p$.
- The LOP holds iff every portfolio with zero payoff has zero price.
- If the LOP is violated then any payoff can be purchased at any price.

- For any payoff in the asset span $z \in \mathcal{M}$ consider the set of possible prices:

$$q(z) \equiv \{\theta'p \text{ for some } \theta \text{ s.t. } z = \theta'X\}.$$

Then we have:

Theorem

The LOP holds if $q(\cdot) : \mathcal{M} \rightarrow \mathbb{R}$ is a linear functional.

- Note that if the LOP holds, then the payoff pricing functional $q(\cdot)$ links the price vector and the payoff matrix via:
 $\theta'p = q(\theta'X) \quad \forall \theta \in R^N$
- If XX' is invertible (i.e., of rank N and there are no redundant assets), then

$$z = \theta'X \quad \underbrace{\Leftrightarrow}_{\text{multiply by } X'} \quad zX' = \theta'XX', \quad (27)$$

which is equivalent to

$$zX'(XX')^{-1} = \theta' \quad (28)$$

and, hence,

$$q(z) = \text{price}(\theta'X) = \theta'p = zX'(XX')^{-1}p \quad (29)$$

- short-sale constraints, funding costs, price impact, CIP, interest rates. Please click on these links:

[CIP Deviations](#)

[CIP Deviations and Convenience Yield](#)

- Institutional frictions imply that some assets are more convenient than others

[Convenience Yields of Near Money Assets](#)

Arbitrage

- A strong **arbitrage** is a trading strategy θ such that $\theta'p < 0$ and $X'\theta \geq 0$.
- A strong arbitrage exists if and only if the LOP does not hold (there is a (strong) arbitrage with zero payoff and negative price).
- An **arbitrage** is a trading strategy θ such that $\theta'p \leq 0$ and $X'\theta \geq 0$ with at least one strict inequality.
- A linear function is (strictly) positive if it assigns a (strictly) positive value to every positive and non-zero element of its domain.

Theorem

There is no arbitrage if and only if the payoff functional is linear and strictly positive.

Theorem

There is no strong arbitrage if and only if the payoff functional is linear and positive.

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Kuhn-Tucker: Solving Optimization with Constraints

Problem:

$$\max\{f(x) : \underbrace{g_i(x) \leq 0}_{\text{inequalities}}, \underbrace{h_i(x) = 0}_{\text{equalities}}\}$$

First Order Conditions (FOC) are:

$$\nabla f(x) = \sum_i \mu_i \nabla g_i + \sum_j \lambda_j \nabla h_j$$

where the **Lagrange Multipliers** satisfy

$$\mu_i \geq 0, \mu_i g_i = 0 \text{ (Complementary slackness)}$$

Example: Transaction costs



$$x'(\mu - (1+r)P(x)) - 0.5 \gamma x' \Sigma x$$

where

$$P(x) = P_A \mathbf{1}_{x>0} + P_B \mathbf{1}_{x<0}$$

- Decompose

$$x = x_+ - x_-, \quad g_1(x_+) = -x_+ \leq 0, \quad g_2(x_-) = -x_- \leq 0.$$

Then,

$$\begin{aligned} f(x_+, x_-) &= x'_+ (\mu - (1+r)P_A) - x'_- (\mu - (1+r)P_B) \quad (30) \\ &\quad - 0.5 A (x_+ - x_-)' \Sigma (x_+ - x_-) \end{aligned}$$

Consumption-Portfolio choice Problem I

- There are $i = 1, \dots, I$ (finite) number of agents who consume at period 0 and 1 with utility function $u_i(c_0, c_1)$, where c_0 is a scalar and c_1 is a S dimensional vector of consumption in each state.
- Typically, we restrict consumption to be positive (in each time and state) and utility to be (strictly) increasing, continuous, and differentiable in every argument. In addition, it is often assumed that utility is strictly concave in every argument.
- Agents receive endowments (ω_0, ω_1) , where ω_0 is a scalar and ω_1 is an S -dimensional vector.
- An agent's consumption-portfolio choice is $\max_{c_0, c_1, \theta} u(c_0, c_1)$ s.t. $c_0 \leq \omega_0 - \theta'p$ and $c_1 \leq \omega_1 + \theta'X$.
In addition, we often impose that $c_0 \geq 0$ and $c_1 \geq 0$.
- $g_0(c, \theta) = c_0 - (\omega_0 - \theta'p)$, $g_s(c, \theta) = c_1(s) - (\omega_1 + \theta'X)$

(a) The first order conditions are:

Consumption-Portfolio choice Problem II

- (b) $\partial_{c_0} u = \lambda(g_0)_{c_0} = \lambda$ and $\lambda \geq 0$, $\lambda(c_0 - (\omega_0 - \theta'p)) = 0$
(c) $\partial_{c_1,s} u = \mu_s$ and $\mu_s \geq 0$, $\mu_s(c_1(s) - (\omega_1 + \theta'X)) = 0$, for all $s = 1, \dots, S$.
(d) $0 = \lambda(g_0)_\theta + \sum_s \mu_s(g_s)_\theta$ is equivalent to $\lambda p = X\mu$.

- Assuming an interior solution exists with $c_0, c_1, \lambda, \mu \gg 0$ we obtain:

$$\boxed{p = X \frac{\mu}{\lambda}} \quad \text{where} \quad \boxed{\frac{\mu}{\lambda} = \frac{\partial_{c_1} u(c_0, c_1)}{\partial_{c_0} u(c_0, c_1)}}$$

- A simpler way to see it is to substitute budget constraints, assuming they are binding:

$$c_0 = \omega_0 - \theta'p, \quad c_1 = \omega_1 + \theta'X. \quad (31)$$

Then, the FOC become

$$0 = \frac{\partial}{\partial \theta} u(\omega_0 - \theta'p, \omega_1 + \theta'X) = -p \partial_{c_0} u(c_0, c_1) + X \partial_{c_1} u(c_0, c_1), \quad (32)$$

Consumption-Portfolio choice Problem III

that is

$$p = X \underbrace{\frac{\partial_{c_1} u(c_0, c_1)}{\partial_{c_0} u(c_0, c_1)}}_{\text{state prices } q}, \quad q_s = \frac{\partial_{c_{1,s}} u(c_0, c_1)}{\partial_{c_0} u(c_0, c_1)} \quad (33)$$

- This connects the consumption portfolio choice problem to state prices and the absence of arbitrage, as we now clarify.
- If there is an interior solution to some agent's optimal consumption-portfolio choice problem, then there exist strictly positive state prices, namely that agent's intertemporal marginal rates of substitutions (**IMRS**).

- Connection between the absence of arbitrage and the existence of a solution to an individual's consumption-portfolio problem:

Theorem

- *If some agent's optimal consumption portfolio problem admits a solution, and her utility function is strictly increasing, then there is no arbitrage.*

Theorem

- *If there is no arbitrage, and if agents' consumption is restricted to be positive, then the optimal consumption portfolio choice problem admits a solution.*

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Extension and valuation functional

- A **valuation functional** $Q(\cdot) : \mathbb{R}^S \rightarrow \mathbb{R}$ is an extension of the payoff functional to the entire contingent claim space $\mathbb{R}^S$, i.e., such that $Q(z) = q(z) \ \forall z \in \mathcal{M}$.
- The fundamental theorem of Asset pricing:

Theorem

There is no arbitrage if and only if there exists a strictly positive valuation functional.

- A weaker form is

Theorem

There is no strong arbitrage if and only if there exists a positive valuation functional.

- Consider an arbitrary contingent claim payoff $z \in \mathbb{R}^S$. Define the upper and lower replicating prices by respectively:
 - $q_u(z) = \min_{\theta} \{\theta' p \text{ s.t. } \theta' X \geq z\}$ and
 - $q_\ell(z) = \max_{\theta} \{\theta' p \text{ s.t. } \theta' X \leq z\}$
- We have the following results on these upper and lower replicating prices:

Theorem

If there is no arbitrage then

- (a) $q_\ell(z) \leq q_u(z) \quad \forall z \in \mathbb{R}^S$
- (b) $q_\ell(z) = q_u(z) = q(z) \quad \forall z \in \mathcal{M}$
- (c) $q_\ell(z) < q_u(z) \quad \forall z \in \mathbb{R}^S \setminus \mathcal{M}$

Proof of the theorem

- Assume that $\mathcal{M}$ is a strict subset of $\mathbb{R}^S$ (else there is nothing to prove).
- We show how to extend the valuation functional in one dimension (subsequent steps are similar until we cover all of $\mathbb{R}^S$).
- Fix a payoff $\hat{z} \notin \mathcal{M}$ and define $\mathbb{N} = \{y : y = z + \lambda \hat{z} \text{ with } z \in \mathcal{M} \text{ and } \lambda \in \mathbb{R}\}$.
- Fix a π such that $q_u(\hat{z}) > \pi > q_\ell(\hat{z})$.
- Define the functional $Q(\cdot) : \mathbb{N} \rightarrow \mathbb{R}$ by $Q(z + \lambda \hat{z}) = q(z) + \lambda \pi$

Theorem

If $q(\cdot) : \mathcal{M} \rightarrow \mathbb{R}$ is a strictly positive linear functional then so is $Q(\cdot) : \mathbb{N} \rightarrow \mathbb{R}$.

- It is clear from the proof that if $\mathcal{M}$ is a strict subset of $\mathbb{R}^S$, then the valuation functional is not unique. In fact, a continuum of extensions exists (depending on the choice of π). Each valuation functional must agree on the payoffs that are in the asset span however.

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Second fundamental theorem

- This result is often called the second fundamental theorem of asset pricing:

Theorem

Suppose there is no arbitrage. Then security markets are complete iff there exists a unique strictly positive valuation functional.

- A linear functional $Q(\cdot) : \mathbb{R}^S \rightarrow \mathbb{R}$ can be identified by its values on the basis vectors $e_1, \dots, e_S$ (e_s has a one on s^{th} row and zeroes everywhere else). Specifically define $q = (q_1, \dots, q_n)$ with $q_s = Q(e_s)$, then $Q(z) = zq \ \forall z \in \mathbb{R}^S$.
- Thus we can price any arbitrary contingent payoff (row) vector z by first computing a set of valid state prices q by solving the system $p = Xq$ for a strictly positive solution and then computing the inner product zq .

- Note that when $S > N$, then we expect more than one solution to that system, and if there are no arbitrage, then at least one of these solutions should be strictly positive. We have

Theorem

There exists a strictly positive valuation functional iff there exists a strictly positive solution q to the system $p = Xq$. Each strictly positive solution q defines a strictly positive valuation functional $Q(z) = zq \ \forall z \in \mathbb{R}^S$.

- If markets are complete, then there should be only one solution, and if there are no arbitrage then that unique solution should be strictly positive.
- If z happens to be in the market span, then any strictly positive q solution will give the same price.
- This last theorem shows that we can construct valuation functionals from state prices and vice-versa.

- In fact we can redefine the upper and lower hedging prices in terms of state prices. $q_u(z) = \max_{q \geq 0} \{zq : p = Xq\}$ and $q_\ell(z) = \min_{q \geq 0} \{zq : p = Xq\}$ Indeed, note that we have shown that for any payoff z not in $\mathcal{M}$ one can find a valuation functional that assigns $Q(z) = \pi$ for any $\pi \in (q_\ell(z), q_u(z))$ (is it a closed or an open interval?). Thus this interval is the range of arbitrage-free prices for this payoff consistent with existing traded securities. Since $Q(z)$ can also be represented in terms of the state price vector q , we must obtain the same price interval as we consider all consistent state price vectors.
- Risk-neutral probabilities ('measures') are yet a third equivalent way to characterize valuation functionals as we will see below.

Optimal consumption portfolio choice problem through state prices

- We note that the optimal consumption portfolio choice problem presented above can be rewritten in terms of the valuation functional (or state prices) as follows:

- An agent's consumption-portfolio choice is $\max_{c_0, c_1, z} u_i(c_0, c_1)$ s.t. $c_0 \leq \omega_0 - q'z$ and $c_1 \leq \omega_1 + z$ and $z \in \mathcal{M}$. (As before, we may also want to explicitly impose the constraints that $c_0 \geq 0$ and $c_1 \geq 0$).
- This recasts the problem in terms of asset payoffs and valuation functional rather than prices and trading strategies. This approach is often preferred for solving equilibrium problems (where security prices themselves are unknown).
- In the case of complete markets, the constraint $z \in \mathcal{M}$ can be dropped, which considerably simplifies the problem. Note that in this case, we obtain from the FOC that $q = \frac{\partial_{c_1} u_i}{\partial_{c_0} u_i}$, which states that marginal rates of substitution for all agents must align with the unique state prices (and valuation functional). (What happens when markets are incomplete?).
- Next we rederive the fundamental theorem of asset pricing starting from the concept of state prices.

Fundamental Theorem of Asset Pricing (Take II)

- Risky Asset prices $p = [p_1, \dots, p_N]'$ has matrix payoff $X = \{X_{i,s}\}$ for $i = 1, \dots, N$ and $s = 1, \dots, S$.
- Recall that a **state price** vector is an S -dimensional vector q such that $p = Xq$, and
- An **arbitrage** is a trading strategy θ such that $\theta'p \leq 0$ and $\theta'X \geq 0$ with at least one strict inequality.
- We shall prove **the first fundamental theorem of asset pricing** using a separation theorem for convex sets:

Theorem

There is no arbitrage if and only if there exists a vector of strictly positive state prices

Mathematical Results

- Separating a convex set from a point

Theorem

Let C be a closed convex set in $\mathbb{R}^n$ which does not contain the origin. Then there exists a real linear functional $L(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\alpha > 0$ such that $\forall x \in C \ L(x) \geq \alpha$. In particular, the hyperplane $L(x) = 0$ does not intersect C .

Proof.

Since C is closed and $0 \notin C$ we know that

$0 < \|x_0\| = \inf_{x \in C \cap B(\lambda)} \|x\|$, where $C \cap B(\lambda)$ is the intersection of the set C with a ball centered at the origin with radius λ chosen sufficiently large so that the intersection is non-empty. (note that Weierstrass' theorem guarantees the existence of x_0 .) Now, for any $x \in C$ we have $tx + (1-t)x_0 \in C$ and therefore Since this holds for arbitrary $t \in [0, 1]$ this implies $\forall x \in C \quad x \cdot x_0 \geq \|x_0\|^2$. Therefore a candidate linear functional is $L(x) = x_0 \cdot x$ and $\alpha = \|x_0\|^2$. $\square$

- Separating two convex sets

Theorem

Let K be a compact convex set in $\mathbb{R}^n$ and $\mathcal{G}$ a subspace of $\mathbb{R}^n$. If $\mathcal{G} \cap K = \emptyset$ then there exists a linear functional $L(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

$$L(x) > 0 \quad \forall x \in K \quad (*)$$

$$L(x) = 0 \quad \forall x \in \mathcal{G} \quad (**)$$

Therefore the subspace $\mathcal{G}$ is included in a hyperplane that does not intersect K .

Proof.

Let $C = K - \mathcal{G} = \{x \in \mathbb{R}^n \text{ s.t. } \exists (y, z) \in K \times \mathcal{G} \text{ and } x = y - z\}$.

Clearly, C is closed and convex and does not contain the origin. It follows from the previous theorem that there exists a linear functional $L(\cdot)$ and $\alpha > 0$ such that $\forall x \in C \quad L(x) \geq \alpha$. Therefore $\forall (y, z) \in K \times \mathcal{G} \quad L(y) \geq L(z) + \alpha$. Since $\mathcal{G}$ is a subspace and $L(\cdot)$ is linear we must have $L(z) = 0$ (why?) and hence $L(y) > 0$. $\square$

- Riesz Representation theorem:

Theorem

Any linear function $L(X) : \mathbb{R}^n \rightarrow \mathbb{R}$ can be represented by a vector A in $\mathbb{R}^n$ such that $L(X) = A'X$.

- Exercise: Prove the fundamental theorem using these mathematical results.

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State Prices, Risk-Neutral Measure, Stochastic Discount Factor, Pricing Kernel

- Suppose there exists a state price vector $q \gg 0$.
- If there exists a set of objective probabilities for the states $\mathcal{P} = [\mathcal{P}_1, \dots, \mathcal{P}_S]$ then we can define the **pricing kernel** or **state price density** M as the random variable such that $p_i = \mathbb{E}[MX_i]$ where $\mathcal{P}_s M_s = q_s \ \forall s$.
- Further we can define the **risk-neutral probability measure** by $\frac{Q_s}{R_f} = \mathcal{P}_s M_s = q_s$ so that $p_i = \mathbb{E}^Q[\frac{X_i}{R_f}]$ where we define $R_f = \frac{1}{\sum_s q_s}$ (is equal to the gross risk-free rate if a risk-free asset exists).
- Note that State prices are often called Arrow-Debreu State prices, and the corresponding securities are Arrow securities.
- M is often called the **stochastic discount factor**.

- When markets are complete, then the risk-neutral measure is unique.
- When Markets are incomplete, then there exists a set of risk-neutral measures $\mathbb{Q} = \{\mathbb{Q} : p_i = \mathbb{E}^{\mathbb{Q}}[\frac{X_i}{R_f}] \ \forall i = 1, \dots, N\}$.
- In fact, we can redefine the upper and lower hedging prices in terms of the set of risk-neutral measures:
 $q_u(z) = \max_{\mathbb{Q} \in \mathbb{Q}} \{\mathbb{E}^{\mathbb{Q}}[\frac{z}{R_f}]\}$, and
 $q_\ell(z) = \min_{\mathbb{Q} \in \mathbb{Q}} \{\mathbb{E}^{\mathbb{Q}}[\frac{z}{R_f}]\}$