Asset Pricing VII

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The Economy

- Consider a *filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,1,2\ldots}, \mathbb{P})$. We assume Ω is finite and time is discrete (think multi-nomial, possibly non-recombining tree).
- There are $d+1$ assets $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$.
- $S_t^0 > 0 \ \forall t$ is the numeraire asset with $S_0^0 = 1$ ($S_t^0 = (1+r)^t$) if risk-free rate is constant).
- All other risky assets pay a dividend D_{t+1}^i if you own one unit of the stock i at time t for $t = 0, \ldots, T$ and $i = 1, \ldots, d$.
- Absence of Arbitrage implies there exists $\mathcal{Q} \sim \mathbb{P}$ under which discounted Gain processes are martingales.

- The following statements are equivalent:
  1. $\frac{S_t^j}{S_t^0} = \mathbb{E}_t^{\mathcal{Q}}\left[\sum_{n=t+1}^{T} \frac{D_n^j}{S_n^0} + \frac{S_T^j}{S_T^0}\right]$
  2. $\frac{S_t^j}{S_t^0} + \sum_{n=1}^{t} \frac{D_n^j}{S_n^0}$ is a $\mathcal{Q}$-martingale
  3. $\frac{S_t^j}{S_t^0} = \mathbb{E}_t^{\mathcal{Q}}\left[\frac{D_{t+1}^j + S_{t+1}^j}{S_n^0}\right]$
  4. $M_t S_t^j = \mathbb{E}_t\left[M_{t+1}(D_{t+1}^j + S_{t+1}^j)\right]$
  5. $M_t S_t^j + \sum_{n=1}^{t} M_n D_n^j$ is a P-martingale
  6. $M_t S_t^j = \mathbb{E}_t\left[\sum_{n=t+1}^{T} M_n D_n^j + M_T S_T^j\right]$
- Where the $M_t = \frac{\xi_t}{S_t^0}$ and $\xi_t = \mathbb{E}_t[\frac{d\mathcal{Q}}{d\mathbb{P}}]$.

The Euler Equation

- Consider an investor who chooses his portfolio allocation and consumption to maximize his expected utility: $\max_{c_n,\pi_n} \mathbb{E}[\sum_{n=0}^{T} \delta^n u(c_n)]$.
- His wealth dynamics are: $W_{t+1} = (W_t - c_t)\pi_t' R_{t+1} + Y_{t+1}$ where
    - π_t is a vector of the proportion of wealth invested in the risky and risk-free asset, i.e., $\pi_t' \mathbf{1} = 1$.
    - $R_{t+1} = \frac{S_{t+1}+D_{t+1}}{S_t}$ is the vector of Gross returns.
    - Hence,

$$
\begin{aligned}
E_t[M_{t+1}W_{t+1}] &= E_t[M_{t+1}((W_t - c_t)\pi_t' R_{t+1} + Y_{t+1})] \\
&= (W_t - c_t)\pi_t' E_t[M_{t+1}R_{t+1}] + E_t[M_{t+1}Y_{t+1}] \\
&= W_t - c_t + E_t[M_{t+1}Y_{t+1}]
\end{aligned} \tag{1}
$$

    - Y_t is the endowment of the agent received at date t.
    - $\delta \leq 1$ is a time preference parameter.

Theorem

A necessary and sufficient condition for optimality (of both the portfolio and consumption decision) is the **Euler Equation**

$$u'(c_t) = \mathbb{E}_t[\delta u'(c_{t+1}) R_i(t+1)] \ \forall i = 0, \ldots, d$$

- For any valid state price density, we have:

$$E_t[M_T W_T + \sum_{n=t}^{T-1} M_n C_n] = M_t W_t + E_t[\sum_{n=t+1}^{T} M_n Y_n]$$

- In particular if at time T we have $C_T = W_T$ and at time 0 we define $W_0 = Y_0$ then we get the intertemporal Budget constraint:

$$E[\sum_{n=0}^{T} M_n C_n] = E[\sum_{n=0}^{T} M_n Y_n]$$

The Complete market case

- The optimal consumption-portfolio problem in a complete market is equivalent to

$$\max_{c_n} \mathbb{E}[\sum_{n=0}^{T} \delta^n u(c_n)] \;\; s.t. E[\sum_{n=0}^{T} M_n c_n] \leq E[\sum_{n=0}^{T} M_n Y_n]$$

- where M_t is the unique pricing kernel.
- The FOC is $\delta^t u'(c_t) = yM_t$ or equivalently $C_t^* = I(y\delta^{-t} M_t)$ where $I(\cdot)$ is the inverse function of $u'(\cdot)$.
- y is the Lagrange multiplier associated with the Budget constraint, i.e., it solves

$$E[\sum_{n=0}^{T} M_n I(y\delta^{-n} M_n)] = E[\sum_{n=0}^{T} M_n Y_n]$$

- The optimal portfolio π_t then satisfies $W_{t+1} = (W_t - C_t^*)\pi_t' R_{t+1} + Y_{t+1}$ given $W_0 = Y_0$ and for any $t > 0$ we define W_t from:

$$E_t[\sum_{n=t}^{T} M_n C_n^*] = M_t W_t + E_t[\sum_{n=t+1}^{T} M_n Y_n]$$

Time Consistency I

- The agent wakes up at time t and realizes that he chose his $C_t^* = I(y\delta^{-t}M_t)$. However, if he now repeats his analysis, he is solving

$$\max_{c_n} \mathbb{E}_t[\sum_{n=t}^{T} \delta^n u(c_n)] \;\; s.t. E[\sum_{n=0}^{T} M_n c_n] \leq W_t M_t + E_t[\sum_{n=0}^{T} M_n Y_n]$$

which is equivalent to

$$\max_{c_n} \mathbb{E}_t[\sum_{n=t}^{T} \delta^{n-t} u(c_n)] \;\; s.t. E_t[\sum_{n=t}^{T} M_{t,n} c_n] \leq W_t + E_t[\sum_{n=t}^{T} M_{t,n} Y_n]$$

and the solution will be

$$\tilde{C}_n^* = I(y_t \delta^{-(n-t)} M_{t,n}).$$

Time Consistency II

For the solutions to coincide, we need

$$y_t \delta^{-(n-t)} M_{t,n} \;=\; y \delta^{-n} M_n$$

which is equivalent to

$$y_t \;=\; y M_t \delta^{-t}\,.$$

By construction, any feasible consumption stream C_t (and hence C_t^* always satisfies the budget constraint

$$E_t[\sum_{n=t}^{T} M_{t,n} c_n] \leq W_t + E_t[\sum_{n=t}^{T} M_{t,n} Y_n]$$

so that

$$E_t[\sum_{n=t}^{T} M_{t,n} I(y_t \delta^{-(n-t)} M_{t,n})] \leq W_t + E_t[\sum_{n=t}^{T} M_{t,n} Y_n]$$

Time Consistency III

The key mechanism that makes all this work is time consistency: We could move from δ^n to δ^{n-t} without affecting the solution. However, if the agent discounts the future using some other function, optimizing

$$\max_{c_n} \mathbb{E}_t[\sum_{n=t}^{T} \delta(t,n)u(c_n)]$$

with a normalization $\delta(t,t) = 1$ at time t, time consistency requires that $\delta(t,n)\delta(n,m) = \delta(t,m)$. This only works when $\delta(t,n) = \delta^{n-t}$

Dynamic Programming

- For certain problems (Markov state vector, Recursive objective function), Dynamic programming is a useful technique to solve optimization problems.
- As an example, consider the problem $\max_{C_n} \mathbb{E}[\sum_{n=0}^{T} u(n, C_n, X_n)]$ subject to a state equation $X_{t+1} = F(t, C_t, X_t, \epsilon_{t+1})$ where ϵ_{t+1} is a random vector of shocks with a conditional distribution at time t.
  For simplicity we assume that the ϵ_t are iid shocks $\forall t$.
- We need to pick the control C_n so as to 'steer' the state X_n optimally to maximize the objective function.

- The trick is to consider controls that are in feedback form: $C_n = C(n, X_n)$. Then, the dynamics of the state vector are clearly Markov: $X_{t+1} = F(t, C(t, X_t), X_t, \epsilon_{t+1})$
- For a particular control $C(t, X_t)$ we can define the value $V^C(t, X_t) = \mathbb{E}_t[\sum_{n=t}^{T} u(n, C(n, X_n), X_n)]$, which is a function of the state by the Markov Property.
- Note that $V^C(t, X_t) + \sum_{n=0}^{t-1} u(n, C(n, X_n), X_n)$ is a martingale (by the law of iterated expectation) so that $V^C(t, X_t) = \mathbb{E}_t[u(t, C(t, X_t), X_t) + V^C(t+1, X_{t+1}^C)] \ \forall t \leq T$, where we set $V(T+1, X) = 0$.
- It is then natural to define the (Hamilton Jacobi Bellman) equation of optimality as $V(t, X_t) = \max_C \mathbb{E}_t[u(t, C, X_t) + V(t+1, X_{t+1}^C)]$.
- Note that when there is a finite horizon, this leads to a natural backward recursion algorithm.

- Now suppose we can find a function $V^*(t, X)$ and a control $C^*(t, X)$ that achieve the optimum, i.e., such that $V^*(T+1, x) = 0$ and for all $t \leq T$ we have

$$V^*(t, X_t) \geq \mathbb{E}_t[u(t, C_t, X_t) + V^*(t+1, X_{t+1})]$$

  and with equality if we pick $C_t = C^*(t, X_t)$.
- Then, note that $V^*(t, X_t) = \mathbb{E}_t[u(t, C^*(X_t), X_t) + V^*(t+1, X^*_{t+1})]$ so that $V^*(0, X_0) = \mathbb{E}_0[\sum_{t=0}^T u(t, C^*(X^*_t), X^*_t)]$.
- Further, since $V^*(t, X_t) \geq \mathbb{E}_t[u(t, C_t, X_t) + V^*(t+1, X^C_{t+1})]\ \forall C_t$, starting at $t = 0$ and iterating forward we obtain that $V^*(0, X_0) \geq \mathbb{E}_0[\sum_{n=0}^T u(n, C_n, X^C_n)]\ \forall C_n$. This establishes the optimality of $C^*(t, X)$ and of $V^*(0, X_0)$.

- Note that the FOC from the HJB equation is:

$$\mathbb{E}_t[u_C(t, C^*, X) + V_X(t+1, F(t, C^*, X, \epsilon_{t+1}))\, F_C(t, C^*, X, \epsilon_{t+1})] = 0 \tag{2}$$

- We can also use the "Envelope Condition:" Differentiating

$$V^*(t, X_t) = \mathbb{E}_t[u(t, C^*(X_t), X_t) + V^*(t+1, F(t, C^*(X_t), X_t, \epsilon_{t+1}))] \tag{3}$$

with respect to X_t, and using (2), we get

$$\begin{aligned} V_X(t, X) = \mathbb{E}_t[&u_X(t, C^*, X) \\ &+ V_X(t+1, F(t, C^*, X, \epsilon_{t+1}))\, F_X(t, C^*, X, \epsilon_{t+1})] \end{aligned} \tag{4}$$

Optimal portfolio choice

- Let's use dynamic programming to solve the optimal portfolio consumption problem of an agent with CRRA utility $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ when returns are i.i.d.
- The value function is $J(t, W_t) = \max_{c_n, \pi_n} \mathbb{E}[\sum_{n=t}^{T} \delta^n u(C_n)]$.
- Subject to $W_{t+1} = (W_t - c_t) R_p(t+1)$.
- We ignore labor income for now, so the investor starts with $W_0 = Y_0$ and gets no future income.
- We also assume that there exists a risk-free rate and solve for π_t, the vector of fractions of wealth invested in risky assets. (so we have substituted the constraint $\pi' \mathbf{1} = 1$). Thus we define $R_p(t+1) = (R_f + \pi_t'(R_{t+1} - R_f \mathbf{1}))$.

- The HJB equation is
  $J(t, W_t) = \max_{\pi,C}\{u(C)\delta^t + \mathbb{E}_t[J(t+1, W_{t+1})]\}$
- The FOC is

$$
\begin{aligned}
0 &= \delta^t u'(C) - \mathbb{E}_t[J_W(t+1, W_{t+1})R_p(t+1)] \\
0 &= \mathbb{E}_t[J_W(t+1, W_{t+1})(R^j_{t+1} - R_f)] \ \forall j = 1, \ldots, d
\end{aligned}
$$

- Using the "envelope condition" the FOC become:

$$
\begin{aligned}
\delta^t u'(C_t) &= J_W(t, W_t) \\
0 &= \mathbb{E}_t[\frac{\delta u'(C_{t+1})}{u'(C_t)}(R^j_{t+1} - R_f)] \ \forall j = 1, \ldots, d
\end{aligned}
$$

We recognize the Euler equation derived previously.

- For a CRRA investor we guess that $J(t,W) = \delta^t A_t^{-\gamma} \frac{W^{1-\gamma}}{1-\gamma}$
- Substituting into the FOC conditions, we get:

$$\begin{aligned} C_t &= A_t W_t \\ 0 &= \mathbb{E}_t[(R_p(t+1))^{-\gamma}(R^j_{t+1} - R_f)] \ \forall j = 1, \ldots, d \end{aligned}$$

  Note that the optimal solution for the portfolio strategy is a constant vector π^*, since the returns are assumed to be i.i.d. Further, that solution is identical to that obtained in the one-period case. That is since returns are i.i.d., agents act myopically with respect to their portfolio choice. Not so, however, for their consumption decision.

- Indeed, substituting the guess into the HJB equation, we obtain $C_t^* = A_t W_t$ where A_t solves a recursive equation:

$$A_t = \frac{A_{t+1}}{A_{t+1} + (\delta B)^{1/\gamma}}$$

  subject to $A_T = 1$ and where $B = \mathbb{E}_t[(R_p^*(t+1))^{1-\gamma}]$ a constant since the optimal portfolio π^* is constant and returns are iid.

- The solution is easily derived $A_{T-t} = \frac{1}{\sum_{n=0}^{t} (\delta B)^{n/\gamma}}$.
- Note that when $T \to \infty$ we obtain a simple solution $C_t^* = AW_t$ with $A = 1 - (\delta B)^{1/\gamma}$ if it is positive (what happens else?).
- In that case the value function is simply $J(t, W) = \delta^t A^{-\gamma} \frac{W^{1-\gamma}}{1-\gamma}$
- One can also attack the problem directly by looking for a stationary solution in the HJB equation subject to a transversality condition $\lim_{T\to\infty} \mathbb{E}[J(T, W_T)] = 0$