Asset Pricing VI

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  - EMM unique $\iff$ Complete Markets
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The economy

- Consider a *filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,1,2\ldots}, \mathbb{P})$. We assume Ω is finite and time is discrete (think multi-nomial, possibly non-recombining tree).
- There are $d+1$ assets $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$.
- $S_t^0 > 0 \ \forall t$ is the numeraire asset with $S_0^0 = 1$ ($S_t^0 = (1+r)^t$) if risk-free rate is constant).
- A trading strategy is an adapted process $\Delta_t = (\Delta_t^0, \Delta_t^1, \ldots)$ (Δ_t is $\mathcal{F}_t$ measurable). The corresponding portfolio value is $V_t(\Delta) = \Delta_t \cdot S_t$.

- A strategy is self-financing if there is no infusion of cash when rebalancing:

$$\Delta_t S_t = \Delta_{t-1} S_t \tag{1}$$

- It follows that $\boxed{V_n(\Delta) = V_0 + \sum_{t=0}^{n-1} \Delta_t dS_t}$ where we define $dS_t = S_{t+1} - S_t$.
- Define $V_t^*(\Delta) = \frac{V_t(\Delta)}{S_t^0}$ the discounted value of the portfolio, and $S_t^* = \frac{S_t}{S_t^0}$ the vector of discounted asset prices. We have the following result:
- If Δ is self-financing for S then Δ is self-financing for S^*, i.e., $\boxed{V_n^*(\Delta) = V_0 + \sum_{t=0}^{n-1} \Delta_t dS_t^*}$
- Since, by definition, $V_n^*(\Delta) = \underbrace{\Delta_n^0}_{risk\ free\ bond\ holdings} + \sum_{i=1}^{d} \underbrace{\Delta_n^i}_{stock\ i\ holdings} S_n^{i*}$ we immediately have:

Theorem

For any adapted process $\underbrace{(\Delta_t^1,\ldots,\Delta_t^d)}_{stock\ \ investments}$ *and* $\mathcal{F}_0$ *measurable* V_0 *there exists a unique adapted process* Δ_t^n *given by*

$$\underbrace{\Delta_n^0}_{risk\ \ free\ \ bond\ \ holdings} \equiv V_n^*(\Delta) - \sum_{i=1}^{d} \underbrace{\Delta_n^i}_{stock\ \ i\ \ holdings} S_n^{i*}\,,$$

such that Δ_t *is self-financing and generates* $V_t(\Delta)$ *with initial value* V_0*.*

- An *Arbitrage* strategy is a self-financing trading strategy such that (1) $V_0(\Delta) \leq 0$, (2) $V_T(\Delta) \geq 0$ and (3) $\mathbb{P}(V_T(\Delta) > 0) > 0$.
- We shall prove **the first fundamental theorem of asset pricing**:

Theorem

There is no arbitrage if and only if there exists a probability measure $\mathcal{Q}$ equivalent to $\mathbb{P}$ such that discounted prices are martingales under $\mathcal{Q}$.

- In the above, we assumed that stocks do not pay any dividends. As an exercise, how would you extend Theorem 2 to the case where:
  - ownership of stock j at any date t entitles you to a dividend D^j_{t+1} at date $t+1$ $\forall j = 1, \ldots, n$. That is when all stocks, except the numeraire asset, pay a dividend.
  - ownership of stock j at any date t entitles you to a dividend D^j_{t+1} at date $t+1$ $\forall j = 0, 1, \ldots, n$. That is when all stocks, including the numeraire asset, pay a dividend.

Mathematics Refresher

- A measure $\mathcal{Q}$ is equivalent to $\mathbb{P}$ (we write $\mathcal{Q} \sim \mathbb{P}$) if $\forall A \in \mathcal{F}$ we have $\{\mathcal{Q}(A) = 0 \iff \mathbb{P}(A) = 0\}$.
- $\mathcal{Q} \sim \mathbb{P}$ implies we can define the random variable $\xi(\omega) = \frac{\mathcal{Q}(\omega)}{\mathbb{P}(\omega)}$ and $\xi(\omega) > 0$ and $\mathbb{E}^{\mathbb{P}}[\xi] = \sum_\omega \xi(\omega)\mathbb{P}(\omega) = \sum_\omega \mathcal{Q}(\omega) = 1$.
- Consider an integrable process $M(t)$ ($\mathbb{E}[M(t)] < \infty \quad \forall t$)
  - An adapted process $M(t)$ is a martingale if

$$\mathbb{E}_s[M(t)] = M(s) \; \forall s \leq t.$$

  - An adapted process $M(t)$ is a super-martingale if

$$\mathbb{E}_s[M(t)] \leq M(s) \; \forall s \leq t.$$

  - An adapted process $M(t)$ is a sub-martingale if

$$\mathbb{E}_s[M(t)] \geq M(s) \; \forall s \leq t.$$

- We have the following results:
  - $M(t)$ is a martingale if and only if

$$M(t) = \mathbb{E}[M(t+1)|\mathcal{F}_t] \; = \; \mathbb{E}_t[M(t+1)]$$

  - Let Δ_t be an adapted process then $X_t = X_0 + \sum_{t=0}^{n-1} \Delta_t dM(t)$ is a martingale, where we define $dM(t) = M(t+1) - M(t)$. Note that $dX_t = \Delta_t dM_t$ (X_t is often called a martingale transform; note the analogy to the stochastic integral).
  - The following result is useful:

Theorem

An adapted process M_t is a martingale if and only if for any adapted process Δ_t we have $\mathbb{E}[\sum_{t=0}^{T} \Delta_t dM_t] = 0$.

Proof of the First Fundamental Theorem of Asset Pricing

- Existence of EMM $\Rightarrow$ AOA
  Consider any self-financing trading strategy $V_n^*(\Delta) = V_0 + \sum_{t=0}^{n-1} \Delta_t dS_t^*$. Then, by definition of EMM, S_t^* is a $\mathcal{Q}$-martingale. Therefore $V_n^*(\Delta)$ is a $\mathcal{Q}$-martingale (why?) and $V_0 = \mathbb{E}^{\mathcal{Q}}[V_T^*(\Delta)]$. It follows that for any strategy such that $V_T \geq 0$ and $\mathbb{P}(V_T > 0) > 0$, we must have $V_0 > 0$ (why?), and there cannot be any arbitrage.

- Converse: AOA $\Rightarrow$ EMM.
  The converse is proved using reduction to one-period models. Consider a self-financing strategy where we hold wealth in the risk-free account, then invest at time t and sell everything at time $t+1$, and then hold proceeds in the risk-free account. This is equivalent to a one-period problem. Thus, AOA implies that there exists a *conditional* stochastic discount factor (SDF) $M_{t,t+1}$ such that

$$S_t^j \;=\; E_t[M_{t,t+1} S_{t+1}^j] \tag{2}$$

  and corresponding conditional EMM is given by $M_{t,t+1}/E_t[M_{t,t+1}]\mathbb{P}(\omega)$ so that

$$S_t^j \;=\; (1/R_t^f)\,\mathbb{E}_t^{\mathcal{Q}}[S_{t+1}^j],\;\; R_t^f = E_t[M_{t,t+1}]^{-1} \tag{3}$$

- What about the multi-period case? How do we compute the one-period SDFs into a multi-period object?

$$M_{t,t+\tau} \;=\; \prod_{s=1}^{\tau} M_{t+s-1,s}$$

and, hence,

$$M_{t,t+\tau} \;=\; M_{t,t+\tau-1} \cdot M_{t+\tau-1,t+\tau} \,. \tag{4}$$

Then,

$$\begin{aligned}
&\mathbb{E}_t[S_{t+\tau} M_{t,t+\tau}] \\
&= \mathbb{E}_t[S_{t+\tau} M_{t+\tau-1,t+\tau}\, M_{t,t+\tau-1}] \\
&\underbrace{=}_{\textit{iterated expectations}} E_t[E_{t+\tau-1}[S_{t+\tau} M_{t+\tau-1,t+\tau}]\, M_{t,t+\tau-1}] \\
&= \mathbb{E}_t[S_{t+\tau-1}\, M_{t,t+\tau-1}] \\
&= \cdots \\
&= \mathbb{E}_t[S_{t+1} M_{t,t+1}] \;=\; S_t
\end{aligned} \tag{5}$$

What about Martingales? Well, let us define interest rates

$$e^{-r_t} = E_t[M_{t,t+1}],\ M_{t,t+1} = e^{-r_t}\,\xi_{t,t+1}\,. \tag{6}$$

where

$$E_t[\xi_{t,t+1}] = E_t[\xi_{t,t+\tau}] = 1\,. \tag{7}$$

Write

$$M_{0,T} = \prod_{\tau=0}^{T-1} e^{-r_\tau}\xi_{\tau,\tau+1} = S^0(T)\xi(T) \tag{8}$$

where

$$\xi_{t,t+1} = M_{t,t+1}/E_t[M_{t,t+1}],\ \xi_{t,t+\tau} = \prod_{s=1}^{\tau}\xi_{t+s-1,t+s} \tag{9}$$

Define a new measure

$$\mathcal{Q} = \frac{\xi(T)}{E[\xi(T)]},\ \xi(T) = \xi_{0,T}\,. \tag{10}$$

where $\xi(t)$ is a *martingale*
Question: How do we compute conditional expectations with a different measure?

Theorem

$$\begin{aligned}\mathbb{E}_t^{\mathcal{Q}}[X] &= \frac{\mathbb{E}_t[\xi_T X]}{\mathbb{E}_t[\xi_T]} = \frac{\mathbb{E}_t[\xi_{0,t}\xi_{t,T}X_T]}{\mathbb{E}_t[\xi_{0,t}\xi_{t,T}]} \\ &= \frac{\xi_{0,t}\mathbb{E}_t[\xi_{t,T}X_T]}{\xi_{0,t}\mathbb{E}_t[\xi_{t,T}]} = \frac{\mathbb{E}_t[\xi_{t,T}X_T]}{\mathbb{E}_t[\xi_{t,T}]} = \mathbb{E}_t[\xi_{t,T}X_T]\end{aligned} \tag{11}$$

Thus,

$$\begin{aligned}S_t &= \mathbb{E}_t[S_T M_{t,T}] \\ &= \mathbb{E}_t[\xi_{t,T}(\prod_{\tau=t}^{T-1} e^{-r_\tau} S_T)] \underbrace{=}_{\mathbb{E}^{\mathcal{Q}} \textit{ absorbs } \xi_{t,T}} \mathbb{E}_t^{\mathcal{Q}}[\prod_{\tau=t}^{T-1} e^{-r_\tau} S_T]\end{aligned} \tag{12}$$

- Note on *admissible* strategies:
  - An *admissible* trading strategy is a self-financing strategy such that $V_t(\Delta) \geq 0 \ \forall t$.
  - It is often customary to restrict the definition of arbitrage to strategies that are also *admissible* to rule out negative wealth along the way.

However, in the discrete finite dimensional setup, this is unnecessary as we have the result (exercise!):

Theorem

There exists an arbitrage for general trading strategies if and only if there exists an arbitrage for admissible trading strategies.

This implies that we might as well rule out general arbitrage trading strategies. However, in continuous time, because of so-called 'doubling strategies,' we will have to restrict the definition of arbitrage to admissible strategies. Specifically, in continuous-time, there will be price processes that rule out arbitrage for admissible strategies while allowing arbitrage in a more general (but economically implausible) sense.

Relation between EMM and Pricing Kernel

- AOA $\iff$ $\exists$ EMM $\mathcal{Q}$ under which any stock price satisfies:

$$
\begin{aligned}
S^j(0) &= \mathbb{E}_0^{\mathcal{Q}}[\frac{S^j(T)}{S^0(T)}] \\
&= \mathbb{E}_0^{\mathbb{P}}[\xi \frac{S^j(T)}{S^0(T)}] \\
&= \mathbb{E}_0^{\mathbb{P}}[M(T)S^j(T)]
\end{aligned}
$$

where we have defined the change of measure random variable $\xi(\omega)$ by

$$\xi(\omega) = \frac{\mathcal{Q}(\omega)}{\mathbb{P}(\omega)}$$

and the *pricing kernel* or *state price density*:

$$M(T) = \frac{\xi}{S^0(T)}$$

- *Bayes Rule* for Conditional expectation states that for any $\mathcal{F}_T$-measurable random variable X we have

$$\mathbb{E}_t^{\mathcal{Q}}[X] = \frac{\mathbb{E}_t^{\mathbb{P}}[\xi X]}{\mathbb{E}_t^{\mathbb{P}}[\xi]}$$

where we use the notation $\mathbb{E}_t[X] = \mathbb{E}[X|\mathcal{F}_t]$.

- Recall the definition of conditional expectation: Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be an integrable r.v. Then $\mathbb{E}[X|\mathcal{F}_t]$, is the unique $\mathcal{F}_t$-measurable random variable Y which satisfies:

$$\int Y(\omega)\mathbf{1}_{\{A\}} d\mathbb{P}(\omega) = \int X(\omega)\mathbf{1}_{\{A\}} d\mathbb{P}(\omega) \quad \forall A \in \mathcal{F}_t$$

So, we need to show that

$$\mathbb{E}^{\mathcal{Q}}\left[\frac{\mathbb{E}_t^{\mathbb{P}}[\xi X]}{\mathbb{E}_t^{\mathbb{P}}[\xi]}\mathbf{1}_{\{A\}}\right] = \mathbb{E}^{\mathcal{Q}}\left[X\mathbf{1}_{\{A\}}\right] \;\; \forall A \in \mathcal{F}_t$$

- To that effect

$$\begin{aligned}
\mathbb{E}^{\mathcal{Q}}\left[\frac{\mathbb{E}_t^{\mathbb{P}}[\xi X]}{\mathbb{E}_t^{\mathbb{P}}[\xi]}\mathbf{1}_{\{A\}}\right] &= \mathbb{E}^{\mathbb{P}}\left[\xi\frac{\mathbb{E}_t^{\mathbb{P}}[\xi X]}{\mathbb{E}_t^{\mathbb{P}}[\xi]}\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}_t^{\mathbb{P}}[\xi]\frac{\mathbb{E}_t^{\mathbb{P}}[\xi X]}{\mathbb{E}_t^{\mathbb{P}}[\xi]}\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}_t^{\mathbb{P}}[\xi X\mathbf{1}_{\{A\}}]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\xi X\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}^{\mathcal{Q}}\left[X\mathbf{1}_{\{A\}}\right]
\end{aligned}$$

- Then, at any time $t \in [0, T]$ we have

$$\begin{aligned}
\frac{S^j(t)}{S^0(t)} &= \mathbb{E}_t^{\mathcal{Q}}\left[\frac{S^j(T)}{S^0(T)}\right] \\
&= \frac{\mathbb{E}_t^{\mathbb{P}}[\xi\frac{S^j(T)}{S^0(T)}]}{\mathbb{E}_t^{\mathbb{P}}[\xi]}
\end{aligned}$$

- This implies $\boxed{M(t)S^j(t) = \mathbb{E}_t^{\mathbb{P}}[M(T)S^j(T)]}$ for all j, where the pricing kernel is defined as

$$M(t) = \frac{\xi(t)}{S^0(t)}$$

and we have defined the conditional likelihood ratio $\xi_t = \mathbb{E}_t^{\mathbb{P}}[\xi]$.

Relation between EMM and Viability

- An economy is viable if the price system supports the optimal portfolio and consumption decision of an agent with a standard (i.e., continuous, increasing, and concave)utility function.
- Specifically, and economy is viable if there exists $U(c)$ such that $\sup_{c \in \mathbb{C}} U(c)$ admits a solution, where the budget feasible consumption set from an initial endowment e denoted by $\mathbb{C}_e$ is the sequence of positive random variables c_t that satisfy:

$$\begin{cases} V_T = c_T & \geq & 0 \\ V_{t+1} & = & V_t + \Delta_t dS_t - c_t \\ V_0 & = & e \end{cases}$$

for some admissible trading strategy Δ and initial endowment e.

- Let us restrict ourselves to time-separable expected utility functions of the type $U(C) = \mathbb{E}^{\mathbb{P}}[\sum_{t=1}^{T} u_t(c_t)]$ with $u_t(c)$ continuous increasing and concave. It is clear that AOA is necessary for a solution to exist (why?). Conversely, AOA guarantees a solution to the utility maximization problem by ensuring that the feasible set is compact (and using Weierstrass's theorem). Exercise!

- Suppose there exists an optimum consumption process $\hat{c}$. Then, a necessary condition for an optimum is

$$\lim_{\delta \to 0} \frac{U(\hat{c} + \delta\tilde{c}) - U(\hat{c})}{\delta} = 0$$

for any $\tilde{c}$ process in $\mathbb{C}_0$. In the time-separable case assuming $u(c)$ is differentiable, this condition simplifies to

$$0 = \mathbb{E}\left[\sum_{t=0}^{T} u_t'(\hat{c}_t)\tilde{c}_t\right] \quad (*)$$

- For example, choose for any $A \in \mathcal{F}_t$

$$\begin{cases} \tilde{c}_s &= 0 & \forall s \neq t, T \\ \tilde{c}_t &= -\delta S_t^j \mathbf{1}_{\{A\}} \\ \tilde{c}_T &= \delta S_T^j \mathbf{1}_{\{A\}} \end{cases}$$

This is clearly in $\mathbb{C}_0$, since all we need is

$$\begin{cases} \Delta_s &= \hat{\Delta}_s & \forall s \neq t, T \\ \Delta_t^j &= \hat{\Delta}_t^j + \delta \mathbf{1}_{\{A\}} \\ \Delta_T^j &= \hat{\Delta}_t^j - \delta \mathbf{1}_{\{A\}} \end{cases}$$

Then (*) above implies the Euler Condition:

$$u'(\hat{c}_t) S_t^j = \mathbb{E}_t^{\mathbb{P}}[u'(\hat{c}_T) S_T^j]$$

which shows that $\boxed{M(t) = u_t'(\hat{c}_t)}$

- If $u_t(\cdot)$ is Concave then the Euler Condition is both a necessary and sufficient condition for optimality of the portfolio and consumption decision.

Contingent Claims

- A contingent claim (CC) is defined by a $\mathcal{F}_T$ measurable payoff h (e.g., a European call $h = |S^j(T) - K|^+$).
- A CC is attainable if there exists a self-financing trading strategy worth h at T.
- The market is complete if every CC is attainable.
- We shall prove the **Second fundamental theorem of asset pricing**:

Theorem

An arbitrage-free market is complete if and only if there exists a unique EMM under which discounted asset prices are martingales.

- Suppose the market is arbitrage-free and complete, but that there are two EMM $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then for any $\mathcal{F}_T$-measurable payoff h we have:

$$\mathbb{E}^{\mathcal{Q}_1}[\frac{h}{S_T^0}] = \mathbb{E}^{\mathcal{Q}_2}[\frac{h}{S_T^0}]$$

  Using $h = S_T^0 \mathbf{1}_{\{\omega\}}$ implies $\mathcal{Q}_1(\omega) = \mathcal{Q}_2(\omega)\ \forall \omega$.

- For the converse, suppose that an arbitrage-free market is incomplete so that there exist F_T-measurable payoffs that are not attainable, say h. Define $\mathcal{G}^* = \{x(\omega) : x(\omega) = e_0 + \sum_{t=0}^{T-1} \Delta_t dS_t^*\}$ the set of attainable payoffs starting from an initial endowment e_0 that is $\mathcal{F}_0$-measurable. Clearly, $\frac{h}{S_T^0} \nsubseteq \mathcal{G}^*$. Therefore, $\mathcal{G}^*$ is a strict subset of $\mathbb{R}^\Omega$. Suppose $\mathcal{Q}_1$ is an EMM and define the inner product $< X, Y >= \mathbb{E}^{\mathcal{Q}_1}[XY]$ on $\mathbb{R}^\Omega \times \mathbb{R}^\Omega$. There exists a random variable $X(\omega)$ that belongs to $\mathcal{G}^{*\perp}$ such that $\mathbb{E}^{\mathcal{Q}_1}[XY] = 0 \ \forall Y \in \mathcal{G}^*$.

- Let us define $\mathcal{Q}_2(\omega) = \mathcal{Q}_1(\omega) * (1 + \frac{X(\omega)}{2\sup_\omega |X(\omega)|})$. note that since $1 \in \mathcal{G}^*$ (there exists a numeraire!), $\mathbb{E}^{\mathcal{Q}_1}[X] = 0$. Therefore $\sum_\omega \mathcal{Q}_2(\omega) = \sum_\omega \mathcal{Q}_1(\omega) + \mathbb{E}^{\mathcal{Q}_1}[\frac{X(\omega)}{2\sup_\omega |X(\omega)|}] = 1$. Also, clearly $\frac{\mathcal{Q}_2(\omega)}{\mathcal{Q}_1(\omega)} > 0$ and thus $\mathcal{Q}_2 \sim \mathcal{Q}_1$. Finally, we note that
$$\mathbb{E}^{\mathcal{Q}_2}[\sum_{t=0}^{T-1} \Delta_t dS_t^*] = \mathbb{E}^{\mathcal{Q}_1}[\sum_{t=0}^{T-1} \Delta_t dS_t^*] = 0$$
Therefore, $\mathcal{Q}_2$ is an EMM distinct from $\mathcal{Q}_1$.

Pricing and hedging in Complete Markets

- If markets are complete, then for any CC with $\mathcal{F}_T$-measurable payoff h there exists a self-financing trading strategy Δ such that $V_T(\Delta) = h(\omega)\ \forall \omega$. Further, $\frac{V_t(\Delta)}{S_t^0} = \mathbb{E}_t^{\mathcal{Q}}[\frac{h}{S_T^0}]\ \forall t$. In particular, $V_t(\Delta)$ is the wealth needed at time t in order to replicate the final payoff of the CC. It is thus natural to define the price of the contingent claim $P_t(h) = V_t(\Delta) := S_t^0 \mathbb{E}_t^{\mathcal{Q}}[\frac{h}{S_T^0}]$. Note that it is a linear pricing rule.

- In general, it is difficult to identify the hedging strategy without specifying the model further. One good example is the binomial model of Cox, Ross, and Rubinstein (see exercise). In continuous time, the *Itô-Doeblin formula* and the *Martingale representation theorem* allow us to go further.
- An American CC specifies a sequence of random variables $(h_t)_{t=0,1,\ldots,T}$ adapted to $\mathcal{F}_t$ that represents the profit upon exercise to the holder of the CC. (for an American Call $h_t = |S_t - K|^+$).
- The buyer of the claim has to find the optimal exercise ("stopping") time τ so as to maximize the value of his claim.
- We motivate the optimal policy using a backward induction argument. Then introduce more formally *stopping times* and *the Snell envelope* of a process.

- Note that
  - at $t = T$: $V_T = h_T$.
  - at $t = T-1$: $V_{T-1} = \max\left[h_{T-1}, S^0_{T-1}\mathbb{E}^{\mathcal{Q}}_{T-1}[\frac{h}{S^0_T}]\right] = \max\left[h_{T-1}, S^0_{T-1}\mathbb{E}^{\mathcal{Q}}_{T-1}[\frac{V_T}{S^0_T}]\right]$
  - by induction at $V_t = \max\left[h_t, S^0_t\mathbb{E}^{\mathcal{Q}}_t[\frac{V_{t+1}}{S^0_{t+1}}]\right]$
- This construction suggests the following results:
  - The discounted value of the American option is a $\mathcal{Q}$-super martingale. Defining $V^*_t = \frac{V_t}{S^0_t}$, we have $V^*_t \geq \mathbb{E}^{\mathcal{Q}}_t[V^*_{t+1}]$.
  - Prior to early exercise, namely as long as $V^*_t > h^*_t$, the discounted value of the option is a $\mathcal{Q}$-martingale, i.e., $V^*_t = \mathbb{E}^{\mathcal{Q}}_t[V^*_{t+1}]$ for all (t, ω) such that $V^*_t(\omega) > h^*_t(\omega)$.
  - It is optimal to exercise the first time $V^*_t \leq h^*_t$.
- To prove these results more formally, we introduce the notion of *stopping time* and *Snell envelope*.

American Contingent claims and Early exercise

- A random variable τ with values in $\{0, 1, \ldots, T\}$ is a stopping time if for any t, $\{\tau \leq t\} \in \mathcal{F}_t$.
- Consider a process X_t adapted to $\mathcal{F}_t$. The stopped process X_t^τ is defined by:

$$X_t^\tau = \begin{cases} X_t & \text{if} \quad t \leq \tau \\ X_\tau & \text{if} \quad t > \tau \end{cases}$$

  Note that we can write $X_t^\tau = X_0 + \sum_{n=0}^{t-1}(1 - \mathbf{1}_{\{\tau \leq n\}})dX_n$.
- If X_t is a martingale (resp. super-martingale) and τ a stopping time then X_t^τ is a martingale (resp. super-martingale).
- The *Snell envelope* of an adapted process (Z_t) is the adapted process U_t defined recursively by

$$\begin{cases} U_T &= Z_T \\ U_t &= \max(Z_t, \mathbb{E}_t[U_{t+1}]) \ \forall t < T \end{cases}$$

- The snell envelope of Z_t is the smallest super-martingale that dominates the process (Z_t).
  Indeed, consider M_t another supermartingale that dominates Z_t. Then $M_T \geq Z_T = U_T$. But if $M_{t+1} \geq U_{t+1}$ then $M_t \geq \mathbb{E}_t[M_{t+1}] \geq \mathbb{E}_t[U_{t+1}]$ and thus $M_t \geq \max[Z_t, \mathbb{E}_t[U_{t+1}]] = U_t$.
- If we define the stopping time $\tau_0 = \inf\{t \geq 0 \,:\, U_t = Z_t\}$ then $U_t^{\tau_0}$ is a martingale. *Note that $dU_t^{\tau_0} = (1 - \mathbf{1}_{\{\tau_0 \leq t\}})dU_t$. And on the set $\{\omega \,:\, \mathbf{1}_{\{\tau_0 \leq t\}} = 0\}$ we have $U_t > Z_t$ and $U_t = \mathbb{E}_t[U_{t+1}]$. So $\mathbb{E}_t[dU_t^{\tau_0}] = 0$.*
- Denote by $\mathcal{T}_{t,T}$ the set of stopping times taking values in $\{t, t+1, \ldots, T\}$. Then we have:

$$U_0 = \mathbb{E}[Z_{\tau_0}] = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[Z_\tau]$$

  For any stopping time, the supermartingale property implies $U_0^\tau \geq \mathbb{E}[U_T^\tau] = \mathbb{E}[U_\tau] \geq \mathbb{E}[Z_\tau]$. For the specific stopping rule τ_0 the martingale property implies $U_0^{\tau_0} = \mathbb{E}[U_T^{\tau_0}] = \mathbb{E}[U_{\tau_0}] = \mathbb{E}[Z_{\tau_0}]$.

- We obtain the general characterization of the optimal stopping rule. A stopping time is optimal if and only if $U_\tau = Z_\tau$ and U_t^τ is a martingale.
- Every supermartingale has a unique (Doob-Meyer) decomposition: $U_t = M_t - A_t$ where M_t is a martingale and A_t is a non-decreasing, predictable process, null at 0. *Set* $U_0 = M_0$*. Taking the difference, we must have* $U_{t+1} - U_t = M_{t+1} - M_t - (A_{t+1} - A_t)$*. Taking expectation, we see that the predictable component is defined recursively via:* $A_{t+1} - A_t = U_t - \mathbb{E}_t[U_{t+1}] \geq 0$ *and the martingale residual:* $M_{t+1} - M_t = U_{t+1} - \mathbb{E}_t[U_{t+1}]$*.*

- The Doob-Meyer decomposition gives an idea of how to derive the replicating portfolio for an American contingent claim. Specifically, we define the discounted value of the American option value U_t^* as the Snell envelope of the process $Z_t = \frac{h_t}{S_t^0}$. From the results above, we have the following:

$$U_t^* = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathcal{Q}}[Z_\tau]$$

Also, $U_t^* = M_t - A_t$ for some $\mathcal{Q}$-martingale M_t and some increasing predictable process A_t. Since the market is complete, there exists a self-financing adapted strategy such that $V_T(\Delta) = S_T^0 M_T$. By definition of the risk-neutral measure we have

$$\begin{aligned} V_t^*(\Delta) &= E_t^{\mathcal{Q}}[V_T^*(\Delta)] \\ &= E_t^{\mathcal{Q}}[M_T] \\ &= M_t \end{aligned}$$

- Thus, $U_t^* = V_t^*(\Delta) - A_t$. What's the interpretation? $V_t(\Delta)$ is the value at any time of a self-financing trading strategy starting from $V_0(\Delta)$. Therefore, following the trading strategy Δ, one is guaranteed to have an amount of money greater or equal to the value of the American option U_t at all times. A_t represents the value of the optimal replicating strategy in excess of the value of the option. (Of course, if the option is optimally exercised, then $A_t = 0 \ \forall t$. So A_t can be interpreted as the gains from selling an option at the arbitrage-free price U_0, replicating it optimally using Δ, and benefitting from a suboptimal exercise policy.)

Pricing and hedging in Incomplete Markets

- Suppose the securities market with d securities $S_t = (S_t^0, \ldots, S_t^d)$ is incomplete. We consider the pricing of some CC with $\mathcal{F}_T$-measurable payoff h.
- Since markets are incomplete, there is a set of EMM $\mathbb{Q}$ under which discounted prices are martingales. It is thus natural to consider that the set of arbitrage-free prices for the CC is $\mathcal{A} = \{\mathbb{E}^{\mathcal{Q}}[\frac{h}{S_T^0}];\ \forall \mathcal{Q} \in \mathbb{Q}\}$.

Proof.

Consider any augmented market with $d+1$ securities $(S_t^0, \ldots, S_t^{d+1})$ where the added security satisfies $S_T^{d+1} = h$. Then, for this augmented market to be arbitrage-free, there must exist a set of EMM $\hat{\mathbb{Q}}$ such that $\forall \mathcal{Q} \in \hat{\mathbb{Q}}$ all discounted prices are martingales. In particular, $\forall \mathcal{Q} \in \hat{\mathbb{Q}}$ $S_t^{*d+1} = \mathbb{E}_t^{\mathcal{Q}}[S_T^{*d+1}] = \mathbb{E}_t^{\mathcal{Q}}[\frac{h}{S_T^0}]$.

Since we clearly have $\hat{\mathbb{Q}} \subset \mathbb{Q}$ (why?) this shows that the set of arbitrage-free prices is a subset of $\mathcal{A}$. For the converse inclusion, take some $Q \in \mathbb{Q}$ and define the price process $S_t^{d+1} = S_t^0 \mathbb{E}_t^{\mathcal{Q}}[\frac{h}{S_T^0}]$.

Then the market $(S_t^0, \ldots, S_t^{d+1})$ thus defined clearly is arbitrage-free (why?). Thus, $\mathcal{A}$ belongs to the set of arbitrage-free prices. □

- Since the set of equivalent martingale measure $\mathbb{Q}$ is a convex set we can characterize $\mathcal{A}$ as an interval. Let us define:

$$\hat{\pi}(h) = \sup_{Q \in \mathbb{Q}} \mathbb{E}^{\mathcal{Q}}[\frac{h}{S_T^0}]$$

$$\check{\pi}(h) = \inf_{Q \in \mathbb{Q}} \mathbb{E}^{\mathcal{Q}}[\frac{h}{S_T^0}]$$

- Then we have
  - If h is attainable then $\check{\pi}(h) = \hat{\pi}(h)$.
  - If h is not attainable then either $\mathcal{A} = \emptyset$ or $\check{\pi}(h) < \hat{\pi}(h)$ and $\mathcal{A} = (\check{\pi}(h), \hat{\pi}(h))$
- Note that $\hat{\pi}(h)$ is the smallest amount at which one can sell the claim and use the proceeds in a dynamic self-financing trading strategy so as to not lose money at maturity, i.e.:

$$\hat{\pi}(h) = \inf\{V_0 \; : \; V_0 + \sum_t \Delta_t dS_t \geq h\}$$

  This is called the *super-replication* cost of the CC.
- Similarly, $\check{\pi}(h)$ is the largest amount that one can afford to pay for the CC while engaging in a dynamic self-financing trading strategy and not losing money at maturity, i.e.:

$$\check{\pi}(h) = \sup\{V_0 \; : \; V_0 + \sum_t \Delta_t dS_t \leq h\}$$

  This is called the *sub-replication* cost of the CC.
- Why is the interval $\mathcal{A}$ an open interval when the claim is not attainable?