Asset Pricing V

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Readings:

- Rubinstein
- Leroy & Werner chap. 15,16

Topics:

- Mean-variance efficiency
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Hilbert Space refresher

- A Hilbert space is a vector space endowed with an innerproduct $< x, y >: H \to \mathbb{R}$, which satisfies for $x, y \in H$:
  - $< x, y >=< y, x >$
  - $< x, x >> 0$
  - $< ax + bz, y >= a < x, y > +b < z, y >$
- The inner product defines a norm $||x|| = \sqrt{< x, x >}$.
- Orthogonality between to vectors x and y is defined by $< x, y >= 0$.
- For any finite-dimensional subspace Z of H and any vector $x \in H$, there exists a unique vector $x_Z \in Z$ and $y \in Z^{\perp}$ such that $x = x_Z + y$. The vector x_Z is the orthogonal projection of x onto the subspace Z. Note that it satisfies $||x - x_Z|| \leq ||x - z|| \;\; \forall z \in Z$.
- Riesz Representation theorem:
  For any continuous linear functional $L(\cdot) : H \to \mathbb{R}$ there exists a unique vector $k_L \in H$ such that $L(x) =< k_L, x > \;\; \forall x \in H$.

- Below we will consider the Hilbert space of payoffs $\mathbb{R}^S$ endowed with the expectations inner product $< x, y >= \mathbb{E}[xy]$.
- Two linear functionals defined on the asset span $\mathcal{M}$ (which may be a subspace of $\mathbb{R}^S$, but since it is a complete subspace, it also qualifies as a Hilbert space) whose kernels are of particular importance are:
    - The pricing kernel k_q, which satisfies $\mathbb{E}[k_q x] = q(x)\ \forall x \in \mathcal{M}$
    - The expectation kernel k_e, which satifies $\mathbb{E}[k_e x] = \mathbb{E}[x]\ \forall x \in \mathcal{M}$.
- Note that if the risk-free asset exists, then $k_e = 1$. Else k_e is the projection of 1 onto the asset span.
- If markets are complete then $k_q = M$, the unique state price density. Else k_q is the projection of M onto the asset span.

Mean Variance Preferences

- A payoff x is mean-variance efficient if there is no other payoff y with the same price and mean but a lower variance.

Theorem

A payoff is mean-variance efficient if it lies in the span of k_e and k_q.

- Note that if you project any payoff onto the span of k_e, k_q, we have $x = ak_e + bk_q + \epsilon$, with ϵ orthogonal to both k_e, k_q. Using the definition of the two kernels implies that ϵ has both zero price and zero expectation. Since x has a larger variance than $ak_e + bk_q$, the latter has to be on the mean-variance frontier.

- We assume for the following that k_e and k_q are not colinear (else all portfolios have the same expected return, which equals the risk-free rate if the risk-free payoff is in $\mathcal{M}$).
- If there are only two non-redundant securities, then the entire asset span is mean-variance efficient.

- Any mv efficient return (which is an mv efficient payoff divided by its price) can be written as

$$
\begin{aligned}
R &= \frac{payoff}{price} \\
&= \frac{ak_e + bk_q}{q(ak_e + bk_q)} = \frac{k_e}{q(k_e)} + \frac{ak_e + bk_q}{aq(k_e) + bq(k_q)} - \frac{k_e}{q(k_e)} \\
&= \underbrace{\frac{k_e}{q(k_e)}}_{R_e} + \frac{ak_e q(k_e) + bk_q q(k_e) - k_e(aq(k_e) + bq(k_q))}{aq(k_e) + bq(k_q)} \\
&= R_e + \frac{bq(k_e)q(k_q)}{aq(k_e) + bq(k_q)}\left(\frac{k_q}{q(k_q)} - \frac{k_e}{q(k_e)}\right) \\
&= R_e + \lambda(R_q - R_e) = R_\lambda
\end{aligned}
\quad (1)
$$

for some unique parameter λ, where $R_e = \frac{k_e}{q(k_e)}$ and $R_q = \frac{k_q}{q(k_q)}$.

- Thus the mv frontier is a **parabola**. Each point on that parabola is characterized by:
$\mathbb{E}[R_\lambda] = \mathbb{E}[R_e] + \lambda(\mathbb{E}[R_q] - \mathbb{E}[R_e])$ and
$\mathbb{V}[R_\lambda] = \mathbb{V}[R_e] + \lambda^2(\mathbb{V}[R_q - R_e]) + 2\lambda\mathbb{C}\mathrm{ov}(R_e, R_q - R_e)$
- If a risk-free rate exists, then $R_e = R_f$ is constant, and the above simplifies.
- Note that $q(k_e) = \mathbb{E}[k_q k_e] = \mathbb{E}[k_q]$. Thus, $\mathbb{E}[R_e] = \frac{E[k_e]}{\mathbb{E}[k_q]}$ which equals $1/R_f$ if a risk-free rate exists. Further, since $q(k_q) = \mathbb{E}[k_q^2] = Var[k_q] + \mathbb{E}[k_q]^2 > \mathbb{E}[k_q]^2$, we have $\mathbb{E}[R_q] = \frac{E[k_q]}{E[k_q^2]} < \frac{1}{\mathbb{E}[k_q]}$, and thus $\mathbb{E}[R_q] < R_f$ if the risk-free rate exists. This shows that R_q lies on the 'inefficient' part of the frontier.
- There always exists a minimum variance portfolio, R_{λ_0}. It satisfies $\partial_\lambda V(R_\lambda) = 0$. We find $\lambda_0 = -\frac{\mathbb{C}\mathrm{ov}(R_e, R_q - R_e)}{\mathbb{V}(R_q - R_e)}$. If there exists a risk-free rate, we have $R_e = const$, and hence $\lambda_0 = 0$ and $R_{\lambda_0} = R_e = R_f$.

- Take any efficient frontier portfolio R_λ with $\lambda \neq \lambda_0$, then one can always find another frontier portfolio R_μ, such that $\mathbb{C}\text{ov}(R_\lambda, R_\mu) = 0$. (simply solve $\mathbb{C}\text{ov}(R_e + \lambda(R_q - R_e), R_e + \mu(R_q - R_e)) = 0$ for μ to get

$$\mu = -\frac{\mathbb{C}\text{ov}(R_e + \lambda(R_q - R_e), R_e)}{\mathbb{C}\text{ov}(R_e + \lambda(R_q - R_e), R_q - R_e)}$$

  Note that the denominator is zero for $\lambda = \lambda_0$. Hence, we need the $\lambda \neq \lambda_0$ condition.)
- Consider any payoff $x_j \in \mathcal{M}$. By projection, we have $x_j = a_j k_e + b_j k_q + \epsilon_j$ where ϵ_j has zero expectation and zero price. Therefore, dividing by $q(x_j) = q(a_j k_e + b_j k_q)$, we find that any traded security return can be written as:

$$R_j = R_\mu + \beta_j(R_\lambda - R_\mu) + \tilde{\epsilon}_j$$

  for some constant β_j. Indeed, since $R_\mu,\ R_\lambda$ form the linear span of $\{k_e, k_q\}$.

Further, taking $\lambda \neq \lambda_0$ and R_μ the zero covariance frontier return we have:

$$\mathbb{C}\mathrm{ov}(R_j, R_\lambda) \;=\; \mathbb{C}\mathrm{ov}(R_\mu + \beta_j(R_\lambda - R_\mu) + \tilde{\epsilon}_j, R_\lambda) = \beta_j \mathbb{V}(R_\lambda)$$

so that $\beta_j = \frac{\mathbb{C}\mathrm{ov}(R_j, R_\lambda)}{\mathbb{V}(R_\lambda)}$.

- Note the implication. As long as second moments are well-defined, there always exists a one-factor beta asset pricing model where the expected return on any security in excess of some benchmark frontier return (R_μ) lines up with the beta of that security with another reference frontier excess return ($R_\lambda - R_\mu$).
- The CAPM obtains if the market portfolio lies on the efficient frontier (and is different from the minimum variance portfolio).

- This leads to the Roll (1977) 'critique(s).' Any test of the CAPM boils down to a test of the mean-variance efficiency of the market portfolio. This makes the test difficult because (a) the true market portfolio is not observable (human capital, real estate, private equity, non-traded assets, . . .), (b) in a finite sample, one can always find an ex-post mean-variance efficient portfolio, which is used as a proxy for the market portfolio would satisfy the CAPM by construction.

Hansen-Jagannathan Bounds

- Note that absence of arbitrage implies there exists a state price density M such that
  - $P_i = \mathbb{E}[MX_i]$.
  - $1 = \mathbb{E}[MR_i]$.
  - $0 = \mathbb{E}[M(R_i - R_j)]$.
- In a representative agent setting we have $M = \frac{\partial u(c_0, c_s)/\partial c_s}{\partial u(c_0, c_s)/\partial c_0}$.
- Using the definition of covariance, we obtain $\mathbb{E}[R_i - R_j] = -\frac{1}{\mathbb{E}[M]}\mathbb{C}\text{ov}(M, R_i - R_j)$

- Now since $\mathbb{C}\mathrm{ov}(M, R_i - R_j) = \rho\sigma_M\sigma_{i-j}$ with $|\rho| \leq 1$ we obtain:
$$\sup_{i,j} |\frac{\mathbb{E}[R_i - R_j]}{\sigma_{i-j}}| \leq \frac{\sigma_M}{\mathbb{E}[M]}$$
In other words, the maximum Sharpe ratio of any zero-cost portfolio that one can find in the economy provides a lower bound to the volatility of the state price density normalized by its mean (recall that if there exists a risk-free rate $\frac{1}{\mathbb{E}[M]} = R_f = 1 + r_f$). This is known as the Hansen Jagannathan bound. The literature on 'asset pricing anomalies' has pushed this bound to very large numbers, which is a challenge for the standard representative agent consumption-based model.

- We can always project M onto the market span, and therefore the absence of arbitrage also implies $0 = \mathbb{E}[R_q(R_i - R_j)]$. It follows that $\mathbb{E}[R_i - R_j] = -\frac{1}{\mathbb{E}[R_q]}\mathbb{C}\mathrm{ov}(R_q, R_i - R_j)$. If we pick $R_j = R_0$ (the zero covariance return for the pricing kernel), then we get the beta pricing model $\mathbb{E}[R_i - R_0] = -\frac{1}{\mathbb{E}[R_q]}\mathbb{C}\mathrm{ov}(R_q, R_i)$. Since it holds for R_q, we also find:

$$\mathbb{E}[R_q - R_0] = -\frac{1}{\mathbb{E}[R_q]}\mathbb{V}(R_q).$$

  Dividing both we obtain:

$$\mathbb{E}[R_i - R_0] = \beta_i(\mathbb{E}[R_q] - \mathbb{E}[R_0])$$

  So, the absence of arbitrage alone guarantees that a one-factor beta pricing model exists where the sole priced factor is a traded return (R_q).

- We note that the Hansen-Jagannathan bound also applies to the pricing kernel (the projection of M onto $\mathcal{M}$). Thus we have

$$\sup_{i,j} \left| \frac{\mathbb{E}[R_i - R_j]}{\sigma_{i-j}} \right| \leq \frac{\sigma_{R_q}}{\mathbb{E}[R_q]}$$

- Now note that above, we showed that

$$\frac{\sigma_{R_q}}{\mathbb{E}[R_q]} = -\frac{\mathbb{E}[R_q - R_0]}{\sigma_{R_q}}$$

- This shows that a specific trading strategy (Short R_q long R_0) will attain the Hansen-Jagannathan bound for the state price density with the lowest variance, which is the projection of M onto the asset span if a risk-free rate is traded. Indeed, in that case, $R_0 = R_f$ and we have that $\frac{\mathbb{E}[R_q - R_0]}{\sigma_{q-0}} = \frac{\mathbb{E}[R_q - R_f]}{\sigma_{R_q}}$. If the risk-free rate is not traded, then the Sharpe ratio on $R_q - R_0$ is $\left|\frac{\mathbb{E}[R_q - R_0]}{\sigma(R_q - R_0)}\right| = \left|\frac{\mathbb{E}[R_q - R_0]}{\sqrt{\sigma_q^2 + \sigma_0^2}}\right| < \left|\frac{\mathbb{E}[R_q - R_0]}{\sigma_q}\right| = \frac{\sigma_{R_q}}{\mathbb{E}[R_q]}$ which shows that the HJ-bound is not attained by the Sharpe ratio on $R_q - R_0$ (or indeed by any zero-cost portfolio's Sharpe ratio, why?) in the absence of a risk-free rate.

Capital Asset Pricing Model

- The CAPM is obtained when the market portfolio lies on the efficient frontier.
- The market portfolio is the portfolio that holds all traded securities in proportion to their relative market capitalizations.
- Define R_m to be the return on the market portfolio. If it is on the efficient frontier (and different from the minimum variance portfolio), then, for any traded security return, we obtain:

$$\mathbb{E}[R_j] = R_z + \beta_j(\mathbb{E}[R_m - R_z]),$$

  where $\beta_j = \frac{\mathbb{C}\text{ov}(R_j, R_m)}{\mathbb{V}(R_m)}$ and R_z is the zero beta asset expected return (which is the risk-free rate if it exists).
- The Zero Beta Rate is different from the risk-free rate!

- Sufficient conditions for the market portfolio to be efficient are that all investors choose a mean-variance frontier portfolio and that at least one investor invests in a portfolio on the efficient part of the frontier and different from the minimum variance portfolio. Then, since the market portfolio is a convex combination of all the portfolios held by all investors, it will be an efficient portfolio.

- Sufficient conditions for investors to hold an mv frontier portfolio is that
    - they have no end-of-period endowment (or that the latter be in the market span) and that they have quadratic preferences.
    - they have a strictly concave utility function, and all endowments and returns are jointly normally distributed.
- As an exercise, you can also prove by aggregating the first-order conditions that the CAPM holds for these two cases. For the second case, use the following result, known as (Rubin)Stein's lemma: for any continuous and differentiable function $f(\cdot)$ and jointly normally distributed random variables X, Y, the following holds $\mathbb{C}\text{ov}(f(X), Y) = \mathbb{E}[f'(X)]\mathbb{C}\text{ov}(X, Y)$.

Factor Models and Arbitrage Pricing Theory

- Suppose there exists K factors $f_1, \dots, f_K$ with $K < n$ the number of securities.
- Without loss of generality assume $\mathbb{E}[f_i] = 0$ and $\mathbb{E}[f_i f_j] = 0$.
- We talk about K-factor Beta pricing if $E[R_i] = \gamma_0 + \sum_k \beta_{i,k}\gamma_k$ where $\beta_{i,k}$ is the regression coefficient of R_j onto f_k and γ_k is the factor premium.

  γ_k can be interpreted as a risk-premium if f_k is a security return.

  γ_0 is the expected return on an asset with zero factor beta (i.e., the risk-free rate if it exists).
- Note that since covariance is linear, if a K-factor beta pricing model exists, then there must also exist a 1-factor beta pricing model.

- Of course, we know that any efficient frontier return and the corresponding zero-beta return can be used to obtain a 1-factor pricing model.
- We obtain the result:

Theorem

K-factor beta pricing obtains if there exists a state price density M that is affine in the factors and $\gamma_0 = \frac{1}{\mathbb{E}[M]}$ *and* $\gamma_k = -\frac{\mathbb{E}[Mf_k]}{\mathbb{E}[M]}$

- In particular, if k_q lies in the factor span, then K-factor beta pricing is obtained.
- Note that if K-factor beta pricing holds then $M = \frac{1}{\gamma_0} - \sum_k \frac{\gamma_k}{\sigma^2_{f_k}\gamma_0} f_k$ is a valid State price density (i.e., $\mathbb{E}[R_i M] = 1$). If f has unbounded support, then M will take on negative values. How can we reconcile this with the absence of arbitrage?
- Note that one can always project returns onto the factor span to obtain: $R_i = \mathbb{E}[R_i] + \sum_k \beta_{ik} f_k + \epsilon_i$ with $\mathbb{E}[\epsilon_i] = \mathbb{E}[\epsilon_i f_k] = 0$.

- Let's define the **pricing error** for security i as

$$\psi_i = \mathbb{E}[R_i] - \gamma_0 - \sum_k \beta_{ki}\gamma_k$$

for the minimum variance pricing kernel (i.e., for $M = k_q$ and thus $\gamma_0 = \frac{1}{\mathbb{E}[k_q]}$ and $\gamma_k = -\frac{\mathbb{E}[k_q f_k]}{\mathbb{E}[k_q]}$).

- Then we have the following bound on the pricing error:

$$|\psi_i| \leq \sigma(\epsilon_i)\frac{1}{\mathbb{E}[k_q]}||k_q - k_q^{\mathcal{F}}||$$

where $k_q^{\mathcal{F}}$ is the projection of k_q onto the factor span.

- To develop the intuition further, suppose that there is an exact factor structure, i.e., $\epsilon_i = 0 \, \forall i$. Then consider any portfolio $\theta' R = \theta' \mathbb{E}[R] + \theta' \beta f$. For any θ such that $\theta' \beta = 0$ absence of arbitrage requires that $\theta' \mathbb{E}[R] = R_f$. This in turns implies that $\mathbb{E}[R] - R_f \mathbf{1} = \beta \gamma$ for some k dimensional vector. Note that we could also have proved that k-factor beta pricing holds simply by using the definition $\mathbb{E}[k_q R_i] = 1 \ \forall i$.

- The Arbitrage Pricing Theory (APT) of Ross starts from an assumption about the factor structure of returns. Namely, it assumes that residuals are uncorrelated $\mathbb{E}[\epsilon_i \epsilon_j] = 0$ in the decomposition $R_i = \mathbb{E}[R_i] + \sum_k \beta_{ik} f_k + \epsilon_i$, to derive limiting results about the pricing errors as the number of securities grows to infinity.
- The intuition is that in a large diversified portfolio, the variance will depend only on factor risk and not on diversifiable risk ($\mathbb{C}\mathrm{ov}(\sum_i \pi_i R_i, \sum_j \pi_j R_j) = factorrisk + \sum_i \pi_i^2 \mathbb{V}(\epsilon_i)$ with the second component going to zero as $n \to \infty$ and $\pi_i \approx 1/n$). It is natural then to think that the pricing kernel will depend only on factor risk in that setting and thus that pricing errors will become small. The APT theory tries to formalize that intuition. Most APT theorems prove that as the number of securities grows 'large,' only a 'small' number of pricing errors can be large and so K-factor beta pricing must hold for all but a 'small' number of securities. The difficulty lies in making these statements precise.

- We have the following result:

Theorem

If returns have a factor structure then

$$\sum_i \psi_i^2 \leq \max_j \sigma(\epsilon_j)^2 \frac{1}{\mathbb{E}[k_q]^2} ||k_q - k_q^{\mathcal{F}}||^2$$

- The upper bound does not depend on n the number of securities (assuming that $\sigma(\epsilon_j)^2 < \overline{\sigma} \ \ \forall j$).
  It follows that as the number of securities goes to infinity, only a 'small' number of them can be mispriced by the K-factor beta pricing model. Further, as expected, when the pricing kernel is 'close' to the K-factor span, the pricing errors are small.