Asset Pricing IV

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Readings:

- Rubinstein
- Leroy & Werner chap. 15,16

Topics:

- Pareto Efficiency and Optimality
- Representative Agent
- First and second Welfare theorem with complete markets
- Aggregation with Linear Risk Tolerance
- Constrained optimality
- Effectively complete markets

Pareto Efficiency

- An allocation $\{c^i\}$ Pareto dominates another $\{\tilde{c}^i\}$ if $u_i(c^i) \geq u_i(\tilde{c}^i)\ \forall i$ where the inequality is strict for at least one agent.
- A feasible consumption allocation (i.e., such that $\sum_i c^i \leq \sum_i \omega^i$) is Pareto optimal if there is no other feasible allocation that Pareto dominates it.

Theorem

An allocation is Pareto optimal if and only if it is the solution of the optimization problem of a social planner with social welfare function given by $U_\mu(\Omega) = \max_{c^i} \sum_i \mu_i u_i(c^i)$ *subject to* $\sum_i c^i \leq \sum_i \omega^i \equiv \Omega$ *for some vector of positive weights* μ*.*

- If the set of feasible allocations is compact and the utility functions are continuous, then the social planner's problem admits a solution.
- The first order conditions of the planner's problem imply that:
  - Marginal rates of substitution across states are equal for all agents.
  - The ratio of marginal utilities of consumption for two agents is independent of the state.

- The central planner's problem implies that in a Pareto Optimal allocation, individual agents' optimal consumptions are given by *sharing rules*:

$$c_s^{i*} = f_s^i(\Omega, \mu)$$

- If the agents have time-separable expected utility (i.e., they maximize $\sum_s \pi_s u_i(c_s^i)$ then so does the social planner: $U_\mu(\Omega) = \sum_s \pi_s u_\mu(\Omega_s)$ where $U_\mu(\Omega) = \max_{c^i} \{\sum \mu_i u_i(c^i)\, s.t. \sum_i c^i = \Omega\}$.

- When agents have time-separable expected utility:
  - Sharing rules (individual's optimal consumption) **only** depend on aggregate consumption in that state and the vector of central planner's weights:

$$c_s^{i*} = f^i(\Omega_s, \mu)$$

  - Sharing rules are increasing in aggregate consumption when agents are risk-averse,
  - Sharing rules are co-monotone: an agent has higher consumption in a state if and only if all other agents have higher consumption in that state.
  - Sharing rules are linear if and only if agents have linear risk tolerance.

Complete markets

- A security market equilibrium is a set of prices p and dividends X such that all I agents (a) solve their individual optimal consumption investment problem, (b) financial markets clear, and (c) consumption markets clear:
  (a) $\max_{\theta^i, c_s^i} u^i(c^i)$ s.t. $c_0 + \theta' p \leq \omega_0^i$ and $c_1^i \leq \omega_1^i + \theta' X$ and $c_0^i, c_1^i \geq 0$.
  (b) $\sum_i \theta^i = 0$.
  (c) $\sum_i c_s^i = \sum_i \omega_s^i$.
- An Arrow Debreu equilibrium is a set of state prices q (which corresponds to the identity payoff matrix), such that all agents (a') solve their optimal consumption allocation problem, and (b') consumption markets clear.
  (a') $\max_{c_s^i} u^i(c^i)$ s.t. $c_0^i + c_1^i q \leq \omega_0^i + \omega_1^i q$ and $c_0, c_1 \geq 0$.
  (b') $\sum_i c_s^i = \sum_i \omega_s^i$.

- The two equilibrium concepts are equivalent in complete markets :

Theorem

If markets are complete (and $u_i' > 0$), then there is a security markets equilibrium if and only if there is an AD equilibrium.

- If markets are complete (and $u_i' > 0$) the first welfare theorem holds:

Theorem

If security markets are complete and $u_i' > 0 \;\; \forall i$ then every equilibrium consumption allocation is Pareto Optimal

- We also have the second welfare theorem (decentralization):

Theorem

If security markets are complete and $u_i' > 0, u_i'' < 0 \;\; \forall i$, then every Pareto optimal allocation is an equilibrium allocation for some distribution of the aggregate endowment.

- In fact, in complete markets, we can construct the equilibrium allocation by solving the no-trade equilibrium for a representative agent endowed with the aggregate endowment.

Theorem

If markets are complete and $u_i' > 0, u'' < 0 \ \forall i$*, then for any Arrow Debreu equilibrium (characterized by* $(c^1, \ldots, c^I, q)$*) there exists a vector* $\mu \in \mathbb{R}^I_+$ *such that* (Ω, q) *is a no trade AD equilibrium for the single representative agent* $U_\mu(\Omega)$ *defined above. Further* $(c^1, \ldots, c^I)$ *solves the planner's problem:*
$U_\mu(\Omega) = \max \sum_i \mu_i u(c^i) \ \ s.t. \ \sum_i c^i = \Omega.$

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- An immediate corollary is that if U_μ is differentiable, then we can use as state prices supporting the equilibrium

$$q_s = \frac{\partial_s U_\mu(\Omega_s)}{\partial_0 U_\mu(\Omega_0)}.$$

- Further, if agents have time-separable expected utility with the objective probability of states $\pi_1, \ldots, \pi_S$ so that $u^i(c_0, c_1, \ldots, c_S) = u^i_0(c_0) + \delta \sum_{s=1}^{S} \pi_s u^i_1(c_s)$, then the representative agent's utility also has that form and we obtain

$$q_s = \frac{\pi_s \delta u'_\mu(\Omega_s)}{u'_\mu(\Omega_0)} = \frac{\pi_s \delta u'_i(c^i_s)}{u'_i(c^i_0)} \;\; \forall i.$$

- using the definition of the pricing kernel $q_s = \pi_s M_s$, we see that M_s is simply the ratio of marginal utilities in each state.

$$M_s = \frac{\delta u'_\mu(\Omega_s)}{u'_\mu(\Omega_0)} = \frac{\delta u'_i(c^i_s)}{u'_i(c^i_0)} \;\; \forall i.$$

Pareto optimal allocation

- If agents are risk-averse and maximize expected utility, then we have the following

Theorem

If agents are risk-averse, then in any Pareto-optimal allocation, sharing rules are increasing in aggregate consumption, i.e., each individual's consumption is an increasing function of the aggregate endowment in each state.

- If agents have linear risk-tolerance, then we can characterize explicitly the form of the sharing rules:

Theorem

Consumption-sharing rules are linear in aggregate endowment if agents have linear risk tolerance with a common slope.

This implies that when agents have linear risk tolerance, any optimal consumption allocation lies in the span of the risk-free asset and a claim to aggregate consumption. In turn, this implies that any optimal allocation can be achieved by a security market equilibrium with only two securities: the risk-free asset and a claim to aggregate consumption. This is called **two-fund separation**. If these two securities are available, then the security market equilibrium will be equivalent to a complete market equilibrium, even if markets are incomplete.

This is an example of *effectively complete markets*.

Equilibrium in incomplete markets: Constrained Optimality

- If markets are incomplete, then the first and second welfare theorem do not hold in general. In incomplete markets, agent IMRS need not be equalized (and, in fact, in general, will differ).
- However, we can define the concept of constrained optimality if we restrict attention to those consumption allocations that are attainable by trading in marketed securities.
- A feasible consumption allocation is constrained optimal if it is attainable by trading in security markets and there does not exist any other feasible allocation, also attainable through security markets, that Pareto dominates it.

Theorem

If $u_i' > 0$ $\forall i$ then every security market equilibrium is constrained optimal.

- Markets are effectively complete if every Pareto optimal allocation can be obtained by trading in security markets.

Theorem

If security markets are effectively complete, and if for every feasible allocation, there exists a Pareto-optimal allocation that weakly Pareto dominates that allocation, then every constrained optimal allocation is Pareto optimal.

- It follows that, under these same conditions, every equilibrium allocation in effectively complete markets is Pareto optimal.
- Further, in effectively complete markets, any complete market equilibrium will also be a security market equilibrium.
- Conversely, if each agent chooses an interior solution in a security market equilibrium, then it also corresponds to a complete market equilibrium.
- We saw one example of effectively complete markets (agents have linear risk tolerances and the risk-free rate and the claim to aggregate consumption are traded).
- See Leroy Werner Chapter 16 for further discussion and examples of effectively complete markets.

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Inelastic Markets I

IN SEARCH OF THE ORIGINS OF FINANCIAL FLUCTUATIONS: THE INELASTIC MARKETS HYPOTHESIS

In this paper, the authors suggest that, in reality, investors are constrained in their ability to express views. Consider a simple demand curve

$$D(p) = \gamma^{-1}(\underbrace{E[d]}_{\textit{views}} - p) \tag{1}$$

and market clearing is

$$D(p) = \underbrace{S}_{\textit{supply}} \tag{2}$$

so that

$$p = E[d] - \gamma S \tag{3}$$

Thus,

- Prices move one-to-one with expectations

Inelastic Markets II

- when risk aversion γ is large, prices are extremely sensitive to supply shocks.
- In reality, this does not happen in real data. One story is that $E[d]$ is, in fact,

$$\alpha\bar{d} \ + \ (1-\alpha)E[d]\,, \tag{4}$$

where α is close to one.

But, here is another possibility. Consider an agent (a fund) who has Assets Under Management (AUM) W_t. He splits it into

$$W_t = \underbrace{x_t p_t}_{\textit{stock investment}} + \underbrace{b_t}_{\textit{cash}} \tag{5}$$

Suppose that the agent follows a 60/40 rule: with $\alpha = 0.6$, the agent always holds exactly αW_t in stocks:

$$x_t p_t = \alpha W_t = \alpha(x_t p_t + b_t), \tag{6}$$

so that

$$x_t = \frac{1}{p_t} \underbrace{\frac{\alpha}{1-\alpha}}_{\textit{multiplier}} b_t \,. \tag{7}$$

Now, suppose this is the representative stock investor who *has to hold the market:* normalizing supply to 1, we get

$$p = \frac{\alpha}{1-\alpha} b_t \tag{8}$$

Flows affect cash b_t. If quantity x_t cannot adjust, prices need to adjust without any capital being moved: Markets are extremely inelastic.