Asset Pricing III

Semyon Malamud

EPFL

October 10, 2022

Table of Contents

Readings:

- Rubinstein
- Leroy & Werner chap. 10,11,12,13

Topics:

- Mean-preserving spreads
- First Order Stochastic Dominance
- Second Order Stochastic Dominance
- Comparative statics on Optimal Portfolios

First Order Stochastic Dominance

- A consumption plan C_a **First order stochastically dominates** C_b iff $\mathbb{E}[u(C_a)] \geq \mathbb{E}[u(C_b)] \; \forall \; u' \geq 0$
- We have the following theorem

Theorem

C_a FSD C_b iff $F_a(x) \leq F_b(x) \; \forall x$ where $F_a(x) = Prob(C_a \leq x)$

- Further we have the second characterization of FSD:

Theorem

C_a FSD C_b iff $C_a \sim^d C_b + \epsilon$ with $\epsilon \geq 0$.

Second Order Stochastic Dominance

- A consumption plan C_a **Second order stochastically dominates** C_b iff $\mathbb{E}[u(C_a)] \geq \mathbb{E}[u(C_b)] \ \forall \ u'' \leq 0$
- We have the following theorem

Theorem

C_a SSD C_b iff $\mathbb{E}[C_a] = \mathbb{E}[C_b]$ and $\int_0^x F_a(z)dz \leq \int_0^x F_b(z)dz \ \forall x$.

- Further we have the second characterization of SSD in terms of mean preserving spread:

Theorem

C_a SSD C_b iff $C_b \sim^d C_a + \epsilon$ with $\mathbb{E}[\epsilon \mid C_a] = 0$.

A few remarks

- Two random variables X, Y are mean-independent if $\mathbb{E}[Y \mid X] = E[Y]$. This is a stronger than if they are uncorrelated, but weaker than if they are independent.
  - Give an example of two random variables that are equal in distribution, but not equal in every state.
  - Give an example of two random variables that are mean-independent but not independent.
  - Give an example of two random variables that are mean-independent but not uncorrelated.
  - Give an example of two random variables that are uncorrelated but not independent.
- Note that if a consumption plan has greater variance it is not necessarily riskier in the sense of SSD. (give an example).
- However, if two consumption plans are normally distributed and have same mean, then lower variance implies SSD.
- If ϵ is mean independent of z and $\mathbb{E}[\epsilon] = 0$ then $z + \lambda\epsilon$ SSD $z + \gamma\epsilon$ for any $\gamma > \lambda$ constants.

Portfolio Choice

- Suppose an agents solve $\max_\theta \mathbb{E}[u(C_1)]$ subject to $C_1 = \theta' X$ and $\theta' p = \omega$.
- N.B.: (i) no period zero consumption, and (ii) either no second period endowment, or the second period endowment is in the market span.
- It is convenient to rewrite the problem in terms of the dollar amount invested in the ith risky security $a_i = \theta_i p_i$ and the gross return $R_i = \frac{X_i}{p_i}$.
- Further, we assume there exists a risk-free security with return $R_f = \frac{1}{p_0}$.
- Note that the problem can we rewritten as $\max_a \mathbb{E}[u(\omega R_f + \sum_{i=1}^n a_i(R_i - R_f))]$.
- The first order condition (assuming an interior solution) is:

$$\mathbb{E}\left[u'\left(\omega R_f + \sum_{i=1}^n a_i(R_i - R_f)\right)(R_i - R_f)\right] = 0$$

- We give a few results charact. the demand for risky assets.

One risky asset, one risk-free asset

- Suppose there is only one risky asset. Then we have

Theorem

$$a \gtreqless 0 \iff \mathbb{E}[R] \gtreqless R_f$$

- Suppose the agent is strictly risk-averse ($u'' < 0$). Define a^* as the optimal investment in the risky asset.

Theorem

If $0 \gtreqless \mathcal{A}'(w)$ *then* $\partial_\omega a^* \gtreqless 0$

If $0 \gtreqless \mathcal{R}'(w)$ *then* $\partial_\omega \frac{a^*}{\omega} \gtreqless 0$

Theorem

If $0 < \mathcal{A}'(w)$ *and* $w > a^* > 0$ *then* $\partial_{R_f} a^* < 0$.

Theorem

If $\mathcal{R}(w) < 1$ *and* $R \geq 0$ *then* $\partial_{R_f} a^* < 0$.

Theorem

If $\mathcal{R}(w) < 1$ and $\mathcal{R}'(w) > 0$ and $\mathcal{A}'(w) < 0$ then a^* decreases if R is replaced by a mean-preserving spread.

- With many risky assets, few results can be obtained without putting more structure on the problem.

Many risky assets, one risk-free asset

- If R^* is the return on the optimal portfolio of a risk-averse agent, and R^* is riskier than R in the sense that $R^* = R + c + \epsilon$ with ϵ a mean-preserving spread, then $\mathbb{E}[R^*] > \mathbb{E}[R]$.
- If $R^* = R_f$ then $\mathbb{E}[R_i] = R_f \ \forall i = 1, \ldots, n$.
- If $R^* > R_f$ then some $a_i > 0$ for some i.

- For special case of linear risk-tolerances we get:

Theorem

If an agent's risk-tolerance is linear $\mathcal{T}(w) = \frac{1}{\mathcal{A}(w)} = \alpha + \gamma w$*, then* $a^*(w) = (\alpha + \gamma w R_f)b$*, where* b *is a vector independent of wealth and* α*. This implies that* $\frac{a_i^*}{a_j^*} = \frac{b_i}{b_j}$ *independent of wealth.*

- Use power utility function $u(w) = \frac{(\alpha+\gamma w)^{1-\frac{1}{\gamma}}}{\gamma-1}$ which nests both CRRA and CARA as special cases.
- In particular, show that $\lim_{\gamma\to 1} \frac{(\alpha+\gamma w)^{1-\frac{1}{\gamma}}-1}{\gamma-1} = \log(\alpha + w)$, and $\lim_{\gamma\to 0} \frac{(1+\frac{\gamma}{\alpha} w)^{1-\frac{1}{\gamma}}}{\gamma-1} = -\exp(-\frac{w}{\alpha})$
- The linearity of the portfolio decision in wealth can be useful to derive equilibrium properties even when markets are incomplete.