EPFL & SFI

Asset Pricing Theory

Problem Set 7: Dynamic Portfolio Choice

1. Portfolio choice: Dynamic Programming

Consider a discrete-time economy where the dynamics of the risky asset and the risk-free asset is given by:

$$\begin{aligned} S_{t+1} &= e^{\mu+\sigma\tilde{w}_{t+1}} \cdot S_t \\ S^0_{t+1} &= e^{r_f} \cdot S^0_t \end{aligned}$$

where $\tilde{w}_t$ $\forall t = 0, 1, 2, 3\ldots$ are i.i.d. binomial random variables with 0.5 probability equal to $+1$ and 0.5 probability equal to -1.

Suppose an agent maximizes her expected utility of intertemporal consumption $E[\sum_{t=0}^{T} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$ by choosing in every period the optimal consumption $\{C_t\}_{t=0,1,2\ldots}$ and optimal fraction of remaining wealth to invest in the risky asset $\{\pi_t\}_{t=0,1,2,\ldots}$ (the fraction $1-\pi_t$ is invested in the risk-free asset) given an initial wealth level W_0.

1. Write the dynamics of wealth for the investor.
2. Setup the HJB equation and derive the first-order conditions for consumption and investment policy.
3. Guess that the optimal value function is of the form $J(t, W) = e^{-\rho t} A_t \frac{W^{1-\gamma}}{1-\gamma}$ and derive the deterministic function A_t. Find the optimal consumption and investment policy.
4. Use the solution you derived to compute the solution to the infinite horizon problem $E[\sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$. Find the optimal consumption and investment policy in that case. Give the conditions on the parameters (the so-called 'transversality condition') under which the infinite horizon problem has a finite value.
5. How does an increase in risk-aversion γ, in the expected return μ, and in the volatility σ affect the optimal investment in the risky security and the optimal consumption? (If you cannot do this analytically, then do it numerically for some choice of parameters).

2. Portfolio choice: Complete Markets

Consider the same economy as in Problem 1 above and consider the infinite horizon problem $\max E[\sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$.

1. Show that in this economy there exists a unique pricing kernel M_t with the property that $\frac{M_{t+1}}{M_t} = e^{m_{t+1}}$, where m_{t+1} is an i.i.d. binomial process you can characterize uniquely (not necessarily in closed-form) in terms of μ, σ, r_f. Conclude that markets are complete.
2. Show that any feasible consumption plan must satisfy the intertemporal budget constraint:
$$E[\sum_{t=0}^{\infty} M_t C_t] = M_0 W_0$$
3. Consider then the problem of maximizing $E[\sum_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{(1-\gamma)}]$, subject to the intertemporal budget constraint. Write the first-order condition for consumption, and show that the optimal consumption of the investor must satisfy:
$$\frac{C_{t+1}}{C_t} = e^{\frac{\rho}{\gamma} - \frac{1}{\gamma} m_{t+1}} \quad (\star)$$
.
4. Assume the agent consumes a constant fraction of wealth $C_t = a\, W_t$ in every period and invests a constant fraction π of the remaining wealth into the risky asset. Show that one can find two constants a^*, π^* so that $C_t = a^* W_t$ is budget feasible and satisfies the dynamics of optimal consumption identified in $(\star)$ above. Conclude that this is the optimal consumption and portfolio choice.