

Asset Pricing Theory

Problem Set 6: Dynamic Arbitrage Pricing

1. First Theorem of Asset Pricing

Consider a discrete-time economy, as considered in the lecture notes, with two risky assets, B_t and S_t . Assume that both are strictly positive at all times and that ownership of S_t at t entitles you to a dividend D_{t+1} paid at $t + 1$.

1. Consider the value of a self-financing portfolio $V_t = \Delta_t^0 B_t + \Delta_t^1 S_t$. Write down its dynamics.
2. Consider the value $V_t^* = \frac{V_t}{B_t}$ of the self-financing portfolio expressed in units of asset B_t , the numeraire asset. Write down its dynamics in terms of the discounted risky asset $S_t^* = S_t/B_t$ and the discounted dividend $D_t^* = D_t/B_t$.
3. Formulate and then prove the first fundamental theorem of asset pricing for this economy. In particular, show that if there is no arbitrage, then there exists a measure Q equivalent to P under which the following holds:
 - $S_t^* = E_t^Q[S_{t+1}^* + D_{t+1}^*]$, and equivalently,
 - for $T > t$: $S_t^* = E_t^Q[\sum_{n=t+1}^T D_n^* + S_T^*]$
4. Show that if there is no arbitrage, then there exists a stochastic discount factor (or pricing kernel) process M_t that is strictly positive and such that :
 - $M_t S_t = E_t[M_{t+1}(S_{t+1} + D_{t+1})]$, and equivalently,
 - $1 = E_t[\frac{M_{t+1}}{M_t} R_{t+1}]$ for the gross return $R_{t+1} = \frac{S_{t+1} + D_{t+1}}{S_t}$, as well as:
 - for $T > t$: $M_t S_t = E_t[\sum_{n=t+1}^T M_n D_n + M_T S_T]$

2. Multiple Periods, Complete Markets: Cox, Ross Rubinstein

The purpose of this exercise is to take you through Cox, Ross, and Rubinstein's derivation of the Black and Scholes formula as a limit of the discrete-time multiperiod binomial model.

Suppose now that period T is subdivided into n periods of $\Delta = T/n$. At each time $t_i = i\Delta \forall i = 0, \dots, n-1$, the dynamics of the risky asset and the risk-free asset are given by:

$$\begin{aligned} S_{i+1} &= \tilde{w}_{i+1}^n \cdot S_i \\ S_{i+1}^0 &= R_n \cdot S_i^0 \end{aligned}$$

where $\tilde{w}_i^n \forall i = 1, \dots, n$ are i.i.d. random variables, which can take each of two values u_n with probability p and d_n with probability $1 - p$.

1. Show that the market (S^0, S) is dynamically complete, in the sense that any time- T random variable $h(S_T)$ can be perfectly replicated by a self-financing dynamic trading strategy in the risk-free and risky asset. Importantly, we impose the no-arbitrage restrictions

$$u_n > R_n > d_n.$$

Here, R_n and $\tilde{w}_i^n$ indicates the dependence on n in the sense that n is the time frequency that controls the approximation to the *continuous time limit*.

Hint: use a recursive argument.

2. Prove that there is a unique equivalent martingale measure. Find the distribution of $\tilde{w}_1^n$ under the equivalent measure and define the associated probabilities π_n^Q and $1 - \pi_n^Q$.
3. We now want to find the price of a contingent claim with payoff:
 $h(S_T) = \max(S_T - K, 0)$, i.e. a European call option. And in particular, we would like to compute the limit of that formula as $n \rightarrow \infty$ for a given T , i.e. $\Delta \rightarrow 0$.
 Let us define $Y_n = \sum_{i=1}^n \log(\tilde{w}_i^n)$. Clearly $S_T = S_0 \cdot \exp(Y_n)$. We have to pick R_n, u_n, d_n so that the distribution of the stock price converges to that of a geometric brownian motion, for which we have: $S_T = S_0 \exp\left((\mu - \frac{\sigma^2}{2})T + \sigma Z(T)\right)$ where $Z(T)$ is a normally distributed random variable with zero mean and variance T (i.e., $Z(T) \sim N(0, T)$).
 Show that if we pick $p = 0.5$, $u_n = \exp(\alpha_n + \sigma_n)$, $d_n = \exp(\alpha_n - \sigma_n)$ for appropriate α_n, σ_n we get the desired convergence. Show that the natural choice of R_n is $\exp(r\Delta)$ where r is the continuously compounded risk-free rate.
Hint: prove it by showing the convergence of the characteristic function of Y_n to that of a standard normal distribution (recall that X is a standard normal if and only if its characteristic function $\phi(t) = E[\exp(itX)] = \exp(-\frac{t^2}{2})$ - for the general theorem on characteristic functions see the book by Williams chap. 16 for example).
4. We now turn to the convergence of the option price. Show that the call option price can be written as follows:

$$C(S_0, 0) = S_0 B(n, \eta, \pi_n^R) - \frac{K}{R_n^n} B(n, \eta, \pi_n^Q)$$

where $B(n, \eta, \pi) = 1 - P\left(\frac{Y_n - E^\pi[Y_n]}{\sqrt{V^\pi[Y_n]}} < \frac{\eta - E^\pi[Y_n]}{\sqrt{V^\pi[Y_n]}}\right)$ Find η and π^Q, π^R the probability of an up realization for $\tilde{w}_1^n$ under two different measures.

Hint: Since markets are complete, the price of any payoff must equal the value of a self-financing trading strategy. The price of a call option $C(S_0, 0)$ can then be expressed as an expectation of its discounted final payoff under the risk-neutral measure (why?). Then notice that the distribution of $\exp(Y_n)$ under the historical measure is $P(\exp(Y_n) = u_n^k d_n^{n-k}) = C_n^k p^k (1-p)^{n-k} \forall k = 0, \dots, n$, where we use the standard notation $C_n^k = \frac{n!}{k!(n-k)!}$.

Notice that the formula could easily be implemented on a computer to find the values of a European call option. Here we are interested in the continuous-time limit of that formula.

5. Show that $X_n = \frac{Y_n - E_n^\pi[Y_n]}{\sqrt{V_n^\pi[Y_n]}}$ converges in distribution towards a centered gaussian random variable.
Hint: Prove the convergence of the characteristic function of X_n . Also remember that Y_n is the sum of i.i.d. random variables.
6. Compute and find the limit as $n \rightarrow \infty$ of $E^{\pi_n^Q}[Y_n]$, $E^{\pi_n^R}[Y_n]$, $V^{\pi_n^Q}[Y_n]$, $V^{\pi_n^R}[Y_n]$ to derive the Black and Scholes formula as the limit of the discrete time price of the European call option.

Hint: Remember Y_n is the sum of i.i.d random variables.

7. Write the resulting limiting Option Pricing Formula.

3. Kolmogorov Equations I

Let X_t be a Markov chain with values x_i and transition probabilities $\pi(x_i, x_j)$. Prove Kolmogorov equations:

- 1.

$$E[g(X_T) | \mathcal{F}_t] = E[g(X_T) | X_t] = G(t, X_t)$$

satisfies

$$G(t, x_i) = \sum_j p(x_i, x_j) G(t+1, x_j)$$

2. define transition matrix Π with $\Pi = (p(x_i, x_j))$. Then, prove

$$G(t, x) = \Pi^{T-t} g(x)$$

3. Let

$$V(x) = \sum_{t=0}^{\infty} e^{-rt} E[X_t | X_0 = x]$$

Prove that

$$V(x) = x + e^{-r} \Pi V(x) \Leftrightarrow V(x) = (\text{Id} - e^{-r} \Pi)^{-1} x$$

4. Kolmogorov Equations II

1. Suppose $X_t = Y_1 \cdots Y_t$ so that

$$X_{t+1} = X_t Y_{t+1} \tag{1}$$

2. Y_t is a Markov process with transition probabilities Π

3. X_t is **not a Markov process**

4. (X_t, Y_t) is a two-dimensional Markov process

5. X_t takes “too many values”

6. Define

$$V(X_t, Y_t) = E_t \left[\sum_{s=0}^{\infty} e^{-rs} X_{t+s} \right] = E \left[\sum_{s=0}^{\infty} e^{-rs} X_{t+s} | X_t, Y_t \right] \tag{2}$$

7. Derive Kolmogorov equation

$$V(X_t, Y_t) = X_t + e^{-r} E_t[V(X_{t+1}, Y_{t+1})]$$

8. Make an Ansatz

$$V(x, y) = xv(y)$$

to get that

$$\begin{aligned}
& \underbrace{V(X_t, Y_t)}_{=X_t v(Y_t)} \\
&= X_t + e^{-r} E_t \underbrace{[V(X_{t+1}, Y_{t+1})]}_{=X_{t+1} v(Y_{t+1})} \\
&= X_t + e^{-r} E_t [X_{t+1} v(Y_{t+1})] \\
&\underbrace{=}_{(1)} X_t + e^{-r} E_t [X_t Y_{t+1} v(Y_{t+1})] \\
&= X_t + e^{-r} E_t [X_t Y_{t+1} v(Y_{t+1})]
\end{aligned} \tag{3}$$

and therefore

$$v(Y_t) = 1 + e^{-r} E_t [Y_{t+1} v(Y_{t+1})] = 1 + e^{-r} \sum_j p(Y_t, y_j) y_j v(y_j),$$

which in vector form becomes

$$v(y) = \mathbf{1} + e^{-r} \Pi \text{diag}(y) v(y)$$

where $\mathbf{1}$ is the vector of ones.

9. Prove

$$v(y) = (\text{Id} - e^{-r} \Pi \text{diag}(y))^{-1} \mathbf{1} = \sum_{\tau=0}^{\infty} e^{-r\tau} (\Pi \text{diag}(y))^\tau \mathbf{1}$$

Theorem This series converges if and only if the *spectral radius*

$$\rho(\Pi \text{diag}(y)) = \max(|\text{eig}(\Pi \text{diag}(y))|)$$

satisfies

$$\rho(\Pi \text{diag}(y)) < e^r \tag{4}$$

5. Kolmogorov Equations III

1. Suppose $X_t = Y_1 \cdots Y_t$
2. Z_t is a Markov process with transition probabilities Π , $Y_t = Y(Z_t)$
3. $r_t = r(Z_t)$ (monetary policy)
4. X_t is **not a Markov process**
5. (X_t, Y_t) is a two-dimensional Markov process
6. X_t takes “too many values”
7. Define

$$V(X_t, Z_t) = E_t \left[\sum_{s=0}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} \right] = E \left[\sum_{s=0}^{\infty} e^{-\sum_{\tau=0}^{s-1} r_{t+\tau}} X_{t+s} | X_t, Z_t \right] \tag{5}$$

8. Derive Kolmogorov equation

$$V(X_t, Z_t) = X_t + e^{-r_t} E_t[V(X_{t+1}, Z_{t+1})]$$

9. We now make the Ansatz

$$V(X_t, Y_t) = X_t v(Y_t)$$

10. Substituting that $X_{t+1} = X_t Y_{t+1}$ and $V(X_t, Y_t) = X_t v(Y_t)$ and $V(X_{t+1}, Y_{t+1}) = X_{t+1} v(Y_{t+1}) = X_t Y_{t+1} v(Y_{t+1})$, we get

$$\begin{aligned} X_t v(Y_t) &= X_t + e^{-r(Z_t)} E_t[X_t Y(Z_{t+1}) v(Z_{t+1})] \Leftrightarrow \\ v(Z_t) &= 1 + e^{-r(Z_t)} E_t[Y(Z_{t+1}) v(Z_{t+1})] = 1 + e^{-r(Z_t)} \sum_j p(Z_t, z_j) Y(z_j) v(z_j), \end{aligned}$$

which in vector form becomes

$$v(z) = \mathbf{1} + \text{diag}(e^{-r(z)}) \Pi \text{diag}(Y(z)) v(z)$$

11. Prove

$$v(z) = (\text{Id} - \text{diag}(e^{-r(z)}) \Pi \text{diag}(Y(z)))^{-1} \mathbf{1} = \sum_{\tau=0}^{\infty} (\text{diag}(e^{-r(z)}) \Pi \text{diag}(Y(z)))^{\tau} \mathbf{1}$$

Theorem This series converges if and only if the *spectral radius*

$$\rho(\Pi \text{diag}(y)) = \max(|\text{eig}(\text{diag}(e^{-r(z)}) \Pi \text{diag}(Y(z)))|) < 1$$

6. Replication and Binomial Trees

Stochastic Integrals

Definition Given a martingale M_t , the process

$$X_t = \sum_{s=0}^{t-1} \pi_s (M_{s+1} - M_s) = \pi \cdot M = \int_0^t \pi_s dM_s$$

is called the stochastic integral of π with w.r.t. M

Lemma X_t is a Martingale

Proof $X_{t+1} = X_t + \pi_t (M_{t+1} - M_t)$ implies

$$E_t[X_{t+1}] = X_t + \pi_t E_t[M_{t+1} - M_t] = X_t$$

Self-Financing Portfolio Gains Processes

- there are two investment opportunities: risk-less with zero interest rate (bank account) and stock with price process M_t
- π_t is the number of shares of the stock purchased at time t
- the total gains process change is

$$X_{t+1} - X_t = \pi_t (M_{t+1} - M_t)$$

- that is

$$X_t = X_0 + \pi \cdot M$$

Replication and Martingale Representation for the Binomial Model

- Let

$$M_t = Y_1 \cdots Y_t$$

where Y_t are i.i.d., $Y_t = u$ or d with prob. p such that $pu + (1-p)d = 1$. Then, M is a martingale.

- $\mathcal{F}_t$ is the natural filtration of Y_t
- X_T is a $\mathcal{F}_T$ -measurable random variable
- then,

$$X_{T-1} = E_{T-1}[X_T]$$

and solving

$$\begin{cases} X_{T-1} + \pi_{T-1} (u - 1) M_{T-1} = X_T(u, X_{T-1}) \\ X_{T-1} + \pi_{T-1} (d - 1) M_{T-1} = X_T(d, X_{T-1}) \end{cases}$$

gives the replicating portfolio π_{T-1}