EPFL & SFI

Asset Pricing Theory

Problem Set 3: Expected Utility and Risk-Aversion

1. Comparative statics

Consider the portfolio choice problem of a risk-averse expected utility investor who can invest in one risky asset and one risk-free asset to maximize his expected utility of terminal wealth. Denote by x the fraction of wealth invested by the agent in the risky asset.

1. Write the terminal wealth of the agent who invests a fraction x of her wealth in the risky asset R and the remaining in the risk-free asset R_f.
2. Derive the first order condition with respect to x. Is the second order condition satisfied?
3. Assume that the risk-premium is strictly positive, that is $\mathbb{E}[R - R_f] > 0$. Show that the optimal x^* is positive.
4. Show that if the investor has decreasing relative risk-aversion below q, that is $R'(w) < 0,\ R(w) < 1$, then $\frac{\partial x^*(w)}{\partial w} > 0$ and that if an agent has constant relative risk-aversion then $\frac{\partial x^*(w)}{\partial w} = 0$.

2. Portfolio choice with linear risk-tolerance

Consider an agent wiht power utility function $u(w) = \frac{(\alpha+\gamma w)^{1-\frac{1}{\gamma}}}{\gamma-1}$

1. Show that this utility nests log-utility and negative exponential utility as special cases. In particular, show that $\lim_{\gamma\to 1} \frac{(\alpha+\gamma w)^{1-\frac{1}{\gamma}}-1}{\gamma-1} = \log(\alpha + w)$, and $\lim_{\gamma\to 0} \frac{(1+\frac{\gamma}{\alpha}w)^{1-\frac{1}{\gamma}}}{\gamma-1} = -\exp(-\frac{w}{\alpha})$
2. Show that the power utility agent has linear risk tolerance, where the risk tolerance is the inverse of the agent's risk-aversion: $T(w) = \frac{1}{A(w)}$.
3. Consider the optimal portfolio choice problem. The power-utility investor can invest in a risk-free asset with a return R_f and n risky assets with a return $R_i\ \forall i = 1, \ldots, n$. Show that the vector a of optimal dollar investments in the risky assets is of the form $a^*(w) = (\alpha + \gamma w R_f)b$, where b is a vector independent of wealth and α.
4. Show that this implies that agents with linear-risk-tolerance all invest in a portfolio of n risky assets whose proportions are independent of wealth, in the sense that $\frac{a_i^*}{\sum_{j=1}^n a_j^*}$ is independent of the agent's wealth. Is it also independent of the coefficient of relative risk-aversion γ? of the coefficient of absolute risk-aversion α?

3. Portfolio Choice with Elliptical Distributions.

Suppose that the vector of returns R has an elliptical density, $kg((x-\mu)'\Sigma^{-1}(x-\mu))$ for some constant k. Show that the moment-generating function satisfies

$$E[e^{\alpha' R}] \ = \ e^{\alpha'\mu}\psi(\alpha'\Sigma\alpha)$$

for some function ψ. Solve the corresponding portfolio choice problem of an investor with exponential utility.

4. Demand with Exponential Families An economic agent observes a signal θ. He has exponential utility, and he invests in an optimal portfolio of risky assets with a payoff X with a density $e^{a(\theta)+\theta X+b(X)}$ for some functions a, b. The optimal demand for the risky asset is $D(\gamma, \theta, p)$ where p is the price. Try to say as much as you can about the properties of this demand function.