

Asset Pricing Theory

Problem Set 2: Expected Utility and Risk-Aversion

1. Expected Utility

- Prove that if we rank lotteries based on expected utility (i.e., for any two lotteries $L^a = \{(x_1, p_1^a), \dots, (x_S, p_S^a)\}$ and $L^b = \{(x_1, p_1^b), \dots, (x_S, p_S^b)\}$ we say that L^a is preferred to L^b if $E[u(L^a)] \geq E[u(L^b)]$) then the preference ordering $\succsim$ thus defined satisfies the four classic axioms (completeness, reflexivity, transitivity, continuity) and the independence axiom.
- Prove that the utility function in the Von Neumann-Morgenstern (VNM) theorem is unique up to an increasing affine transformation. Specifically, show that if a preference relation on a set of simple lotteries is represented by a utility function u such that for any two lotteries $L^a = \{(x_1, p_1^a), \dots, (x_S, p_S^a)\}$ and $L^b = \{(x_1, p_1^b), \dots, (x_S, p_S^b)\}$

$$L^a \succsim L^b \iff E[u(L^a)] \geq E[u(L^b)]$$

Then, another function v represents the same ordering if and only if there exist two constants $a > 0$ and b such that $v = au + b$

2. Allais Paradox

Consider the following gambles:

- A 6 million EUR with probability 0.9 and 0 else
- B 12 million EUR with probability 0.45 and 0 else
- C 12 million EUR with probability 0.001 and 0 else
- D 6 million EUR with probability 0.002 and 0 else

- Show that an expected utility maximizer who prefers A to B must also prefer D to C
- Show that if someone prefers A to B and C to D, she violates the independence axiom.

3. Insurance premium with Negative Exponential Utility Case

- Consider an agent with negative exponential utility $u(x) = -e^{-\gamma x}$. Suppose his terminal wealth is $W_1 = W_0 + \epsilon$, where ϵ is normally distributed $N(\mu, \sigma^2)$ and W_0 is constant. Compute the expected utility for this agent $\mathbb{E}[u(W_1)]$. Deduce the premium π the agent would be willing to pay to fully insure against this risk, i.e., such $u(W_0 - \pi) = \mathbb{E}[u(W_1)]$. N.B.: $W_0 - \pi$ is known as the **certainty equivalent** of the risky terminal wealth W_1 .
- How does the premium change with μ, σ, W_0 respectively?
- Suppose now that $W_0 \sim N(\mu_0, \sigma_0^2)$ is also normally distributed and correlated with the risk that you are insuring, i.e., $\text{Corr}(W_0, \epsilon) = \rho$. Calculate the insurance premium in that case (i.e., such $\mathbb{E}[u(W_0 - \pi)] = \mathbb{E}[u(W_1)]$). How does the amount π the agent is willing to pay to depend on μ_0, σ_0, ρ ?

4. Risk aversion

Consider two expected utility agents with respective utility functions U_a and U_b

- Show that agent a is more risk-averse than agent b in the sense that a dislikes any risky gamble that b dislikes, if and only if $U_a(w) = \phi(U_b(w))$ for some function $\phi(x)$ that is strictly increasing and concave.
- Show that this is also equivalent to agent a having a larger risk-aversion coefficient than agent b . The coefficient of risk-aversion is defined as $A_i(x) = -\frac{U_i''(x)}{U_i'(x)} \quad \forall i = a, b$

5. Risk

- We say that two variables X and Y are mean-independent if $E[X|Y] = E[X]$ and $E[Y|X] = E[Y]$. Give an example of two random variables that are mean-independent but not independent.
- Give an example of two random variables that are uncorrelated but not mean-independent.
- We say that X first order stochastically dominates Y if $E[u(X)] \geq E[u(Y)]$ for any monotone increasing u ; and X second order stochastically dominates Y if $E[u(X)] \geq E[u(Y)]$ for any monotone increasing and concave u .
- Show that if C_a and C_b are normally distributed, and have the same mean, then C_a second-order stochastically dominates C_b if and only if $Var(C_a) \leq Var(C_b)$.
- Show that if two consumption plans C_a and C_b have equal expectation and $Var(C_a) < Var(C_b)$ this does not necessarily imply that C_a second order stochastically dominates C_b (give an example).
- Consider two investors with strictly increasing utility functions u_1 and u_2 who have same level of initial wealth w_0 and face a risk ϵ . Define ϕ by $u_1(x) = \phi(u_2(x))$. Prove that the insurance premium agent one would be willing to pay to fully insure against the risk is greater than that of agent two if and only if $\phi''(x) < 0$.
- Consider risky gambles with payoff $C_\lambda = \mu + \lambda\epsilon$ where $\mathbb{E}[\epsilon|\mu] = 0$. Show that C_λ is always preferred by a risk-averse investor to C_γ as long as $\gamma > \lambda$.
- Consider risky gambles $C_\gamma = \gamma\epsilon_1 + (1 - \gamma)\epsilon_2 \quad \forall \gamma \in (0, 1)$ where the $\epsilon_i \quad i = 1, 2$ are iid. Show that $\gamma = 1/2$ is preferred by any risk-averse investor.

6. Optimal portfolio choice

Given endowment $w_0 = \text{const.}$, $w_1 = \text{random}$, invest x in a bond with interest rate r , gives consumption

$$c_0 = w_0 - x, \quad c_1 = w_1 + rx$$

Problem: maximize

$$u(c_0) + e^{-\rho} E[u(c_1)]$$

over x . Let $u(c) = c^{1-\gamma}/(1-\gamma) - 1$ and $u(c) = \log c$ when $\gamma = 1$.

- solve the problem when $w_1 = w$ is constant

The first order condition is

$$(w_0 - x)^{-\gamma} = e^{-\rho}(w + rx)^{-\gamma} r \Rightarrow x = \frac{w_0 - w(e^{-\rho}r)^{-1/\gamma}}{1 + r(e^{-\rho}r)^{-1/\gamma}}$$

- solve the problem when w_1 takes two values, w_H, w_L with pro. $p, 1 - p$, and $\gamma = 1$.
- Solve the previous problem for exponential utility $u(c) = -e^{-ac}$.

Hint: Here, it is possible to get a closed-form solution without using perturbation theory.