EPFL & SFI

Asset Pricing Theory

Problem Set 1: Absence of Arbitrage

1. Arbitrage and Negative State Prices in complete markets

Consider the economy with three assets and three payoff states. asset 1 has price $P_1 = 2.5$ and payoff vector $X_1 = [3, 2, 1]$. Asset 2 has $P_2 = 15$ and $X_2 = [4, 5, 6]$. And asset 3 has $P_3 = 4$ and $X_3 = [2, 3, 1]$.

- Is this a complete market economy?
- Determine a vector of state prices. Is it unique?
- Is the economy arbitrage-free? If not, construct an arbitrage trading strategy.

hint: you may need to inverse a matrix using Matlab or alike...

2. Arbitrage and Negative State Prices in Incomplete Markets

- Construct an example of an incomplete market economy (e.g., write down the payoffs of 2 risky securities in three states) where negative state prices are supporting the economy, but there are no-arbitrage opportunities when restricted to trading the existing securities.
- In your example economy, give the payoff pricing functional $q(z)$ defined for any $z \in \mathcal{M}$. Then, choose a payoff $\hat{z}$ that is not spanned by the existing securities. Derive the upper and lower replication prices of $\hat{z}$. Finally, give a valuation functional $Q(y)$ defined for any $y \in \mathbb{R}^3$ using the payoff $\hat{z}$. Is that valuation functional unique?
- Suppose you add a third security to your example that completes the market (i.e., its terminal payoff is not a linear combination of the two other securities' payoff). Can you still have negative state prices and no-arbitrage? What is then the relation between the payoff pricing functional $q(\cdot)$ and the valuation functional(s) $Q(\cdot)$?

3 One Period, Complete Markets

Suppose there is only one risky asset S and one risk-free asset S^0. The risky asset will be worth $\tilde{w} \cdot S$ at the end of period T, where $\tilde{w}$ is the only source of risk in the economy. $\tilde{w}$ can take only 2 values u with probability p and d with probability $1 - p$. The risk-free asset will be worth $R \cdot S^0$. Assume $u > d$.

Let us define the state price of the up state by q_u and the state price of the down state by q_d.

1. Show that there is a unique state price vector consistent with the two prices (S^0, S) and that the market is arbitrage-free if and only if $u > R > d$.
2. Show that one can define an **equivalent martingale measure** Q_0 (characterized by the probability $\pi^{Q_0} = Q_0(\tilde{w} = u)$) such that the stock price is equal to the expected value of its discounted future cash-flows, i.e., $S = E^{Q_0}[\frac{\tilde{w}S}{R}]$.
3. What is the relation between the state price q_u and the risk-neutral probability π^{Q_0}?
4. Show that markets are complete in the sense that any contingent claim with promised payoff of $h(\tilde{w} \cdot S)$ at time T can be perfectly replicated by a portfolio investing in the stock and the bond.

5. Show that the value of this portfolio is equal to the expectation under the equivalent martingale measure Q_0 of the discounted final payoff of the contingent claim.
6. Conclude that the augmented market (S^0, S, h) is arbitrage-free if and only if Q_0 is an equivalent martingale measure for the redundant security as well.

4 One Period, Incomplete Markets

Suppose now that $\tilde{w}$ can take on multiple (possibly a continuum of) values with support $[u, d]$ at date T. From the previous problem clearly absence of arbitrage requires $u > R > d$ which we will assume in the following.

1. Show that, in general, there is a continuum of possible equivalent martingale measures for the market (S^0, S).

   Hint: for example, consider $\tilde{w}$ taking on three possible values: $u > m > d$ with probability p_u, p_m, p_d.
2. Consider the augmented market (S^0, S, h) and assume $h(\cdot)$ is convex and increasing (it could be a European call option for example). Show that the price h_0 of the contingent claim paying $h(\tilde{w} \cdot S)$ is bounded above by the value in a market where $\tilde{w}$ takes on values u or d only and below by the value in a market where $\tilde{w} = R$ *a.s.*. In other words, denoting $\mathcal{Q}$ the set of all equivalent martingale measures, show that

$$\begin{aligned}\sup_{Q_i \in \mathcal{Q}} \frac{1}{R} E^{Q_i}\left(h(\tilde{w} \cdot S)\right) &\leq \frac{1}{R}\left(\pi^{Q_0} h(u \cdot S) + (1 - \pi^{Q_0}) h(d \cdot S)\right) \\ \inf_{Q_i \in \mathcal{Q}} \frac{1}{R} E^{Q_i}\left(h(\tilde{w} \cdot S)\right) &\geq \frac{h(R \cdot S)}{R}\end{aligned}$$

   where π^{Q_0} is the complete market's Q_0 probability identified above.

   Hint: use a line going through $(d \cdot S, h(d \cdot S))$ and $(u \cdot S, h(u \cdot S))$, and the fact that $h(\cdot)$ is convex for the first inequality, and the line tangent to $h(\cdot)$ at the point $(R \cdot S, h(R \cdot S))$ for the second inequality.

   Interpret your result in terms of the volatility of an option, i.e., when is an option most valuable?

5 One Period, Incomplete Markets in the Real World: SPY Options

Column names:

- ticker: Ticker
- stkpx: Stock price (price of the underlying)
- expirdate: option expiration date;
- yte: years to expiry
- strike: Option strike
- cvolu: Call volume
- coi: Call Open Interest
- pvolu: Put Volume
- poi: Put Open Interest
- cbidpx: Call bid price
- cvalue: call "smart mid" (cleaned mid price)

- caskpx: Call Ask Price
- pbidpx: Put Bid Price
- pvalue: Put smart mid
- paskpx: Put Ask Price
- cbidiv: Call Bid Implied Vol (IV)
- cmidiv: Call Mid IV
- caskiv: Call Ask IV
- smoothsmvvol: Smoothed IV
- pbidiv: Put Bid IV
- pmidiv: Put Mid IV
- paskiv: Put Ask IV
- irate: Annualized Interest rate
- divrate: Dividend rate
- residualratedata, delta, gamma, theta, vega, rho, phi, driftlesstheta, extvol, extctheo, extptheo: Option Greeks
- spot_px: Nan
- trade_date: Date on Trading

Investigate the Data. Does the Law of One Price Hold? Is the Data Arbitrage Free? Use the Breeden-Litzenberger formula to find state prices. Are they positive? How do you construct them when there are bid-ask spreads?

No Arbitrage With Option Prices

- $C_A(K) \geq C_B(K),\ P_A(K) \geq P_B(K)$: bid and ask prices of calls and puts with strike K. Let also r_+ be the lending rate and r_- the borrowing rate. $S_A > S_B$ are the ask and bid prices of the stock.

  We always need to verify no-arbitrage conditions; otherwise, utility maximization will have an infinite solution.

  - Positive bid-ask spreads

    $$P_A(i) \ \geq \ P_B(i),\ C_A(i) \ \geq \ C_B(i) \tag{1}$$

  - put-call parity arbitrage: since

    $$(x-K)^+ - (K-x)^+ \ = \ x - K,$$

    we can build arbitrage portfolios: if we buy the call, sell the put, sell the underlying, and buy K units of the bond, we get zero at T, and hence, I cannot get money from this: we must have

    $$C_A(i) - P_B(i) \ \geq \ S_B - K_i(1-r_-) \tag{2}$$

    Similarly, we need to have

    $$C_B(i) - P_A(i) \ \leq \ S_A - K_i(1-r_+) \tag{3}$$

– We have that $(K-x)^+$ and $(x-K)^+$ are convex functions in K. Thus,

$$(x-K_i)^+\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i} \;+\; (x-K_{i+2})^+\frac{K_{i+1}-K_i}{K_{i+2}-K_i} \;>\; (x-K_{i+1})^+\,.$$

As a result, a violation of the inequality

$$P_A(K_i)\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i} \;+\; P_A(K_{i+2})\frac{K_{i+1}-K_i}{K_{i+2}-K_i} \;>\; P_B(K_{i+1}) \tag{4}$$

is an arbitrage opportunity. However, already if we have

$$P_A(K_i)\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i} \;+\; P_A(K_{i+2})\frac{K_{i+1}-K_i}{K_{i+2}-K_i} \;<\; P_A(K_{i+1})\,, \tag{5}$$

this is a **quasi-arbitrage opportunity.** Namely, if the dealers are positing ask prices that satisfy 5, any market participant can offer the price

$$\hat{P}_A(K_{i+1}) \;=\; P_A(K_i)\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i} \;+\; P_A(K_{i+2})\frac{K_{i+1}-K_i}{K_{i+2}-K_i}\,. \tag{6}$$

Indeed, if we simply by the portfolio of puts: $P_A(K_i)\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i}+P_A(K_{i+2})\frac{K_{i+1}-K_i}{K_{i+2}-K_i}$, this is a **super-hedge** for our position. We then sell the $K_{i+1}-$put at this price and consume the difference. Similarly, for calls, we ought to have

$$C_A(K_i)\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i} \;+\; C_A(K_{i+2})\frac{K_{i+1}-K_i}{K_{i+2}-K_i} \;>\; C_B(K_{i+1}) \tag{7}$$

and we can always offer an $C_A(K_{i+1})$ given by

$$\hat{C}_A(K_{i+1}) \;=\; C_A(K_i)\frac{K_{i+2}-K_{i+1}}{K_{i+2}-K_i} \;+\; C_A(K_{i+2})\frac{K_{i+1}-K_i}{K_{i+2}-K_i}$$

- **Breeden-Litzenberger** Let $K_1,\cdots,K_N$ be the grid of strikes. Pick $K_0<K_1,\ K_{N+1}>K_N$. Define $C(j)=C(K_j)$ and

$$C(0) \;=\; \tilde{S}-\tilde{B}K_0\ ,\ C(N+1)=C(N+2)=0\,.$$

where $\tilde{S}$ is the stock price and $\tilde{B}$ is the bond price. Define the $(N+2)$-dimensional vector $q=(q_i)_{i=0}^{N+1}$ via

$$q_0 \;=\; \tilde{B}+\frac{\tilde{B}K_0-\tilde{S}+\tilde{C}(1)}{K_1-K_0}$$

and

$$q_j \;=\; \Big(C(j+1)(K_{j+1}-K_j)^{-1}+C(j-1)(K_j-K_{j-1})^{-1}-C(j)[(K_{j+1}-K_j)^{-1}+(K_j-K_{j-1})^{-1}]\Big)\,,\ i\ge 1$$

Show that q is a vector of state prices in the sense that

$$C(j) \;=\; \sum_i q_i(K_i-K_j)^+ \tag{8}$$

Do the same calculation for puts.

6 CIP Arbitrage

Study and derive the formula for covered interest parity arbitrage. Describe exactly how you can make money today using this trade. Read

`https://www.dropbox.com/s/tzm2ytm48591jiw/Manuscript_Avdjiev_Du_Koch_Shin_AER_Insights_Revision.pdf?dl=0`