

Asset Pricing II

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1. Expected Utility and Risk Aversion

Overview

Readings:

- Rubinstein p.24-48 & p.87-95
- Kreps Chap. 1, 4, 5
- Back Chap. 1

Topics:

- Von Neumann and Morgenstern Expected Utility
- Risk-Aversion
- Arrow Pratt measures of risk
- Typical Utility Functions

Expected Utility: background

- Historically (see Rubinstein), much interest in valuation under uncertainty is motivated by gambling. The working assumption is that the value of a gamble is its expected payoff (Pascal, Fermat, Huygens in the 17th century).
- Bernoulli (1738) proposes Expected Utility (specifically the expectation of the logarithm of the payoff) as an appropriate valuation operator to circumvent his 'St. Petersburg paradox': Consider the gamble of flipping a coin indefinitely to receive 2^n if you flip heads for the first time in round n . The paradox is that its expected value is $\sum_{n=1}^{\infty} \frac{1}{2}^n 2^n = \infty$. Yet you may not want to pay an infinite amount for this gamble. With Bernoulli's solution the value of the gamble is $\sum_{n=1}^{\infty} \frac{1}{2}^n \log(2^n) = 0.2 \log 2 \approx 1.4 < \infty$. But see the discussion in Rubinstein (p. 40-41) for Menger's 'super St Petersburg' paradox.

- Von Neuman and Morgenstern (1944,1947), in their treatise on game theory, propose an axiomatization of the idea that rational agents should make choices under uncertainty by maximizing the expected utility criterion (see the discussion in Rubinstein (p. 88-95) for appropriate historical references). Here, we present them very succinctly.

Axiomatic system

- Why use an axiomatic system to describe behavior?
 - As a normative guide to behavior (I believe these axioms - how should I behave)
 - As a positive explanation for the observed behavior
- What is a good axiomatic system:
 - Parsimonious (as few independent axioms as possible)
 - Consistent
 - Realistic
- Von Neuman and Morgenstern's axiomatization takes objective probabilities of states as input (assumed common knowledge) $\sim$ frequentist.
- Savage (1954) proposes an axiomatization that only depends on objective states and actions. Given his axioms, agents' preferences can be represented by the expected utility with subjective probabilities $\sim$ Bayesian.

Axioms of behavior under uncertainty

- We assume, for simplicity, that the set of possible outcomes $X : \{x_s\}_{s=1,\dots,S}$ is finite and known by all agents.
- A simple lottery is a probability distribution on these outcomes: $p = (p_1, \dots, p_S)$ with $p_s \geq 0$ and $\sum_s p_s = 1$.
- Assume that agents have a preference relation $\succeq$ on the set of lotteries, which satisfies the classic axioms of utility theory under certainty:

A1 Completeness

A2 Reflexivity

A3 Transitivity

In addition, we require

A4 Continuity: $p^a \succeq p^b \succeq p^c$ implies

$\exists \alpha \in [0, 1] \text{ s.t. } p^b \sim \alpha p^a + (1 - \alpha)p^c$

A5 Independence: $p^a \succeq p^b$ iff

$\alpha p^a + (1 - \alpha)p^c \succeq \alpha p^b + (1 - \alpha)p^c \quad \forall \alpha \in [0, 1]$

- From Classic Utility theory, we know that axioms A1-A4 guarantee the existence of a preference ordering on the set of lotteries, i.e., there exists a Utility function $V(p) : \mathcal{P} \rightarrow \mathbb{R}$ such that $p^a \succeq p^b$ iff $V(p^a) \geq V(p^b)$. Note that if the underlying choice set is countable, then such a function is easy to construct (how?). For uncountably infinite sets, an additional axiom is required (cf. discussion in Kreps).

- The independence axiom A5 then is the cornerstone of classical VNM decision theory under uncertainty. It guarantees that the preference order can be represented by a utility function that is linear in probabilities. Specifically,

Theorem

Under axioms A1-A5, there exists scalars u_s such that for any two lotteries p^a and p^b we have $p^a \succeq p^b$ iff $\sum_s p_s^a u_s \geq \sum_s p_s^b u_s$.

- It is also straightforward to see that the expected utility representation is unique up to an increasing affine transformation.
- The result generalizes easily to lotteries with payoffs on multiple dates (how?). Note that it does not imply the additively separable form often used in the literature.
- The generalization to uncountably infinite X is non-trivial (See discussion in Kreps).

Critique of Expected Utility: The Allais Paradox

- Compare lotteries a and b. Which do you prefer? Then compare c and d.

<i>payoff</i>	p^a	p^b	p^c	p^d
\$5M	0	.1	.1	0
\$1M	1	.89	0	.11
\$0	0	.01	.9	.89

Most individuals rank $p^a \succeq p^b$ and $p^c \succeq p^d$. But this is inconsistent with the independence axiom. Prove it.

Risk Aversion

- An investor is risk-averse if he prefers to avoid zero-mean risks at all levels of wealth, i.e., $u(W_0) \geq E[u(W_0 + \tilde{\epsilon})]$ for any random variable $\tilde{\epsilon}$ such that $E[\tilde{\epsilon}] = 0$.
- An equivalent definition is that an agent is risk-averse iff $u(E[\tilde{W}_1]) \geq E[u(\tilde{W}_1)]$
- This is equivalent (by Jensen's inequality) to the statement: An individual is risk-averse if his utility function is concave.
- u_1 is more risk-averse than u_2 if u_1 dislikes all the risks that u_2 dislikes at all initial wealth levels (assumed to be identical for both). This is equivalent to:
 - u_1 is an increasing concave transformation of u_2 (meaning $u_1 = \phi \circ u_2$ with ϕ increasing and concave).
 - $A_1(x) > A_2(x) \forall x$ where $A_i(x) = -\frac{U_i''(x)}{U_i'(x)}$ is the **Arrow-Pratt measure of Absolute Risk-aversion**.

- We define a risk-premium π to be the amount the risk-averse agent is willing to give up to avoid a risk:

$$\mathbb{E}[u(W_0 + \epsilon)] = u(W_0 - \pi)$$

- $W_0 - \pi$ is also called the **certainty equivalent** wealth for the gamble ϵ .
- Show that if u_1 is more risk-averse than u_2 then $\pi_1(W_0, \epsilon) > \pi_2(W_0, \epsilon)$. (obvious for small risks; see below).
- Consider now the behavior of the risk-premium for 'small' ϵ (Back p.7 makes this rigorous). Using Taylor expansion, we obtain the Arrow-Pratt approximation to the risk-premium:

$$\pi(W_0) \approx \frac{1}{2} \sigma_\epsilon^2 A(W_0)$$

- Similarly, if we consider small **multiplicative** risks, i.e. , $\mathbb{E}[u(W_0(1 + \epsilon))] = u(W_0(1 - \hat{\pi}))$ then the approximation becomes:

$$\hat{\pi}(W_0) \approx \frac{1}{2}\sigma_\epsilon^2 R(W_0)$$

where $R(x) = -\frac{xU_i''(x)}{U_i'(x)}$ is the **Arrow-Pratt measure of relative risk-aversion**.

- In the CARA-Normal setting (combination of negative exponential utility $u(W) = -e^{-\alpha W}$ and normally distributed returns with additive risks), the approximation is exact (does not depend on 'small' ϵ): $\pi = \frac{1}{2}\alpha\sigma_\epsilon^2$

Some classic Utility functions

- Widely used utility functions
 - Quadratic $u(w) = -\frac{1}{2}(w - \zeta)^2$
 - Log $u(w) = \log w$
 - Constant relative risk-aversion (CRRA) $u(w) = \frac{w^{1-\rho}}{1-\rho}$ for $\rho \neq 1$
 - Negative exponential (CARA) $u(w) = -e^{-\alpha w}$
 - Shifted Log: $u(w) = \log(w - \zeta)$
 - Shifted Constant relative risk-aversion (CRRA)
$$u(w) = \frac{\rho}{1-\rho} \left(\frac{w-\zeta}{\rho} \right)^{1-\rho} \text{ for } \rho \neq 1$$
- All of them belong to the so-called Hyperbolic absolute risk aversion (HARA) since their absolute risk-aversion is a hyperbola: $A(W) = \frac{1}{a+bW}$

- They form the so-called **Linear Risk-Tolerance** family of utility function:

$$\tau(W) = \frac{1}{A(W)} = a + bW$$

- . It will be central to many aggregation results.
- See discussion in Back p.12 for interpretation of the parameters.
- You should calculate A, R, τ for all these utility functions. Also, note that the log utility function can be recovered as the limiting case of the CRRA utility function when $\rho = 1$ (applying l'Hopital's rule to $u(w) = \frac{w^{1-\rho}-1}{1-\rho}$).

Consumption Euler Equation I

- If consumers maximize

$$\begin{aligned} U(C_0, C_1) &= u(C_0) + e^{-\rho} \underbrace{E[u(C_1)]}_{\text{subjective probabilities}} \\ &= u(C_0) + e^{-\rho} \sum_{s=1}^S \underbrace{p_s}_{\text{subjective probability of state } s} U(C_{1,s}) \end{aligned} \quad (1)$$

under the budget constraint

$$C_0 = w_0 - \theta' p, \quad C_1 = w_1 + \theta' d \quad (2)$$

where d = payoff, then the first order condition is

$$-u'(C_0)p + e^{-\rho} E[u'(C_1)d] = 0. \quad (3)$$

Consumption Euler Equation II

Defining

$$M = e^{-\rho} \frac{u'(C_1)}{u'(C_0)} = MRIS,$$

we get

$$p = e^{-\rho} E[u'(C_1)d] / u'(C_0) = E[Md] = \sum_{s=1}^S \underbrace{\pi_s M_s}_{=q_s} d_s \quad (4)$$

In the multi-period case,

$$U(C) = \sum_{t=0}^T e^{-\rho t} E[u(C_t)]. \quad (5)$$

Marginal argument: we can always move a bit of consumption from t to $t + 1$.

Consumption Euler Equation III

Consider an optimal feasible consumption plan C_t and a deviation whereby on an $\mathcal{F}_t$ measurable state A , we have $\tilde{C}_t = C_t - \epsilon \mathbf{1}_A$ and at $t+1$ we have $\tilde{C}_{t+1}(\omega) = C_{t+1}(\omega) + \epsilon \hat{R}_{t+1}(\omega) \mathbf{1}_A$, where

$$\hat{R}_{t+1} = \frac{P_{t+1} + d_{t+1}}{P_t}. \quad (6)$$

Note that $\tilde{C}_t$ is feasible and only differs from C_t on A :

$$-e^{-\rho t} E[U'(C_t) \mathbf{1}_A] + e^{-\rho(t+1)} E[U'(C_{t+1}) \hat{R}_{t+1}(\omega) \mathbf{1}_A] = 0 \quad (7)$$

that is

$$E_t \left[e^{-\rho} \frac{U'(C_{t+1})}{U'(C_t)} \hat{R}_{t+1} \right] = 1. \quad (8)$$

Thus, for time-separable utility, the stochastic discount factor is

$$M_{t,t+1} = e^{-\rho} \frac{U'(C_{t+1})}{U'(C_t)} \quad (9)$$

Consumption Euler Equation IV

Thus, **state price are separable**: If the time- t state of the world is ω_t , then

$$M_{t,t+1} = M(\omega_t, \omega_{t+1}) = e^{-\rho} \frac{U'(C(\omega_{t+1}))}{U'(C(\omega_t))} \quad (10)$$

The Ross Recovery Theorem

- X_t is a Markov chain capturing the “true” state of the world
- Asset prices satisfy

$$P_t = E_t[M_{t,t+1} (P_{t+1} + d_{t+1})]$$

- In the Markov world,

$$P(i) = \sum_j \pi_{i,j} M(i,j) (P(j) + d(j)) = \sum_j q(i,j) (P(j) + d(j))$$

- $q(i,j)$ are the “state prices”. They are “kind of observable”
- **Question:** Can we **recover** $\Pi = (\pi_{i,j})$ from $Q = (q_{i,j})$?
- **This is the “holy grail” of asset pricing:** Can we recover (**beliefs about**) physical probabilities from asset prices

- Ross' insight: when $M(i,j) = \delta U(j)/U(i)$ then we can
- **Trick:**

$$Q = \delta \operatorname{diag}(U)^{-1} \Pi \operatorname{diag}(U)$$

and hence Q is **similar** to Π

$$\operatorname{diag}(U) Q \operatorname{diag}(U)^{-1} = \delta \Pi$$

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$$\operatorname{diag}(U) Q \operatorname{diag}(U)^{-1} \mathbf{1} = \delta \Pi \mathbf{1} = \delta \mathbf{1}$$

Theorem

Let

$$Qz = \delta z$$

be the unique positive right eigenvector of Q (by the Perron-Frobenius Theorem, it is unique and strictly positive). Then, $U = z^{-1}$ and

$$\Pi = \delta^{-1} \operatorname{diag}(U) Q \operatorname{diag}(U)^{-1}.$$

Kreps-Porteus-Epstein-Zin Utility I

- Recursive utility seeks to separate risk-aversion from EIS. The continuation utility of a consumption stream is now defined recursively via:

$$\begin{aligned} V_t^\rho &= (1 - e^{-\beta}) C_t^\rho + e^{-\beta} \mathbb{E}_t[V_{t+1}^\alpha]^{\rho/\alpha} \quad \forall \rho \neq 0 \\ \log V_t &= (1 - e^{-\beta}) \log C_t + e^{-\beta} \log \mathbb{E}_t[V_{t+1}^\alpha]^{1/\alpha} \quad \text{if } \rho = 0 \end{aligned}$$

- One verifies that if $\rho = \alpha$, then V_t is of the time-separable CRRA form. Further, Epstein and Zin (1989) show that for gambles that are risk-free, the continuation utility is of the form constant EIS (i.e., identical to the CRRA time separable model) with an EIS coefficient $\psi = \frac{1}{1-\rho}$. Instead, if agents consider one-period risky gambles, then the risk-premium they are willing to pay to avoid such a gamble is identical to that a CRRA utility agent would pay with a CRRA coefficient equal to $\gamma = 1 - \alpha$. In that sense, this utility separates the coefficient of EIS from that of relative risk aversion. It turns out it also leads to a preference for early resolution of uncertainty, as we now show. The following is based on Epstein, Farhi, and Strzalecki (2014).

- Suppose that

$$C_{t+1} = C_t X_{t+1}, \log X_{t+1} \sim N(\mu, \sigma^2), \text{ i.i.d.},$$

so that C_t is a *geometric random walk* and let

$$c_t = \log C_t.$$

What is the continuation utility of such a consumption stream? Let's first focus on the case $\rho = 0$, where calculations are simple. We have

$$c_{t+1}|c_t \sim N(\mu + c_t, \sigma^2), E_t[e^{ac_{t+1}}] = e^{a(\mu+c_t)+0.5a^2\sigma^2}$$

Making an *Ansatz* $\log V_t = a + bc_t$ and then plugging into the recursion

$$\log V_t = (1 - e^{-\beta}) \log C_t + e^{-\beta} \log \mathbb{E}_t[V_{t+1}^\alpha]^{1/\alpha},$$

we obtain

$$a + bc_t = (1 - e^{-\beta})c_t + e^{-\beta} \log \mathbb{E}_t[e^{a\alpha + b\alpha c_{t+1}}]^{1/\alpha}$$

or

$$a + bc_t = (1 - e^{-\beta})c_t + e^{-\beta}(a + b(c_t + \mu) + b^2\alpha\sigma^2/2)$$

Equating terms in c_t and constants, we find two equations that are easily solved for a, b . We find $b = 1$ and

$$a = e^{-\beta} \frac{(\mu + \alpha\sigma^2/2)}{1 - e^{-\beta}} \text{ and thus}$$

$$\log V_0 = c_0 + e^{-\beta} \frac{\mu}{1 - e^{-\beta}} + \alpha e^{-\beta} \frac{\sigma^2/2}{(1 - e^{-\beta})}$$

Now, when $\rho \neq 0$, we make the same Ansatz and substitute it into

$$V_t^\rho = (1 - e^{-\beta})C_t^\rho + e^{-\beta} \mathbb{E}_t[V_{t+1}^\alpha]^\rho/\alpha$$

to get

$$e^{\rho a + \rho b c_t} = (1 - e^{-\beta})e^{\rho c_t} + e^{-\beta} \mathbb{E}_t[e^{\alpha a + \alpha b c_{t+1}}]^{\rho/\alpha}$$

It follows

$$\begin{aligned} e^{\rho a + \rho b c_t} &= (1 - e^{-\beta})e^{\rho c_t} + e^{-\beta} [e^{\alpha a + \alpha b(c_t + \mu) + \frac{1}{2}\alpha^2 b^2 \sigma^2}]^{\rho/\alpha} \\ &= (1 - e^{-\beta})e^{\rho c_t} + e^{\rho b c_t} e^{-\beta} e^{\rho a + \rho b(\mu) + \frac{1}{2}\alpha \rho b^2 \sigma^2} \end{aligned} \quad (11)$$

Dividing this identity by $e^{\rho c_t}$, we get

$$e^{\rho a + \rho(b-1)c_t} = (1 - e^{-\beta}) + e^{\rho(b-1)c_t} e^{-\beta} e^{\rho a + \rho b \mu + \frac{1}{2}\alpha \rho b^2 \sigma^2} \quad (12)$$

Matching the terms, we find that $b = 1$ and

$$e^{\rho a} = (1 - e^{-\beta}) + e^{\rho a - \beta + \rho \mu + \frac{1}{2}\rho \alpha \sigma^2},$$

so that

$$e^{-\rho a} = \frac{1 - e^{-\beta + \rho \mu + \frac{1}{2}\rho \alpha \sigma^2}}{(1 - e^{-\beta})}$$

- To see the effect of **early resolution** of uncertainty, consider the case where all consumption risk is resolved next period. So in period 1 the agents knows the whole future path $c_1, \dots, c_\infty$. This utility at time 1 is simply

$$\begin{aligned} V_1^\rho &= (1 - e^{-\beta})C_1^\rho + e^{-\beta}V_2^\rho \\ &= (1 - e^{-\beta})(C_1^\rho + e^{-\beta}C_2^\rho + e^{-2\beta}C_3^\rho + \dots) \end{aligned}$$

- For general ρ , we do not know the distribution of this future utility stream (since it is an infinite stream of log-normal random variables. However, when $\rho = 0$, then this expression simplifies: We have

$$\log C_t = c_t = c_{t-1} + \Delta c_t, \quad \Delta c_t = \log X_t,$$

and therefore

$$\begin{aligned}\log V_1^{\text{early resol}} &= (1 - e^{-\beta}) \log C_1 + e^{-\beta} \log V_2^{\text{early resol}} \\ &= c_1 + e^{-\beta} (\log V_2^{\text{early resol}} - c_1) \\ &= (c_0 + \Delta c_1) + e^{-\beta} ((1 - e^{-\beta}) \log C_2 + e^{-\beta} \log V_3^{\text{early resol}} - c_1) \\ &= c_0 + \Delta c_1 + \Delta c_2 + e^{-\beta} (\log V_3^{\text{early resol}} - c_2) \\ &= \dots \\ &= c_0 + \Delta c_1 + e^{-\beta} \Delta c_2 + e^{-2\beta} \Delta c_3 + \dots\end{aligned}$$

which is normally distributed with mean

$$\begin{aligned}M_1 &= E[c_0 + \Delta c_1 + e^{-\beta} \Delta c_2 + e^{-2\beta} \Delta c_3 + \dots | c_0] \\ &= c_0 + \sum_i e^{-\beta i} E[\Delta c_i] \\ &= c_0 \frac{\mu}{1 - e^{-\beta}}\end{aligned}\tag{13}$$

and variance

$$\begin{aligned}\mathbb{V}_1 &= \text{Var}[c_0 + \Delta c_1 + e^{-\beta} \Delta c_2 + e^{-2\beta} \Delta c_3 + \dots | c_0] \\ &= \sum_i e^{-2\beta i} \text{Var}[\Delta c_i] \\ &= \sum_i e^{-2\beta i} \sigma^2 = \frac{\sigma^2}{1 - e^{-2\beta}}\end{aligned}\tag{14}$$

- We can thus find the continuation utility at date 0 from this early resolution consumption plan:

$$E_0[(V_1^{\text{early resol}})^\alpha] = E_0[e^{\alpha \log V_1^{\text{early resol}}}] = e^{\alpha M_1 + 0.5\alpha^2 \mathbb{V}_1}$$

and

$$\begin{aligned} \log V_0^{\text{early resol}} &= (1 - e^{-\beta}) \log C_0 + e^{-\beta} \log E_0[(V_1^{\text{early resol}})^\alpha]^{1/\alpha} \\ &= (1 - e^{-\beta}) c_0 + e^{-\beta} (M_1 + \tfrac{1}{2} \alpha \mathbb{V}_1) \\ &= c_0 + e^{-\beta} \frac{\mu}{1 - e^{-\beta}} + \alpha e^{-\beta} \frac{\sigma^2/2}{(1 - e^{-2\beta})} \end{aligned}$$

If we compare both utility streams we see that

$$\log V_0^{\text{early resol}} - \log V_0 = -\alpha e^{-\beta} \sigma^2/2 \frac{e^{-\beta}}{(1 - e^{-2\beta})}. \text{ Thus:}$$

- if $\sigma = 0$ they are both equal.
- if $\alpha = 0 = \rho$ then both are equal.
- if $\sigma > 0$ then early resolution is (i) preferred if $\alpha < 0 = \rho$, but (ii) disliked if $\alpha > 0 = \rho$.

- There is another interesting alternative consumption plan to consider. Suppose instead that the agent is proposed a consumption stream, where in period one he will get one random draw Δc_1 and that for all future dates $\Delta c_i = \Delta c_1$, that is future consumption shocks are perfectly correlated; there is no **time-diversification** in consumption shocks. Let's call the utility associated with this consumption plan $\mathcal{V}$, then as before (see (13)):

$$\begin{aligned}
 \log \mathcal{V}_1 &= (1 - e^{-\beta}) \log C_1 + e^{-\beta} \log \mathcal{V}_2 \\
 &= c_0 + \Delta c_1 + e^{-\beta} \Delta c_2 + e^{-2\beta} \Delta c_3 + \dots \\
 &= \{ \text{by assumption, } \Delta c_t = \Delta c_1 \} = c_0 + \Delta c_1 \sum_i e^{-\beta i} \\
 &= c_0 + \Delta c_1 / (1 - e^{-\beta}),
 \end{aligned}$$

which is normally distributed with mean $M_1 = c_0 + \frac{\mu}{1-e^{-\beta}}$ and variance $\mathbb{V}_1^c = \frac{\sigma^2}{(1-e^{-\beta})^2}$. Note that

$$E[\log V_1^{\text{early resol}}] = E[\log \mathcal{V}_1]$$

but

$$\text{Var}[\log \mathcal{V}_1] > \text{Var}[\log V_1^{\text{early resol}}] : \quad (15)$$

$V_1^{\text{early resol}}$ has more time – diversification

- We can thus find the continuation utility at date 0 from this "correlated" consumption plan. It solves

$$\begin{aligned}\log \mathcal{V}_0 &= (1 - e^{-\beta})c_0 + e^{-\beta}(M_1 + \frac{1}{2}\alpha\mathbb{V}_1^c) \\ &= c_0 + e^{-\beta}\frac{\mu}{1 - e^{-\beta}} + \alpha e^{-\beta}\frac{\sigma^2/2}{(1 - e^{-\beta})^2}\end{aligned}$$

If we compare both utility streams we see that

$\log \mathcal{V}_0 - \log V_0 = -\alpha e^{-2\beta} \frac{\sigma^2/2}{(1-e^{-\beta})^2}$. Thus:

- if $\sigma = 0$ they are both equal.
- if $\alpha = 0 = \rho$ then both are equal.
- if $\sigma > 0$ then (i) $\mathcal{V}_0 \geq V_0$ if $\alpha < 0 = \rho$, but (ii) $\mathcal{V}_0 \leq V_0$ if $\alpha > 0 = \rho$.

- Note that the difference $\mathcal{V}_0 - V_0$ contains two components, one due to early resolution and one due to time-diversification. We can isolate the pure **time-diversification** component by comparing $\mathcal{V}_0$ to $V_0^{\text{early resol}}$. in particular, note:

$$\begin{aligned}
 \log V_0^{\text{early resol}} - \log \mathcal{V}_0 &= c_0 + e^{-\beta} \frac{\mu}{1 - e^{-\beta}} + \alpha e^{-\beta} \frac{\sigma^2/2}{(1 - e^{-2\beta})} \\
 &- \left(c_0 + e^{-\beta} \frac{\mu}{1 - e^{-\beta}} + \alpha e^{-\beta} \frac{\sigma^2/2}{(1 - e^{-\beta})^2} \right) \\
 &= -2\alpha e^{-\beta} \frac{\sigma^2/2}{(1 - e^{-\beta})^2(1 + e^{-\beta})}
 \end{aligned} \tag{16}$$

Thus:

- if $\sigma = 0$ they are both equal.
 - if $\alpha = 0 = \rho$ then both are equal.
 - if $\sigma > 0$ then **time-diversification** is preferred if $\alpha < 0 = \rho$, but (ii) disliked if $\alpha > 0 = \rho$.
- This result can be generalized to $\rho \neq 0$. In general, investors prefer **early resolution** of uncertainty and **time-diversification** if $\alpha > \rho$.

- As we will see next, a preference for the early resolution of uncertainty is crucial to explain asset pricing puzzles in the Bansal-Yaron paper.
- Epstein et al. (2015) extends the analysis above to discuss the magnitude of the preference for early resolution. They argue that for the preference parameters chosen by BY, agents would be willing to give up an unreasonably (based on introspection) a large amount of wealth (or per-period consumption) to move from V to $V^{early\ resol}$. So, in a sense, they argue that while the RRA and EIS coefficients seem "reasonable", the combination of both within the EZ utility leads to an unreasonably high aversion to late resolution *for the parameters and endowment process chosen by BY*. So perhaps, the puzzle is still alive...

The stochastic discount factor in the EZ economy

- Suppose we want to value long-term assets in a BY economy.
- Let's derive the Euler equation for the EZ agent.
- Consider an optimal feasible consumption plan C_t and a deviation whereby on an $\mathcal{F}_t$ measurable state A , we have $\tilde{C}_t = C_t - \epsilon \mathbf{1}_A$ and at $t + 1$ we have $\tilde{C}_{t+1}(\omega) = C_{t+1}(\omega) + \epsilon R_{t+1}(\omega) \mathbf{1}_A$. Note that $\tilde{C}_t$ is feasible and only differs from C_t on A .
- Then $\tilde{V}_t = V_t - \frac{\partial V_t}{\partial C_t} \epsilon \mathbf{1}_A + \sum_{\omega} \frac{\partial V_t}{\partial C_{t+1}(\omega)} \epsilon \mathbf{1}_A R_{t+1}(\omega)$
- $\pi_t(\omega)$ the *conditional probabilities*

- Setting $\lim_{\epsilon \rightarrow 0} \frac{\tilde{V}_t - V_t}{\epsilon} = 0$ we get the relation:

$$\sum_{\omega} \frac{\frac{\partial V_t}{\partial C_{t+1}(\omega)}}{\frac{\partial V_t}{\partial C_t}} \mathbf{1}_A R_{t+1}(\omega) = 1 \quad \forall A \in \mathcal{F}_t$$

This implies:

$$\mathbb{E}_t \left[\frac{M_{t+1}}{M_t} R_{t+1} \right] = 1$$

where the stochastic discount factor M_t is defined by

$$\underbrace{\pi_t(\omega)}_{\text{physical prob}} \underbrace{\frac{M_{t+1}}{M_t}}_{\text{SDF}} = \frac{\frac{\partial V_t}{\partial C_{t+1}(\omega)}}{\frac{\partial V_t}{\partial C_t}}$$

$\frac{\partial V_t}{\partial C_{t+1}(\omega)}$ contains physical probabilities because V contains expectations

- Now we use the chain rule to rewrite

$$\pi_t(\omega) \frac{M_{t+1}}{M_t} = \frac{\frac{\partial V_t}{\partial C_{t+1}(\omega)}}{\frac{\partial V_t}{\partial C_t}} = \frac{\partial V_t}{\partial V_{t+1}(\omega)} \frac{MC_{t+1}(\omega)}{MC_t}$$

with

$$MC_t = \frac{\partial V_t}{\partial C_t}, \quad MC_{t+1} = \frac{\partial V_{t+1}}{\partial C_{t+1}}$$

- Recall that

$$V_t^\rho = (1 - e^{-\beta})C_t^\rho + e^{-\beta}\mathbb{E}_t[V_{t+1}^\alpha]^{\rho/\alpha} \quad \forall \rho \neq 0$$



$$MC_t = \frac{\partial V_t}{\partial C_t} = (1 - e^{-\beta}) \left(\frac{V_t}{C_t} \right)^{1-\rho}$$
$$\frac{\partial V_t}{\partial V_{t+1}(\omega)} = e^{-\beta} V_t^{1-\rho} \mathbb{E}_t[V_{t+1}^\alpha]^{\frac{\rho-\alpha}{\alpha}} \pi_t(\omega) V_{t+1}^{\alpha-1}(\omega)$$

Combining we get a state price deflator:

$$\frac{M_{t+1}}{M_t} = e^{-\beta} \left(\frac{C_{t+1}}{C_t} \right)^{(\rho-1)} \left(\frac{V_{t+1}}{\mathbb{E}_t[V_{t+1}^\alpha]^{(1/\alpha)}} \right)^{\alpha-\rho}$$