

Asset Pricing Theory

Final Exam

1. One period complete market Equilibrium

Consider an economy with two dates 0 and 1. At date 1 there are two states s_1, s_2, s_2 that occur with equal probabilities $1/3$. There are three securities traded, with payoff vectors $X_i = (1^i, 2^i, 3^i)$. All securities are in zero net supply. There are two agents a_1, a_2 with endowments $\omega_{k,t}, k = 1, 2$. Note that $\omega_{k,1}$ is random. We denote by Ω_t the aggregate endowment.

Agent a_k maximizes expected utility of consumption of the form $-e^{-\gamma_k c_{k,0}} + \delta_k E[-e^{-\gamma_k c_{k,1}}]$

1. Prove that markets are complete
2. Write the single inter-temporal budget constraint in terms of the state-price density M .
3. Solve the utility maximization problem of each agent in terms of M . **Hint:** You also need to solve for the Lagrange multiplier λ for the single budget constraint. This can be done in closed form! When you write the budget constraint, it will become an equation for λ .
4. write down the market clearing for consumption.
5. solve for the equilibrium state price density M in terms of Ω_1 and the Lagrange multipliers λ_k . **Do not solve for λ_k !!**
6. find the actual portfolio of the basic securities that the agents hold in equilibrium. **Hint:** do not forget the role of their endowments!

Proof. We have

$$u'(c_0) = \lambda$$

and

$$\delta u'(c_1) = \lambda M$$

that is

$$e^{-\gamma c_0} = \lambda$$

and

$$\delta e^{-\gamma c_1} = \lambda M$$

so that

$$c_1 = -\gamma^{-1} \log(\lambda \delta^{-1} M), \quad c_0 = -\gamma^{-1} \log(\lambda)$$

and the budget constraint is

$$-\gamma^{-1} \log(\lambda) - E[\gamma^{-1} \log(\lambda M) M] = \omega_0 + E[\omega_1 M]$$

which implies

$$\log \lambda = \frac{-\omega_0 - E[\omega_1 M] - \gamma^{-1} E[M \log M]}{\gamma^{-1} (1 + E[M])}$$

Market clearing implies

$$\sum_k -\gamma_k^{-1} \log(\lambda_k \delta_k^{-1} M) = \Omega_1$$

which gives

$$\sum_k \gamma_k^{-1} \log M = -\Omega_1 - \sum_k \gamma_k^{-1} \log(\lambda_k \delta_k^{-1})$$

which gives

$$M = e^{-\gamma(\Omega_1 + K)}$$

where

$$\gamma = \left(\sum_k \gamma_k^{-1} \right)^{-1}$$

and

$$K = \sum_k \gamma_k^{-1} \log(\lambda_k \delta_k^{-1})$$

Substituting into the budget constraints gives

$$\log \lambda_k = \frac{-\omega_{0,k} - E[\omega_{1,k} e^{-\gamma(\Omega_1 + K)}] - \gamma^{-1} E[e^{-\gamma(\Omega_1 + K)} \log e^{-\gamma(\Omega_1 + K)}]}{\gamma^{-1}(1 + E[e^{-\gamma(\Omega_1 + K)}])}$$

2. Representative Agent Suppose that the aggregate endowment of an economy follows a multiplicative process: $Y_t = X_1 \cdots X_t$ where X_t follows a Markov chain $X_t = x_1, x_2$ with a transition probability matrix $\Pi \in \mathbb{R}^{2 \times 2}$. The economy is populated by a representative agent with a CRRA utility $u(c) = c^{1-\gamma}/(1-\gamma)$ and a discount factor $e^{-\rho}$, who consumes the aggregate endowment Y_t so that his consumption coincides with Y_t .

- Find the state price density in terms of Y_t
- Use the Kolmogorov equation from the Stochastic Calculus Class to find the price of a security that is a claim on the infinite stream (like a stock) of dividends equal to Y_t^α for some α . (No need to actually invert the matrix, just write the expression in terms of the inverse of a matrix)
- Use Kolmogorov equation to find the time $-t$ price of a risk-free bond with T periods to expiry in terms of a power of a matrix. **Hint:** It will not be Π , but it will be related.

Proof. We know that the pricing kernel is given by $M_t = e^{-\rho t} u'(c_t) = e^{-\rho t} u'(Y_t) = e^{-\rho t} Y_t^{-\gamma}$. We have

$$\begin{aligned}
 P_t &= E_t \left[\sum_{\tau=t}^{\infty} (M_\tau / M_t) Y_\tau^\alpha \right] = E_t \left[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)} Y_\tau^{\alpha-\gamma} Y_t^\gamma \right] \\
 &\quad \underbrace{=}_{\text{trick: } Y_t \text{ is multiplicative, hence } P_t \text{ scales with } Y_t^\alpha} \\
 &= Y_t^\alpha E_t \left[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)} \left(\frac{Y_\tau}{Y_t} \right)^{\alpha-\gamma} \right]
 \end{aligned} \tag{1}$$

Let us define

$$p(X_t) = E_t \left[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)} \left(\frac{Y_\tau}{Y_t} \right)^{\alpha-\gamma} \right] \tag{2}$$

Let us derive the Kolmogorov equation for p :

$$\begin{aligned}
 p(X_t) &= E_t \left[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)} \left(\frac{Y_\tau}{Y_t} \right)^{\alpha-\gamma} \right] = 1 + E_t \left[\sum_{\tau=t+1}^{\infty} e^{-\rho(\tau-t)} \left(\frac{Y_\tau}{Y_t} \right)^{\alpha-\gamma} \right] \\
 &= 1 + E_t \left[(Y_{t+1}/Y_t)^{\alpha-\gamma} \sum_{\tau=t+1}^{\infty} e^{-\rho(\tau-t)} \left(\frac{Y_\tau}{Y_{t+1}} \right)^{\alpha-\gamma} \right] \quad \underbrace{\text{law of iterated expectations}} \\
 &= 1 + E_t \left[(Y_{t+1}/Y_t)^{\alpha-\gamma} E_{t+1} \left[\sum_{\tau=t+1}^{\infty} e^{-\rho(\tau-t)} \left(\frac{Y_\tau}{Y_{t+1}} \right)^{\alpha-\gamma} \right] \right] \\
 &= \{Y_{t+1}/Y_t = X_{t+1}\} \\
 &= 1 + E_t \left[X_{t+1}^{\alpha-\gamma} p(X_{t+1}) \right]
 \end{aligned} \tag{3}$$

Denote $p(x_i) = p_i$, and we get the system

$$p_i = 1 + E_t \left[X_{t+1}^{\alpha-\gamma} p(X_{t+1}) | X_t = x_i \right] = 1 + \sum_j \pi_{i,j} x_j^{\alpha-\gamma} p_j \tag{4}$$

Define the matrix $A = (\pi_{i,j} x_j^{\alpha-\gamma})_{i,j=1}^2 = \Pi \text{diag}(x_i^{\alpha-\gamma})$. Then, the system is

$$p = \mathbf{1} + Ap \tag{5}$$

and the solution is

$$p = (I - A)^{-1} \mathbf{1}. \tag{6}$$

3. Dynamic Equilibrium with Complete Markets

The economy is populated by two agents maximizing

$$E\left[\sum_{t=0}^{\infty} e^{-\rho_k t} c_{k,t}^{1-\gamma} / (1-\gamma)\right], \quad k = 1, 2$$

Agent 1 endowment follows $Y_t = X_1 \cdots X_t$ where X_t are i.i.d., $\log X_t = N(\mu, \sigma^2)$.

Agent 2 is endowed with ω_0 units of cash at time zero. Markets are complete. Aggregate endowment is $\Omega_t = Y_t + \mathbf{1}_{t=0}\omega_0$.

- Since markets are complete, there is a unique state price density process M_t . Write the single inter-temporal budget constraint and find the optimal consumption stream $c_{k,t}$ as a function of M_t . Also express the Lagrange multiplier λ_k in terms of M
- Substitute in into the consumption market clearing, $\sum_k c_{k,t} = \Omega_t$ and solve for the equilibrium M_t
- Find the equilibrium price of the endowment claim (the claim on the infinite stream of Y_t) and the one period risk free rates $r_t = -\log E_t[M_{t+1}/M_t]$.
- Find the wealth $W_{k,t}$ of agent k at time t based on the inter-temporal budget constraint

Proof. We have

$$e^{-\rho_k t} c_{k,t}^{-\gamma} = \lambda_k M_t$$

which gives

$$c_{k,t} = e^{-\rho_k \gamma^{-1} t} \lambda_k^{-1/\gamma} M_t^{-1/\gamma}$$

so that the budget constraint

$$E\left[\sum_t M_t c_{k,t}\right] = E\left[\sum_k \omega_{k,t} M_t\right]$$

where $\omega_{k,t}$ is the time- t endowment of agent k . This gives

$$E\left[\sum_t M_t e^{-\rho_k \gamma^{-1} t} \lambda_k^{-1/\gamma} M_t^{-1/\gamma}\right] = E\left[\sum_k \omega_{k,t} M_t\right]$$

implying the expression for the Lagrange multiplier:

$$\lambda_k^{-1/\gamma} = \frac{E[\sum_k \omega_{k,t} M_t]}{E[\sum_t e^{-\rho_k \gamma^{-1} t} M_t^{1-1/\gamma}]}$$

At any given point in time, we must have

$$\underbrace{E_t\left[\sum_{\tau} (c_{k,\tau} - \omega_{k,\tau}) M_t\right]}_{\text{future excess consumption}} = W_{k,t}$$

Market clearing

$$\sum_k e^{-\rho_k \gamma^{-1} t} \lambda_k^{-1/\gamma} M_t^{-1/\gamma} = \Omega_t$$

implies

$$M_t = \Omega_t^{-\gamma} \left(\sum_k e^{-\rho_k \gamma^{-1} t} \lambda_k^{-1/\gamma}\right)^{1/\gamma}$$