EPFL & SFI

**Asset Pricing Theory**

# Final Exam

## 1. One period complete market Equilibrium

Consider an economy with two dates, 0 and 1. At date 1, three states $s_1, s_2, s_2$ occur with equal probabilities 1/3. Two securities are traded, with payoff vectors $X_1 = (1, 1, 1)$ and $X_2 = (1, 2, 10)$. All securities are in zero net supply. There are two agents 1, 2 with endowments $\omega_{k,t}, k = 1, 2$, where the time-zero endowments are $\omega_{k,0} = k$ for $k = 1, 2$, and the time-1 endowments are given $\omega_{1,1} = (2, 3, 11)$ and $\omega_{2,1} = (0, 1, 9)$ (across the three states $s_1, s_2, s_2$). We denote by $\Omega_t$ the aggregate endowment.

Agent $k$ maximizes expected utility of consumption of the form $c_{k,0}^{1-\gamma} + \delta_k E[c_{k,1}^{1-\gamma}]$.

1. Prove that markets are incomplete, but are *effectively complete*. Hence, we can solve for equilibrium as if the markets were complete. Please explain the logic behind this argument.
2. Write the single inter-temporal budget constraint in terms of the state-price density $M$.
3. Solve the utility maximization problem of each agent in terms of $M$.
4. write down the market clearing for consumption.
5. solve for the equilibrium state price density $M$ in terms of $\Omega_1$ and the Lagrange multipliers $\lambda_k$. **Do not solve for $\lambda_k$!!**
6. find the actual portfolio of the basic securities that the agents hold in equilibrium. **Hint:** do not forget the role of their endowments!

**2. Representative Agent** Suppose that the aggregate endowment of an economy follows a multiplicative process: $Y_t = \sum_{\tau=1}^{t} X_\tau$ where $X_t$ follows a Markov chain $X_t = x_1, x_2$ with a transition probability matrix $\Pi \in \mathbb{R}^{2\times 2}$. The economy is populated by a representative agent with a CARA utility $u(c) = -e^{-c\gamma}$ and a time discount factor $e^{-\rho}$,

$$E[\sum_{t=0}^{\infty} u(c_t)e^{-\rho t}]$$

who consumes the aggregate endowment $Y_t$ so that his consumption coincides with $Y_t$.

- Find the state price density in terms of $Y_t$
- Use the Kolmogorov equation to find the price of a security that is a claim on the infinite stream (like a stock) of dividends equal to $\alpha X_t^{15}$ for some $\alpha$. (No need to invert the matrix, write the expression in terms of the inverse of a matrix)
- Use the Kolmogorov equation to find the time -$t$ price of a risk-free bond with $T$ periods to expiry in terms of a power of a matrix. **Hint:** It will not be $\Pi$, but it will be related. **Last Hint:** Define a new Markov process $Z_t = e^{-\gamma X_t}$ and express everything in terms of this process.

## 3. Dynamic Equilibrium with Complete Markets with Heterogeneous Beliefs

The economy is populated by two agents maximizing

$$E[\sum_{t=0}^{\infty} e^{-\rho t} Z_t^{\alpha_k} c_{k,t}^{1-\gamma}/(1-\gamma)],\ k=1,2,\ \alpha_1 > \alpha_2,$$

where

$$Z_t \ = \ X_1 \cdots X_t$$

and where $X_t$ are i.i.d., $\log X_t = N(\mu, \sigma^2)$. Each agent $k$ has the same constant endowment $Y_{k,t} = 1$ for all $k = 1, 2$ and all $t$. There are two securities traded in the market: a risk-free asset with a rate $R_{f,t}$ at time $t$ and a stock paying the dividend $Z_t^3$ at time $t$.

- Since markets are complete; there is a unique state price density process $M_t$. Write the single inter-temporal budget constraint and find the optimal consumption stream $c_{k,t}$ as a function of $M_t$. Also express the Lagrange multiplier $\lambda_k$ in terms of $M$
- Substitute in into the consumption market clearing, $\sum_k c_{k,t} = \sum_{k=1}^{2} Y_{k,t}$ and solve for the equilibrium $M_t$
- Find the equilibrium price of the stock (the claim on the infinite stream of $Z_t^3$) and the one-period risk-free rates $R_{f,t} =$
- Find the wealth $W_{k,t}$ of agent $k$ at time $t$ based on the inter-temporal budget constraint (do not forget the endowments!) and write the expression for optimal portfolios of the two agents.