EPFL & SFI

Asset Pricing Theory

Final Exam

1. One period Equilibrium

Consider an economy with two dates 0 and 1. At date 1 there are two states $s = (a, b)$ that occur with equal probabilities 1/2. There is only one security traded: the risk free bond that pays 1 in each state. It is in zero net supply in equilibrium. There are two agents $i = J, K$ who maximize their expected utility of consumption of the form $\log(c_0) + E[\log(c_1(s))]$ by trading the risk free bond with each other at date 0.

Agents have initial endowments of $e_J(0)$ and $e_K(0)$. Their time 1 endowment is respectively $e_J(1) = (w, W)$ and $e_K(1) = (z, z)$.

1. Derive explicitly the optimal investment into the risk free bond for each agent
2. Prove that these investments are monotone decreasing in the price of the bond
3. Write down the market clearing equation and use 2. to conclude that it has a unique solution
4. Under what conditions on the endowments is the equilibrium allocation Pareto efficient?

2. Dynamic equilibrium Suppose that the aggregate endowment of an economy follows a geometric random walk: $Y_t = X_1 \cdots X_t$ where X_t are i.i.d. positive variables. Markets are dynamically complete. The economy is populated by N agents; agent i is endowed with share α_i of the aggregate endowment and maximizes

$$E[\sum_{t=0}^{\infty} \delta_i^t \log(c_{i,t})]$$

Find:

- the unique equilibrium state price density M_t explicitly
- equilibrium consumption allocation $c_{i,t}$
- assuming that $\delta_1 > \cdots > \delta_N$, show that $\lim_{t\to\infty} c_{i,t}/Y_t = 0$ for all agents but one. Which one?
- explicit formula for the claim on the aggregate endowment (the price P_t of the asset whose dividend stream coincides with the aggregate endowment)
- explicit formula for the price of the risk-free bond $B_{t,T}$ that expires at time T and pays 1 at that time.

3. Dynamic Saving Problem with Exponential Utility and Random Endowment

An agent maximizes

$$E[\sum_{t=0}^{\infty} -\delta^t e^{-\gamma c_t}]$$

and has initial wealth w_0 and a random endowment ϵ_t that is i.i.d. over time. He can only save: invest x_t into a risk free bond at time t that returns $x_t e^r$ at time $t+1$, so that the budget constraint is

$$c_t \ = \ w_t - x_t + \epsilon_t, \ w_{t+1} = e^r x_t \, .$$

Due to the stationary setting, the value function only depends on wealth:

$$V(t, w_t) \ = \ \max E_t[\sum_{\tau=t}^{\infty} -\delta^\tau e^{-\gamma c_t}]$$

- Derive the Bellman equation (maximization is only over x) for $V(t, w_t)$ linking it with $V(t+1, w_{t+1})$
- make an Ansatz $V(t, w_t) = -\delta^t e^{-A w_t}$ with some unknown constant A which does not depend on t
- substitute this Ansatz into the Bellman equation
- use the envelope theorem to show that $c_t = B w_t + K$ for some explicit constants B, K that depend on A.
- Finish deriving a fixed point equation for A (same A will appear on the left- and right-hand sides of the equation)