

Asset Pricing Theory

Final Exam

1. One period Equilibrium

Consider an economy with two dates 0 and 1. At date 1 there are three states $s = (a, b, c)$ that occur with equal probabilities $1/3$. There are three securities traded. One is a stock with time 0 price S with payoff at date 1 of $D = (1, 2, 3)$. The other is a derivative with a payoff of $(0, 0, 1)$. The third is a risk-free bond with time 0 price B and which pays 1 in every future state. All securities are in zero net supply in equilibrium. There are two agents $i = J, K$ who maximize their expected utility of consumption of the form $c_0^{1/2} + E[c_1(s)^{1/2}]$ by trading in the three securities with each other at date 0.

Both agents have an initial endowment of $e_J(0) = 2$ and $e_K(0) = 1$. Their time 1 endowment is respectively $e_J(1) = (2, 3, 4)$ and $e_K(1) = (1, 3, 5)$.

1. Derive the optimal consumption allocation, the price of the stock and the risk-free rate in this economy.
2. Give state prices and risk-neutral probabilities for this economy. Are these unique? Is this economy arbitrage-free? Is this a complete market economy? Is the equilibrium Pareto Optimal? (justify each of your answer).
3. Suppose now that agents cannot trade the derivative (the payoff $(0, 0, 1)$). Compute the equilibrium consumption, trades in the stock, and its price. Is the market still complete? Is the equilibrium Pareto optimal? Explain.

2. Mean-Variance portfolio and random endowments

Consider a set of n risky assets with payoffs $R_i \ \forall i = 1, \dots, n$ that are jointly Gaussian and a risk-free asset with return R_f . Consider an economy populated by m investors who seek to optimize

$$E[-e^{-\alpha_i(W_{1,i} + \xi_i)}]$$

where $\xi = (\xi_i)_{i=1}^m$ is a vector of random endowments that is jointly Gaussian with returns so that

$$\text{Cov}(\xi_i, \xi_j) = \sigma_{i,j}^\xi, \quad \text{Cov}(\xi_i, R_k) = \sigma_{i,k}^{\xi,R}$$

and $W_{1,i}$ satisfies the standard budget constraint. Find the optimal portfolio for each investor and then write down market clearing to derive equilibrium asset prices. Under what conditions does the standard CAPM hold?

3. Dynamic Portfolio Choice in Complete Markets

Consider a Markov chain with three possible states, $s = 1, 2, 3$, and transition probabilities $p(i, j)$, $i, j = 1, 2, 3$. There are two risky assets, with prices $P_i(s)$, $i = 1, 2$ that depend on the Markov state. There is also a risk-free asset with rate $r = 0$.

- Under what conditions the market is complete?
- Under what conditions the market is arbitrage free?
- When the market is complete, compute the unique state price density M_t
- Given the state price density, solve the optimal portfolio choice problem for an investor maximizing

$$E\left[\sum_{t=0}^T \delta^t c_t^{1-\gamma} / (1-\gamma)\right]$$

who has an initial wealth of w_0 . **Hint:** first, reduce the problem to an Arrow-Debreu problem and solve for the optimal consumption. Then, compute the wealth process (=present value of future optimal consumption; it is fine to keep it as an expectation; just write it down). Then, find the replicating portfolio.