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On the Aggregation of Information in Competitive Markets

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1. INTRODUCTION

This paper analyzes how a competitive market serves to communicate information between the market participants. The communication process in a market is usually described by the phrase that the equilibrium price "reflects all the available information in the market" and communicates it to the participants (Fama, 1970).

If there is only one piece of information to consider, the meaning of this phrase is unambiguous. Thus the simple news of higher firm profits will be "reflected" in a higher share price because the demand for shares has gone up.

Problems arise when there are many agents with *different* pieces of information, and the vector of market prices has a smaller dimension than the vector of information available to different agents. In this case the equilibrium price vector corresponds to some *aggregate* of all the individual pieces of information. Then the question arises how this aggregate is formed.

In two recent papers, Grossman (1976, 1978) proposed a remarkable solution to this problem. In a model of the capital market, he argues that the equilibrium price aggregates the available information perfectly. If agents $i = 1, 2, \ldots, n$ take their information from the market price, the economy achieves the same allocation as if each trader knew the whole vector $(I_1, \ldots, I_n)$ of informations available to individual agents. Any aspect of the information vector $(I_1, \ldots, I_n)$ that is not reflected in the price is not woth communicating because it would merely be treated as noise.

However, this approach involves some conceptual difficulties. In Grossman's model the aggregation of information through the price depends only on the statistical properties of the information vector $(I_1, \ldots, I_n)$ and is independent of agents' preferences. This is quite surprising. Intuitively one would expect that the weight with which agent i's information I_i affects the equilivrium price should depend on the strength of agent i's reaction to this information, which in turn should depend on his preferences. Presumably it should make a difference whether the news of an increase in a firm's profits is passed to somebody who is almost risk neutral and responds by buying a

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large number of shares, or whether this piece of news is passed to a risk averter who hardly responds at all.

In Grossman's model this intuition fails because agents do not use their own information at all. The point is that if the price reveals all that is worth knowing about the vector $(I_1, \ldots, I_n)$, agents will neglect their individual informations and look at the price only. Any aspect of the information I_i that is not contained in the price is regarded as noise. If agent i uses the information contained in the price, he can afford to disregard the information I_i.

However, if agents do not act on their own informations at all, it is unclear why the price should reflect the informations $I_1, \ldots, I_n$ in the first place. Grossman's proposition does not seem to capture the original idea that individual agents react to their individual informations and *therefore* the equilibrium price reflects some aggregate of the informations $I_1, \ldots, I_n$. Instead, it must be presumed that the "auctioneer" somehow happens to know the vector $I_1, \ldots, I_n$, in which case perfect aggregation of information through the price induces agents to disregard their individual informations and therefore is consistent with market clearing.

Against this objection, Grossman (1975) suggests that in fact there does not exist an equilibrium in which the information that is contained in the equilibrium price can itself be traced back to the demand decisions of those agents who had obtained the information originally.[1]

The equilibrium in which price aggregates information perfectly is unique.

Grossman's argument on this point makes heavy use of agents' awareness that their own information and the information contained in the price are statistically dependent. In particular, agents take account of the covariance between "noise" in their own information and "noise" in the price. It is this covariance which makes them neglect their own information when they pay attention to the information contained in the price.

But then, Grossman's agents are slightly schizophrenic. The covariance between "noise" in individual information and "noise" in the price is nonzero because the number of agents is finite, and each agent exerts a nonnegligible influence on the price.[2] Therefore one should expect that agents who are aware of this covariance will also notice the effect they have on the price. Yet, Grossman's agents are price takers. They do not attempt to manipulate the price and the information content of the price.

In order to avoid these difficulties, the present paper will study the aggregation of information in a large market, in which individual agents have no

[1] This is proved as a theorem under the additional restriction that the equilibrium price depend linearly on the vector of information.

[2] The alternative interpretation of a finite number of types and a unit mass continuum of agents of each type would fail to capture the notion that *each* agent has different information.

influence on the price. This analysis will lead to an alternative view of the aggregation of information through the price.

First, it will be seen that in a large market the relative importance of the information available to agent i depends on his preferences. In particular, I_i is relatively the more important, the less risk averse agent i is.

Second, in a large market, the equilibrium price will reflect only those elements of information that are common to a large number of agents. Because an individual agent does not affect the price, his information enters the price only to the extent that it is shared by other agents.

This second result suggests that the market is a good aggregator of information, if there are many agents with many independent sources of information. In this case, "noise" in the information available to any individual agent is filtered out and does not affect the price.

The plan of the paper is as follows: Sections 2 and 3 introduce a generalized version of Grossman's model with a finite number of agents. In particular, I shall allow for the presence of exogenous noise in the price system.[3] Section 4 analyses equilibrium with a finite number of agents and amplifies the foregoing critique of Grossman's approach. The aggregation of information in a large market is studied in Section 5 by means of a competitive sequence of economies. This section contains the main results of the paper.

2. The Basic Model

I shall use the following version of Grossman's model. There are n agents $i = 1 \cdots n$. Each agent i allocates his initial wealth w_{0i} between a riskless and a risky asset. For each unit purchased, the riskless asset yields 1 unit, the risky asset $\tilde{X}$ units of a single consumption good, where $\tilde{X}$ is a random variable. Using the riskless asset as numeraire, let p be the price of the risky asset. If agent i holds z_i units of the risky asset, his portfolio yields the return

$$\tilde{w}_{1i} = w_{0i} + z_i(\tilde{X} - p).$$

Agents' preferences are described by the following assumption:

A.1. For $i = 1 \cdots n$, agent i maximizes the expected utility of consumption $E_i u_i(\tilde{w}_{1i})$. The utility function u_i exhibits constant absolute risk aversion $\rho_i \in (0, \infty)$.

Under this assumption agent i's demand for the risky asset is independent of his initial wealth w_{0i} (Pratt, 1965). It depends only on the price p and the

[3] On the importance of such exogenous noise, see Grossman, 1977; Grossman and Stiglitz, 1976.

expectations operator E_i, which in turn is determined by the agent's information I_i. If the supply of the risky asset is Z, the market clearing condition takes the form

$$Z = \sum_{i=1}^{n} z_i(p, I_i). \tag{1}$$

The information I_i, on which agent i bases his expectation of $\tilde{X}$, consists of the market price p and his private information y_i. The latter is taken to be the realization of a random variable $\tilde{y}_i$, which communicates the true return $\tilde{X}$ perturbed by some noise $\tilde{\epsilon}_i$:

$$\tilde{y}_i = \tilde{X} + \tilde{\epsilon}_i. \tag{2}$$

Furthermore, the supply Z is taken to be the realization of a random variable $\tilde{Z}$. The following distributional assumption will be imposed:

A.2. The random vector $(\tilde{X}, \tilde{Z}, \tilde{\epsilon}_1 \cdots \tilde{\epsilon}_n)$ has a normal distribution with mean $(\overline{X}, \overline{Z}, 0 \cdots 0)$ and nonsingular variance–covariance matrix: $\sum = (\sigma^2, \Delta^2, s_1{}^2 \cdots s_n{}^2)\, I_{n+2}$, where I_{n+2} is the $(n+2)$-dimensional identity.

Substituting for $I_i = (y_i, p)$ in (1), one finds that the market clearing price p is given as the realization of a random variable $\tilde{p}$ which depends on $\tilde{Z}$ and the vector of signals $(\tilde{y}_1 \cdots \tilde{y}_n)$. The precise form of this relationjhip depends on the functions $z_i(\cdot)$, in particular on the way in which the information I_i affects the expectations operator E_i. Imposing the hypothesis that expectations are rational, I shall require:

A.3. For $i = 1 \cdots n$, agent i knows the actual joint distribution of the triple $(\tilde{X}, \tilde{y}_i, \tilde{p})$. For any information $I_i = (y_i, p)$, he derives the expectations operator E_i from the actual conditional distribution of $\tilde{X}$ given y_i, p.

It should be noted that under Assumption A.3 expectations formation and market clearing cannot be treated separately. The use that agent i makes of his information depends on the joint distribution of $(\tilde{X}, \tilde{y}_i, \tilde{p})$. This in turn depends on the price–supply–signals relation that is imposed by market clearing. Therefore individual expectations formation and demand cannot be analyzed by themselves; from the beginning, the system as a whole must be considered, because the market clearing condition determines the information that agents draw from the market price.

Formally, the determination of equilibrium under the rational expectations assumption A.3 can be treated as a fixed-point problem in the space of functions relating the asset price to supplies and signals. Given any function f: $\mathbb{R}^{n+1} \to \mathbb{R}$, suppose, initially, that agents act on the hypothesis that $\tilde{p} = f(\tilde{Z}, \tilde{y}_1 \cdots \tilde{y}_n)$. Then agent i's asset demand depends on the price p, the signal y_i *and* the function f, which determines the joint distribution of the triple $(\tilde{X}, \tilde{y}_i, \tilde{p})$ and therefore the conditional distribution of $\tilde{X}$ given any realization $I_i = (y_i, p)$. Rewriting the market clearing condition (1) in the form

$$Z = \sum_{i=1}^{n} \bar{z}_i(p, y_i; f), \tag{1a}$$

one finds that the market clearing price depends on f as well as on the vector $(Z, y_1 \cdots y_n)$. With the function f, one can associate a new function Tf: $\mathbb{R}^{n+1} \to \mathbb{R}$, such that $Tf(Z, y_1 \cdots y_n)$ is the market clearing price for the realization $(Z, y_1 \cdots y_n)$ given that agents base their expectations on the hypothesis that $\tilde{p} = f(\tilde{Z}, \tilde{y}_1 \cdots \tilde{y}_n)$. In this formulation, expectations are rational if f is a fixed point of the mapping T, i.e., if $f(Z, y_1 \cdots y_n) = Tf(Z, y_1 \cdots y_n)$ for all $(Z, y_1 \cdots y_n) \in \mathbb{R}^{n+1}$.

Under Assumptions A.1 and A.2, the given fixed-point problem has a *linear* solution. To determine this solution, one proceeds as follows. Consider an arbitrary linear relation:

$$\tilde{p} = \pi_0 + \sum_{i=1}^{n} \pi_i \tilde{y}_i - \gamma \tilde{Z}. \tag{3}$$

Define $\pi \equiv \sum_{i=1}^{n} \pi_i$. Given (2), (3), and Assumption A.2, the triple $(\tilde{X}, \tilde{y}_i, \tilde{p})$ has a normal distribution with mean $(\bar{X}, \bar{X}, \pi_0 + \pi\bar{X} - \gamma\bar{Z})$ and variance–covariance matrix:

$$V_i = \begin{pmatrix} \sigma^2 & \sigma^2 & \pi\sigma^2 \\ \sigma^2 & \sigma^2 + s_i^2 & \pi\sigma^2 + \pi_i s_i^2 \\ \pi\sigma^2 & \pi\sigma^2 + \pi_i s_i^2 & \pi^2\sigma^2 + \sum_{k=1}^{n} \pi_k^2 s_k^2 + \gamma^2 \Delta^2 \end{pmatrix}.$$

From normal distribution theory, the posterior distribution of $\tilde{X}$ given a realization (y_i, p) is again normal, with mean and variance of the form

$$E(\tilde{X} \mid y_i, p) = \alpha_{0i} + \alpha_{1i} y_i + \alpha_{2i} p, \tag{4a}$$

$$\operatorname{Var}(\tilde{X} \mid y_i, p) = \beta_i, \tag{4b}$$

where the values of α_{0i}, α_{1i}, and β_i depend on the matrix V_i.

If expectations are based on relation (3), normality implies that asset demands under expected utility maximization depend only on the posterior mean and variance of returns, $E(\tilde{X} \mid y_i, p)$ and $\operatorname{Var}(\tilde{X} \mid y_i, p)$. In the particular case of constant absolute risk aversion given by Assumption A.1, one has (for a derivation, see, e.g., Grossman (1976, p. 575 f.))

$$\bar{z}_i(p, y_i; \pi_0, \pi_1 \cdots \pi_n, \gamma) = \frac{E(\tilde{X} \mid y_i, p) - p}{\rho_i \operatorname{Var}(\tilde{X} \mid y_i, p)} = \frac{\alpha_{0i} + \alpha_{1i} y_i + (\alpha_{2i} - 1) p}{\rho_i \beta_i}.$$

Substituting for $\bar{z}_i$ in (1a) and solving for p, one has

$$p = \left[\sum_{i=1}^{n} \frac{1 - \alpha_{2i}}{\rho_i \beta_i}\right]^{-1} \left[\sum_{i=1}^{n} \frac{\alpha_{0i} + \alpha_{1i} y_i}{\rho_i \beta_i} - Z\right].^4 \tag{1b}$$

Thus the market clearing price–supply–signals relation (1b) that is induced by the hypothesis (3) is again linear. Expectations based on (3) are rational, if and only if the coefficients π_0, π_i, γ in (3) are the same as the corresponding coefficients in (1b). This yields the following conditions:

$$\gamma = \left[\sum_{i=1}^{n} \frac{1 - \alpha_{2i}}{\rho_i \beta_i}\right]^{-1}, \tag{5a}$$

$$\pi_i = \frac{\gamma}{\rho_i} \frac{\alpha_{1i}}{\beta_i}, \qquad i = 1 \cdots n, \tag{5b}$$

$$\pi_0 = \gamma \sum_{i=1}^{n} \frac{\alpha_{0i}}{\rho_i \beta_i}. \tag{5c}$$

To analyze Eqs. (5), remember that the coefficients α_{1i}, α_{2i}, α_{01}, and β_i arise from the formulas for the conditional mean and variance of $\tilde{X}$ given y_i, p. Under Assumption A.3, these coefficients will in turn depend on π_0, π_i, γ as they affect the joint distribution of $(\tilde{X}, \tilde{y}_i, \tilde{p})$. Before writing out the formulas for these coefficients, I need the following:

LEMMA 2.1. *Assume* A.1–A.3. *Then the equilibrium price–supply–signals relation* (3) *has* $\gamma \neq 0$, *and the matrices* V_i, $i = 1 \cdots n$, *are nonsingular.*

Proof. Suppose that $\gamma = 0$. Then (5b) and (5c) imply $\pi_i = 0$, $i = 1 \cdots n$, and $\pi_0 = 0$, hence $\tilde{p} \equiv 0$. In this case, the coefficients in (4a) and (4b) are calculated as:

$$\alpha_{0i} = s_i^2/(\sigma^2 + s_i^2), \ \alpha_{1i} = \sigma^2/(\sigma^2 + s_i^2), \ \alpha_{2i} = 0,$$

$\beta_i = \sigma^2 s_i^2/(\sigma^2 + s_i^2)$. Then (5a) implies $\gamma \neq 0$, a contradiction. Nonsingularity of the matrices V_i follows directly from the fact that $\gamma \neq 0$.

Q.E.D.

Given that the matrices V_i are nonsingular, the coefficients α_{1i}, α_{2i}, α_{0i} and β_i are given by the formulas (Raiffa and Schlaifer, 1961, p. 250)

[4] Given that $\Delta^2 > 0$, the implicit assumption that $\sum_{i=1}^{n} [(1 - \alpha_{2i})/\rho_i\beta_i] \neq 0$ is harmless. If $\sum_{i=1}^{n} [(1 - \alpha_{2i})/\rho_i\beta_i] = 0$, variations in p have no effect on excess demand. Then there is no way to clear the market for different realizations of $\tilde{Z}$. Since $\tilde{Z}$ has positive variance, by Assumption A.2, one cannot have an equilibrium with rational expectations. The existence of a rational expectations equilibrium with $\sum_{i=1}^{n} [(1 - \alpha_{2i})/\rho_i\beta_i] \neq 0$ is discussed separately in Proposition 3.3.

$$\alpha_{1i} = \frac{\sigma^2}{b_i}\left[\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i \pi s_i{}^2\right], \tag{6a}$$

$$\alpha_{2i} = \frac{\sigma^2}{b_i}(\pi - \pi_i)\, s_i{}^2, \tag{6b}$$

$$\alpha_{0i} = \bar{X}\,\frac{s_i{}^2}{b_i}\left[\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i{}^2 s_i{}^2\right] - \alpha_{2i}(\pi_0 - \gamma\bar{Z}), \tag{6c}$$

$$\beta_i = \frac{\sigma^2 s_i{}^2}{b_i}\left[\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i{}^2 s_i{}^2\right], \tag{6d}$$

$$b_i = (\sigma^2 + s_i{}^2)\left[\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i{}^2 s_i{}^2\right] + \sigma^2 s_i{}^2 (\pi - \pi_i)^2. \tag{6e}$$

Now Eqs. (5) and (6) form a nonlinear system of equations in π_{0i}, π_{1i}, γ, α_{1i}, α_{2i}, α_{0i}, β_i. The solutions to this system correspond to the equilibria of the economy under Assumptions A.1–A.3. The problem is to analyze (5) and (6) in detail so as to obtain some insight into the way in which the price p aggregates and communicates the information contained in the signals $y_1 \cdots y_n$.

3. Preliminary Results

To analyze equilibrium under Assumptions A.1–A.3, I shall first use Eqs. (6) to eliminate the coefficients α_{1i}, α_{2i}, α_{0i} and β_i from (5). As a result one has

$$\pi_i = \frac{\gamma}{\rho_i s_i{}^2}\,\frac{\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i \pi s_i{}^2}{\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i{}^2 s_i{}^2}, \qquad i = 1 \cdots n, \tag{7a}$$

$$\frac{1}{\gamma} = \sum_{i=1}^{n} \frac{\sigma^2 + s_i{}^2}{\rho_i \sigma^2 s_i{}^2} + \sum_{i=1}^{n} \frac{(\pi - \pi_i)^2 - (\pi - \pi_i)}{\rho_i\left[\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i{}^2 s_i{}^2\right]}, \tag{7b}$$

$$\pi_0 = \frac{\bar{X}}{\sigma^2}\sum_{i=1}^{n}\frac{1}{\rho_i} - \gamma(\pi_0 - \gamma\bar{Z})\sum_{i=1}^{n}\frac{\pi - \pi_i}{\rho_i\left[\sum_{k=1}^{n} \pi_k{}^2 s_k{}^2 + \gamma^2 \Delta^2 - \pi_i{}^2 s_i{}^2\right]}. \tag{7c}$$

Now Eqs. (7a) can be analyzed independently of (7b) and (7c). To see this, define the new variables $Q_i \equiv \pi_i/\gamma$, $i = 1 \cdots n$, $Q \equiv \pi/\gamma \equiv \sum_{i=1}^n \pi_i/\gamma$, and reqrite (7a) as

$$Q_i = \frac{1}{\rho_i s_i^2} \frac{\sum_{k=1}^n Q_k^2 s_k^2 + \Delta^2 - Q_i s_i^2 \sum_{k=1}^n Q_k}{\sum_{k=1}^n Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}, \qquad i = 1 \cdots n. \tag{8a}$$

Equations (8a) serve to determine the variables $Q_1 \cdots Q_n$ regardless of γ and π_0. One has:

LEMMA 3.1. *Under Assumptions* A.1 *and* A.2, *Eqs.* (8a) *have a solution. Any solution* $Q_1 \cdots Q_n$ *to* (8a) *satisfies*

$$0 < Q_i < 1/\rho_i s_i^2.$$

Proof. I shall prove the last statement first. Let $Q_1 \cdots Q_n$ be a solution to (8a) and define the index set $I = \{i \mid Q_i \leqslant 0\}$. Suppose that $I \neq \varnothing$. Then there exists $i_0 \in I$, such that for all $k \in I$, $Q_{i_0} s_{i_0}^2 \geqslant Q_k s_k^2$. Therefore, for $\Delta^2 > 0$,

$$\begin{aligned}\sum_{k=1}^n Q_k^2 s_k^2 + \Delta^2 - Q_{i_0} Q s_{i_0}^2 &> \sum_{k \in I} Q_k^2 s_k^2 - Q_{i_0} s_{i_0}^2 \sum_{k \in I} Q_k \\ &= \sum_{k \in I} Q_k (Q_k s_k^2 - Q_{i_0} s_{i_0}^2) \geqslant 0.\end{aligned}$$

Now (8a) implies $Q_{i_0} > 0$, hence $i_0 \notin I$, a contradiction. Therefore, $Q_i > 0$ for all i, implying $Q_i \sum_{k=1}^n Q_k > Q_i^2$. Then (8a) implies $Q_i < 1/\rho_i s_i^2$, $i = 1 \cdots n$.

The existence of a solution to (8a) is shown by a fixed-point argument. Let $Y \equiv \prod_{i=1}^n [0, 1/\rho_i s_i^2] \subset \mathbb{R}_+^n$ and define mappings T_0: $Y \to \mathbb{R}^n$, T_1: $Y \to Y$ by the conditions

$$(T_0 Q)_i = \frac{1}{\rho_i s_i^2} \frac{\sum_{k=1}^n Q_k^2 s_k^2 + \Delta^2 - Q_i s_i^2 \sum_{k=1}^n Q_k}{\sum_{k=1}^n Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}, \qquad i = 1 \cdots n,$$

$$\begin{aligned}(T_1 Q)_i &= 0 && \text{if } (T_0 Q)_i < 0 \\ (T_1 Q)_i &= (T_0 Q)_i && \text{if } 0 \leqslant (T_0 Q)_i \leqslant 1/\rho_i s_i^2, \qquad i = 1 \cdots n, \\ (T_1 Q)_i &= 1/\rho_i s_i^2 && \text{if } (T_0 Q)_i > 1/\rho_i s_i^2.\end{aligned}$$

Under Assumptions A.1 and A.2, Y is compact, and T_1 is continuous. By Brouwer's theorem, T_1 has a fixed point Q^*. It remains to show that $0 < Q_i^* < 1/\rho_i s_i^2$ for all i, so that $Q^* = T_0 Q_0^*$. By way of contradiction, suppose first that $Q_i^* = 0$ for some i. Then $(T_0Q^*)_i = 1/\rho_i s_i^2$, hence $(T_1Q^*)_i = 1/\rho_i s_i^2 \neq Q_i^*$, a contradiction. Hence $Q_i^* > 0$ for all i, implying $(T_0Q^*)_i < 1/\rho_i s_i^2$ and hence $(T_1Q^*)_i < 1/\rho_i s_i^2$ for all i. Thus Q^* is a fixed point of T_0 and, hence a solution to (8a). Q.E.D.

For future reference, it is useful to state explicitly the following obvious:

COROLLARY 3.2. *Any solution* $Q_1 \cdots Q_n$ *to* (8a) *satisfies*

$$\Delta^2 > Q_i s_i^2 \sum_{k=1}^{n} Q_k - \sum_{k=1}^{n} Q_k^2 s_k^2, \qquad i = 1 \cdots n.$$

Further, one now has:

PROPOSITION 3.3. *Under Assumptions* A.1 *and* A.2, *Eqs.* (5) *and* (6) *have a solution.*

Proof. It suffices to show that (7) has a solution. Substituting for $\pi_i = \gamma Q_i$, $i = 1 \cdots n$, and solving for γ and π_0, one rewrites (7b) and (7c) as

$$\gamma = \frac{1 + \sum_{i=1}^{n} \dfrac{Q - Q_i}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}{\sum_{i=1}^{n} \dfrac{\sigma^2 + s_i^2}{\rho_i \sigma^2 s_i^2} + \sum_{i=1}^{n} \dfrac{(Q - Q_i)^2}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}, \tag{8b}$$

$$\pi_0 = \frac{\dfrac{\bar{X}}{\sigma^2} \sum_{i=1}^{n} \dfrac{1}{\rho_i} + \bar{Z} \sum_{i=1}^{n} \dfrac{Q - Q_i}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}{\sum_{i=1}^{n} \dfrac{\sigma^2 + s_i^2}{\rho_i \sigma^2 s_i^2} + \sum_{i=1}^{n} \dfrac{(Q - Q_i)^2}{\rho_i \left(\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2\right)}}. \tag{8c}$$

The proposition now follows immediately from Lemma 3.1. Q.E.D.

Most of the subsequent analysis will be based on Lemma 3.1. But for some purposes, I shall need a sharper bound on the Q_i. This is given by the following lemma, which is proved in the Appendix.

LEMMA 3.4. *Let* $Q_1 \cdots Q_n$ *be a solution to* (8a), *and let* $Q \equiv \sum_{k=1}^{n} Q_k$. *Then*

$$\frac{Q_i}{Q} > \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} + \sum_{k=1}^{n} \frac{s_i^2}{s_k^2}\right]^{-1}, \qquad i = 1 \cdots n.$$

4. The Aggregation of Information in a Small Competitive Market

I shall now turn to the question how the equilibrium price under Assumptions A.1–A.3 aggregates information. In particular, I shall be concerned with the relative strength of the weights π_i, π_j with which the signals y_i and y_j affect the equilibrium price. From (8a) and the definition $Q_i \equiv \pi_i/\gamma$, one has

$$\frac{\pi_i}{\pi_j} = \frac{Q_i}{Q_j} = \frac{\rho_j s_j^2}{\rho_i s_i^2} \frac{\dfrac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i Q s_i^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}}{\dfrac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_j Q s_j^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_j^2 s_j^2}}. \tag{9}$$

In the Appendix, I prove the following:

Proposition 4.1. *Assume* A.1–A.3 *and let* π_0, π_i, $i = 1 \cdots n$, γ *be the coefficients of an equilibrium price function.*

(a) *If* $\rho_i \geqslant \rho_j$ *and* $s_i^2 \geqslant s_j^2$, *then* $\pi_i \leqslant \pi_j$.
(b) *If* $\rho_i \geqslant \rho_j$ *and* $s_i^2 = s_j^2$, *then* $\rho_i \pi_i \geqslant \rho_j \pi_j$.
(c) *If* $\rho_i = \rho_j$ *and* $s_i^2 \geqslant s_j^2$, *then* $s_i^2 \pi_i \leqslant s_j^2 \pi_j$.

If one of the inequalities in the premises of statements (a)–(c) *is strict, the inequality in the corresponding conclusion is also strict.*

Proposition 4.1 shows the dependence of π_i/π_j on the risk aversions ρ_i, ρ_j and the variances s_i^2, s_j^2. The first part of the proposition expresses the simple fact that a precise signal available to a risk neutral agent affects the price more than an imprecise signal available to a risk averse agent. This is so because the sensitivity of agent i's demand to the signal y_i increases with the precision of the signal and decreases with the agent's risk aversion.

The other statements of Proposition 2 give some indication of the extent of the dependence of π_i/π_j on risk aversions and the variances s_i^2, s_j^2. To see the significance of these statements, consider the situation when agents condition *only* on their own signals y_i without drawing information from the price. In this situation one has $\pi_i \rho_i s_i^2 = \pi_j \rho_j s_j^2$ for all i, j, i.e., π_i is just inversely proportional to the factor $\rho_i s_i^2$. In contrast, if agents condition on both signals and price, the ratio π_i/π_j is relatively *less* sensitive to agents' risk aversions and relatively *more* sensitive to the variances s_i^2, s_j^2 (Proposition 4.1b, c)).

These effects arise mainly because agents take account of the covariance between their own signal and the price.[5] The signal y_i affects agent i's expectation about $\tilde{X}$ both directly and indirectly, through the price p. To avoid "double counting" and compensate for his indirect reaction to y_i through the price, he reduces his direct reaction to y_i. For this reason, the coefficient π_i depends less strongly on agent i's risk aversion.

In contrast, the variances s_i^2 and s_j^2 become relatively *more* important because they enter into the covariance between signals and the price. The larger the variance s_i^2, the larger the covariance between noise in y_i and noise in p, and the greater the need to scale down one's direct reaction to the signal y_i.

If the exogenous noise in supplies is large, this covariance effect is insignificant. In this case, variations in price reflect variations in supply rather than variations in signals. Therefore agents draw only little information from the price. In the limit as $\Delta^2 \to \infty$ the system goes back to the situation when agents condition only on their own signals. Formally, one has

PROPOSITION 4.2. *Assume* A.1–A.3 *and let* $\Delta^2 \to \infty$. *Then*:

(a) *the equilibrium price converges to*

$$\tilde{p}_\infty = \frac{1}{\sigma^2 B + A}\left[A\bar{X} + \sigma^2 \sum_{i=1}^{n} \frac{\tilde{y}_i}{\rho_i s_i^2} - \sigma^2 \tilde{Z}\right],$$

where

$$A \equiv \sum_{i=1}^{n} \frac{1}{\rho_i}, \qquad B \equiv \sum_{i=1}^{n} \frac{1}{\rho_i s_i^2};$$

(b) *the conditional expectation of* $\tilde{X}$ *given* y_i, p *converges to* $(s_i^2\bar{X} + \sigma^2 y_i)/(\sigma^2 + s_i^2)$;

(c) *the conditional variance of* $\tilde{X}$ *given* y_i, p *converges to* $\sigma^2 s_i^2/(\sigma^2 + s_i^2)$.

Proof. Immediate from (8a)–(8c) and (6). Q.E.D.

The other limit as $\Delta^2 \to 0$ is of greater interest. Here one has

PROPOSITION 4.3. *Assume* A.1–A.3 *and let* $\Delta^2 \to 0$. *Then*

(a) *the equilibrium price converges to*:

$$\tilde{p}_0 = \frac{1}{\sigma^2 C + 1}\left[\bar{X} + \sigma^2 \sum_{i=1}^{n} \frac{\tilde{y}_i}{s_i^2} - \frac{\sigma^2 \bar{Z}}{A}\right],$$

[5] Formally, the entry $\mathrm{cov}(y_i, p) = \pi\sigma^2 + \pi_i s_i^2$ in the matrix V_i is responsible for the terms $\pi_i \pi s_i^2$ and $\pi_i^2 s_i^2$ in (7), which lead to the inequality $\pi_i/\gamma < 1/\rho_i s_i^2$.

where

$$A \equiv \sum_{i=1}^{n} \frac{1}{\rho_i}, \qquad C \equiv \sum_{i=1}^{n} \frac{1}{s_i^2};$$

(b) *the conditional expectation of* $\tilde{X}$ *given* (y_i, p) *converges to*

$$\frac{\sigma^2 \bar{Z}}{A(\sigma^2 C + 1)} + p;$$

(c) *the conditional variance of* $\tilde{X}$ *given* (y_i, p) *converges to* $\sigma^2/(\sigma^2 C + 1)$.

Proof. Given $\Delta^2 \to 0$, let $Q_1(\Delta^2) \cdots Q_n(\Delta^2)$ be a solution to (8a) and with $Q(\Delta^2) = \sum_{i=1}^{n} Q_i(\Delta^2)$. From Corollary 3.2 one has

$$\frac{\Delta^2}{Q(\Delta^2)^2} \geqslant \sum_{k=1}^{n} \frac{Q_k(\Delta^2)}{Q(\Delta^2)} \left[\frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2 - \frac{Q_k(\Delta^2)}{Q(\Delta^2)} s_k^2\right], \qquad i = 1 \cdots n,$$

and therefore, by Lemma 3.4,

$$\frac{\Delta^2}{Q(\Delta^2)^2} \geqslant q \max_{i,k} \left[\frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2 - \frac{Q_k(\Delta^2)}{Q(\Delta^2)} s_k^2\right], \tag{10a}$$

where

$$q \equiv \min_j \left[\sum_{k=1}^{n} \frac{\rho_j s_j^2}{\rho_k s_k^2} + \sum_{k=1}^{n} \frac{s_j^2}{s_k^2}\right]^{-1} > 0$$

is the smallest of the lower bounds in Lemma 3.4. Further, from (8a) and Lemma 3.4, one has

$$\lim_{\Delta^2 \to 0} Q_i(\Delta^2) = 0 \qquad \text{if and only if:} \tag{10b}$$

$$\lim_{\Delta^2 \to 0} \left[\sum_{k=1}^{n} \frac{Q_k(\Delta^2)^2}{Q(\Delta^2)^2} s_k^2 + \frac{\Delta^2}{Q(\Delta^2)^2} - \frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2\right] = 0.$$

The proposition now follows from two intermediate steps:

Step 1. $\lim_{\Delta^2 \to 0} Q_i(\Delta^2) = 0$, $i = 1 \cdots n$.

By way of contradiction, suppose that for some i, there exists $\epsilon > 0$ and a subsequence $\{\Delta_m^2\} \to 0$, such that $Q_i(\Delta_m^2) \geqslant \epsilon$ for all m. Since $Q(\Delta_m^2) > Q_i(\Delta_m^2)$, one has $\lim_{\Delta^2 \to 0} \Delta_m^2/Q(\Delta_m^2) = 0$, and (10a) implies

$$\lim_{m \to \infty} \left[\frac{Q_i(\Delta_m^2)\, s_i^2 - Q_k(\Delta_m^2)\, s_k^2}{Q(\Delta_m^2)}\right] = 0$$

for all i, k. Given that

$$\sum_{i=1}^{n} \frac{Q_i(\Delta_m^2)}{Q(\Delta_m^2)} = 1,$$

one has

$$\lim_{m\to\infty} \frac{Q_i(\Delta_m^2)}{Q(\Delta_m^2)} = \frac{1}{Cs_i^2}.$$

In consequence,

$$\lim_{m\to\infty} \sum_{k=1}^{n} \left[\frac{Q_k(\Delta_m^2)^2}{Q(\Delta_m^2)^2} s_k^2 + \frac{\Delta_m^2}{Q(\Delta_m^2)^2} - \frac{Q_i(\Delta_m^2)}{Q(\Delta_m^2)} s_i^2\right] = 0,$$

and, by (10b), $\lim_{m\to\infty} Q_i(\Delta_m^2) = 0$, contrary to the assumption that $Q_i(\Delta_m^2) \geqslant \epsilon > 0$ for all m. Hence $\lim_{\Delta^2\to 0} Q_i(\Delta^2) = 0$.

Step 2. $\lim_{\Delta^2\to 0} [Q_i(\Delta^2)/Q(\Delta^2)] = 1/Cs_i^2$.
From (10b) and the fact that $Q_i(\Delta^2) \to 0$, for all i, one has

$$\lim_{\Delta^2\to 0} \left[\sum_{k=1}^{n} \frac{Q_k(\Delta^2)^2}{Q(\Delta^2)^2} s_k^2 + \frac{\Delta^2}{Q(\Delta^2)^2} - \frac{Q_i(\Delta^2)}{Q(\Delta^2)} s_i^2\right] = 0, \qquad \text{for all } i,$$

hence

$$\lim_{\Delta^2\to 0} \left[\frac{Q_i(\Delta^2)\, s_i^2}{Q(\Delta^2)} - \frac{Q_j(\Delta^2)\, s_j^2}{Q(\Delta^2)}\right] = 0, \qquad \text{for all } i, j.$$

The desired result now follows from the fact that

$$\sum_{i=1}^{n} \frac{Q_i(\Delta^2)}{Q(\Delta^2)} = 1.$$

These two steps and (10b) further imply that $\lim_{\Delta^2\to 0} \Delta^2/Q(\Delta^2)^2 = 0$. Using all these results to take limits in (8b) and (8c), one has

$$\lim_{\Delta^2\to 0} \gamma(\Delta^2) = \lim_{\Delta^2\to 0} \pi_0(\Delta^2) = \infty;$$

$$\lim_{\Delta^2\to 0} \gamma(\Delta^2)\, Q(\Delta^2) = \frac{C\sigma^2}{1 + C\sigma^2},$$

and therefore,

$$\lim_{\Delta^2\to 0} \pi_i(\Delta^2) = \lim_{\Delta^2\to 0} \gamma(\Delta^2)\, Q(\Delta^2) \frac{Q_i(\Delta^2)}{Q(\Delta^2)} = \frac{\sigma^2}{1 + C\sigma^2} \cdot \frac{1}{s_i^2};$$

$$\lim_{\Delta^2\to 0} [\pi_0(\Delta^2) - \gamma(\Delta^2)\, \bar{Z}] = \frac{\bar{X} - \sigma^2\bar{Z}/A}{(\sigma^2 C + 1)},$$

as was to be shown. Statements (b) and (c) of the proposition are obtained in a similar way by taking limits in (6). Q.E.D.

Proposition 4.3 is basically Grossman's result. As the supply-induced noise in the market disappears, the equilibrium price of the risky asset becomes a sufficient statistic for the vector of signals. Variations in the price reflect and communicate variations in the precision weighted sum $\sum_{i=1}^{n} y_i/s_i^2$ of the signals.

The key to this result is the disappearance of the risk aversion coefficients ρ_i from the weights π_i. As Δ^2 becomes small, it hardly makes a difference whether a favorable signal is given to a risk averse or a risk neutral agent.

This is an extreme consequence of the agent's awareness of the covariance between the price and their own signal. As Δ^2 becomes small, the price $\tilde{p}$ becomes a more reliable predictor of the return $\tilde{X}$. Therefore agents pay more attention to the price. Because of the price–signal covariance, this is compen-compensated by paying less attention to their own signal. In consequence, the weight π_i of the signal y_i becomes less dependent on the particular properties of agent i. This effect is to some extent self-enforcing: As Δ^2 becomes small, price becomes a more reliable predictor of $\tilde{X}$, both because it reveals $\sum_{i=1}^{n} \pi_i y_i$ more precisely *and* because, with disappearance of the coefficients ρ_i from the weights π_i, $\sum_{i=1}^{n} \pi_i y_i$ becomes a more efficient predictor of $\tilde{X}$. In the limit as $\Delta^2 \to 0$, the price reveals $\sum_{i=1}^{n} \pi_i y_i$ perfectly, and, moreover, $\sum_{i=1}^{n} \pi_i y_i$ becomes a sufficient statistic for the vector $(y_1 \cdots y_n)$.

However, if $\Delta^2 = 0$, the model is no longer well specified. If the price is a sufficient statistic, agents no longer even look at their own information, because the pair (p, y_i) is no better than the price alone. But if demands are independent of the signals y_i, there is no reason why the price should vary with the sum $\sum_{i=1}^{n} y_i/s_i^2$.

One can also see this difficulty by considering price formation in a Walrasian tâtonnement. The auctioneer begins the auction by calling a price p "au hasard." Individual agents know their own signals and announce their desired asset demands knowing that trade at p will take place if and only if p clears the market. Therefore the demand announcement must take account of the information carried by p, *if* it happened to clear the market. From the formula

$$\bar{z}_i(p, y_i\,;\, \pi_0, \pi_1 \cdots \pi_n, \gamma) = \frac{E(\tilde{X} \mid y_i, p) - p}{\rho_i \operatorname{Var}(\tilde{X} \mid y_i, p)}$$

and Proposition 4.3, this revealed demand is calculated as $\bar{z}_i(p, y_i\,;\, \pi_0, \pi_1 \cdots \pi_n, \gamma) = \bar{Z}/\rho_i A$, regardless of p, y_i. But then $\sum_{i=1}^{n} \bar{z}_i(p, y_i, \pi_0, \pi_1 \cdots \pi_n, \gamma) = \bar{Z}$ regardless of $p, y_1 \cdots y_n$. No matter where the auctioneer begins the auction, and what signals agents have received, the market will clear. There is no mechanism by which the auctioneer can, for given $y_1 \cdots y_n$, find the price prescribed by Grossman's formula.

In summary, Grossman's result gives an *approximate* description of communication through the market when the supply-induced noise is small. It should *not* be regarded as an exact result for the case $\Delta^2 = 0$. In this case, the communication process simply is not well defined.

5. The Aggregation of Information in a Large Market

The preceding model of communication in a finite market is a bit schizophrenic. On the one hand agents are aware of the covariance between the price and their own signals and actions. On the other hand they behave as price takers. To remove this difficulty I shall now look at the aggregation of information in a competitive sequence of economies.

Let $(\Omega, \mathscr{F}, \nu)$ be a probability space and $\mathscr{N}$ the set of normal random variables on Ω. An *economy* $\mathscr{E}$ is defined by a finite set $\mathscr{A}$ of economic agents, a mapping $e: \mathscr{A} \to \mathbb{R}_+^2$, and two random variables $\tilde{X} \in \mathscr{N}$, $\tilde{Z} \in \mathscr{N}$ with the following interpretation:

(i) $\tilde{X}$ is the per-unit return, $\tilde{Z}$ the supply of the risky asset.

(ii) For $i \in \mathscr{A}$, $\text{proj}_1\, e(i) = \rho_i$, agent i's coefficient of risk aversion under A.1.

(iii) For $i \in \mathscr{A}$, $\text{proj}_2\, e(i) = s_i^2$, the variance of the error $\tilde{\epsilon}_i = \tilde{y}_i - \tilde{X}$ in agent i's signal, where the vector $(\tilde{X}, \tilde{Z}, \tilde{\epsilon}_i \cdots \tilde{\epsilon}_n)$ satisfies Assumption A.2.

The *characteristic distribution* of the economy $\mathscr{E} = (\mathscr{A}, e, \tilde{X}, \tilde{Z})$ is a measure $\mu_{\mathscr{E}} \in \mathscr{M}(\mathbb{R}_+^2)$, defined by the usual formula, $\mu_{\mathscr{E}}(B) = \# e^{-1}(B)/\#\mathscr{A}$, for every Borel subset B of $\mathbb{R}_+^2$.

A sequence of economies $\mathscr{E}_n = \{\mathscr{A}^n, e^n, \tilde{X}^n, \tilde{Z}^n\}$ will be called *competitive*, if it satisfies the following conditions:

(B.1) $\#\mathscr{A}^n = n \to \infty$.

(B.2) There exist two random variables $\tilde{X} \in \mathscr{N}$, $\tilde{Z} \in \mathscr{N}$, such that for all n, $\tilde{X}^n = \tilde{X}$ and $\tilde{Z}^n/n = \tilde{Z}$.

(B.3) There exists a closed rectangle $[\underline{\rho}, \bar{\rho}] \times [\underline{s}^2, \bar{s}^2]$ in the interior of $\mathbb{R}_+^2$, such that for all n, $e^n(\mathscr{A}^n) \subset [\underline{\rho}, \bar{\rho}] \times (\underline{s}^2, \bar{s}^2]$.

(B.4) The sequence of characteristic distributions $\{\mu_{\mathscr{E}_n}\}$ converges weakly to a measure $\mu \in \mathscr{M}(\mathbb{R}_+^2)$.

If one writes $\rho_i(n)$, $s_i^2(n)$ for the characteristics $e^n(i)$ of agent $i \in \mathscr{A}^n$, one obtains:

PROPOSITION 5.1. *Let $\{\mathscr{E}_n\}$ be a competitive sequence of economies. Any solution $Q_i(n)$, $i \in \mathscr{A}^n$ of* Eqs. (8a) *for economy $\mathscr{E}_n$ satisfies*

$$Q_i(n) = \frac{1}{\rho_i(n)\, s_i^2(n)} + o(1/n), \qquad i \in \mathscr{A}^n.$$

Proof. See the Appendix.

Further let $\bar{Z}(n)$, $\Delta^2(n)$ denote the mean and variance of total supply $\tilde{Z}^n$, and let Z^*, Δ^{*2} denote the mean and variance of per capita supply $\tilde{Z}$. Clearly, (B.2) implies $\bar{Z}(n) = nZ^*$ and $\Delta^2 = n^2 \Delta^{*2}$.

With obvious notation, write the market-clearing price for economy $\mathscr{E}_n$ under the rational expectations hypothesis A.3 as

$$\tilde{p}^n = \pi_0(n) + \sum_{i \in \mathscr{A}^n} \pi_i(n)\, \tilde{y}_i^n - \gamma(n)\, \tilde{Z}^n.$$

In the Appendix, I prove:

PROPOSITION 5.2. *Let $\{\mathscr{E}_n\}$ be a competitive sequence of economies and let $\{\tilde{p}^n\}$ be a corresponding sequence of market-clearing prices under* A.3. *Then the sequence $\{\tilde{p}\}$ converges in probability to $\tilde{p}^* = \pi_0^* + \pi^* \tilde{X} - \gamma^* \tilde{Z}$, where*

$$\pi_0^* = \frac{\bar{X}\Delta^{*2}A^* + \sigma^2 Z^* A^* B^*}{A^*\Delta^{*2} + \sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}},$$

$$\pi^* = \frac{\sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}}{A^*\Delta^{*2} + \sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}},$$

$$\gamma^* = \frac{\sigma^2 \Delta^{*2} + \sigma^2 A^* B^*}{A^*\Delta^{*2} + \sigma^2 B^* \Delta^{*2} + \sigma^2 A^* B^{*2}},$$

$$A^* \equiv \int \frac{1}{\rho}\, d\mu, \qquad B^* \equiv \int \frac{1}{\rho s^2}\, d\mu.$$

As the number of agents becomes large, the weights $\pi_i(n)$ converge to zero. Individual agents can no longer affect the price. This has two important consequences: First, the relative weights

$$\frac{\pi_i(n)}{\pi_j(n)} = \frac{Q_i(n)}{Q_j(n)}$$

of signals given to two agents i, j become approximately equal to $\rho_j(n)\, s_j^2(n)/\rho_i(n)\, s_i^2(n)$. As $\pi_i(n)$ converges to zero, the covariance between the noise $\tilde{\varepsilon}_i^n$ and the equilibrium price $\tilde{p}^n$ disappears. The relative importance of an agent's information becomes inversely proportional to his degree of risk aversion. The aggregation of information in a large market depends on agents' preferences as well as on the precision of their signals.

Second, as the number of agents increases, the equilibrium price comes to depend only on the actual return $\tilde{X}$, which is the common element in all the signals $\tilde{y}_i$. The noises $\tilde{\epsilon}_i^n$ disappear from the equilibrium price, by the weak law of large numbers. This is an instance of the well known result that individual uncertainty does not affect the equilibrium price in a large economy (see, e.g., Malinvaud (1972)). In the more general case where the common element in the individual signals $\tilde{y}_i$ involves some noise $\tilde{\delta}$, so that $\tilde{y}_i = \tilde{X} + \tilde{\delta} + \tilde{\epsilon}_i$, the equilibrium price will depend on $\tilde{\delta}$ as well as $\tilde{X}$, but the $\tilde{\epsilon}_i$ will still cancel out.

In summary, Propositions 5.1 and 5.2 suggest that communication through the market relies on the presence of a large number of independent sources of information. Because of differences in preferences, the equilibrium price is not, in general, an "efficient" aggregator of information. However, this inefficiency is irrelevant, if the market draws on many independent sources of information, so that individual errors cancel out.

Even so, individuals cannot actually read $\tilde{X}$ off the equilibrium price. Because of the noise in supplies, they cannot distinguish whether a high price is due to a high realization of $\tilde{X}$ or a low realization of $\tilde{Z}$. Therefore, they find it worthwhile to draw information from their own signal as well as the price. Formally one has:

PROPOSITION 5.3. *Let $\tilde{p}^*$ be the equilibrium price as in Proposition 5.2, and let $\tilde{y} = \tilde{X} + \tilde{\epsilon} \in \mathcal{N}$ be a signal with $E\tilde{\epsilon} = 0$, $E\tilde{\epsilon}^2 = s^2$, $E\tilde{\epsilon}\tilde{Z} = 0$. Given a realization (y, p) of the random pair $(\tilde{y}, \tilde{p}^*)$ the posterior mean and variance of returns are*

$$E(\tilde{X} \mid y, p) = \frac{1}{D}\left[\frac{\Delta^{*2}}{B^{*2}} s^2\bar{X} + \frac{\Delta^{*2}}{B^{*2}} \sigma^2 y + \sigma^2 s^2 \frac{p - \pi_0^* + \gamma^* Z^*}{\pi^*}\right],$$

$$\operatorname{Var}(\tilde{X} \mid y, p) = \frac{1}{D}\frac{\Delta^{*2}}{B^{*2}} s^2\sigma^2,$$

where

$$D = \frac{\Delta^{*2}}{B^{*2}}\sigma^2 + \frac{\Delta^{*2}}{B^{*2}} s^2 + \sigma^2 s^2,$$

and π_0^*, π^*, γ^*, B^* are the coefficients defined in Proposition 5.2.

Proposition 5.3 shows the importance of preferences for the communication of information through the market. The distribution of degrees of risk aversion and of signal variances determines the coefficient $B^* = \int (1/\rho s^2)\, d\mu$. This coefficient is equal to the ratio π^*/γ^* of the weights of $\tilde{X}$ and $\tilde{Z}$ in the equilibrium price. For given variances σ^2, Δ^{*2}, this ratio determines the relative contributions of $\tilde{X}$ and $\tilde{Z}$ to variations in the equilibrium price. The point is that the strength of agents' reactions to their signals—and implicitly to $\tilde{X}$—is inversely related to both their degree of risk aversion and the level of noise in their signal. If agents are less risk averse, variations in $\tilde{X}$ will lead to larger variations in asset demands, which in turn induce larger variations in the equilibrium price.

The larger the coefficient B^*, the more reliable the market as a means of communication. If variations in the equilibrium price arise mainly from variations in the return, there is little chance that a given price change will reflect a change in supply rather than a change in return. Therefore, agents can rely heavily on the price as a source of information.

From this point of view, the ratio Δ^{*2}/B^{*2} in Proposition 5.3 is a natural measure of the level of noise in the market communication process. The smaller the ratio Δ^{*2}/B^{*2}, the more attention agents pay to the price and the less to their own signals. Then also the posterior variance of returns decreases. There is no noise in the market, if either $\Delta^{*2} = 0$ *or* $B^{*2} = \infty$. The latter situation arises when a positive set of agents is risk neutral and reacts so strongly to its information that variations in supplies have no effect on the price ($\gamma^* = 0$). In this case *all* agents can actually infer the realization of returns from the price.[6]

APPENDIX

Proof of Lemma 3.4

By Lemma 3.1,

$$Q < \sum_{k=1}^{n} \frac{1}{\rho_k s_k^2}.$$

Hence, from (8a),

$$\frac{Q_i}{Q} \geqslant \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \cdot \frac{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i Q s_i^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}$$

$$= \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \left[1 - \frac{Q_i(Q - Q_i)\, s_i^2}{\sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2 - Q_i^2 s_i^2}\right]$$

[6] However, one again has the paradox that agents inferring the realization of $\tilde{X}$ from the price pay no attention to their own signals, so that the question is, Why should the price reveal the realization of $\tilde{X}$ in the first place.

$$> \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \left[1 - \frac{Q_i(Q - Q_i)\, s_i^2}{\sum_{k \neq i} Q_k^2 s_k^2}\right]$$

$$\geqslant \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2}\right]^{-1} \left[1 - \frac{Q_i}{Q - Q_i} \sum_{k \neq i} \frac{s_i^2}{s_k^2}\right] = \frac{1 - \dfrac{Q_i}{Q} \displaystyle\sum_{k=1}^{n} \frac{s_i^2}{s_k^2}}{\displaystyle\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} \left(1 - \frac{Q_i}{Q}\right)},$$

where the last inequality is based on the fact that $\sum_{k \neq i} Q_k^2 s_k^2 \geqslant (Q - Q_i)^2 / \sum_{k \neq i} 1/s_k^2$.[7]

Multiplying by

$$\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} \left(1 - \frac{Q_i}{Q}\right)$$

and rearranging terms, one has

$$\frac{Q_i}{Q} \left[\sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} + \sum_{k=1}^{n} \frac{s_i^2}{s_k^2}\right] > 1 + \frac{Q_i^2}{Q^2} \sum_{k=1}^{n} \frac{\rho_i s_i^2}{\rho_k s_k^2} > 1,$$

and the lemma follows immediately. Q.E.D.

Proof of Proposition 4.1

Define $L = \sum_{k=1}^{n} Q_k^2 s_k^2 + \Delta^2$ and rewrite (9) as

$$\frac{Q_i \rho_i s_i^2}{Q_j \rho_j s_j^2} = \frac{(L - Q_i Q s_i^2)(L - Q_j^2 s_j^2)}{(L - Q_i^2 s_i^2)(L - Q_j Q s_j^2)}. \tag{9'}$$

Further, define

$$M \equiv (L - Q_i Q s_i^2)(L - Q_j^2 s_j^2) - (L - Q_i^2 s_i^2)(L - Q_j Q s_j^2)$$

and note that, by Lemma 3.1 and Corollary 3.2,

$$Q_i \rho_i s_i^2 \gtrless Q_j \rho_j s_j^2 \qquad \text{as} \quad M \gtrless 0. \tag{9''}$$

(a) Let $\rho_i \geqslant \rho_j$, $s_i^2 \geqslant s_j^2$ and suppose that $Q_i > Q_j$. Then one calculates

$$M = Q_j(Q - Q_j)(s_j^2 - s_i^2)L + (Q_j - Q_i)[(Q - Q_i - Q_j)\, s_i^2 L + Q_i Q_j Q s_i^2 s_j^2] < 0.$$

Now (9″) implies $Q_i \rho_i s_i^2 < Q_j \rho_j s_j^2$, hence $Q_i < Q_j$, a contradiction. Therefore, $Q_i \not> Q_j$, hence $\pi_i/\gamma \leqslant \pi_j/\gamma$. From (8b), $\gamma > 0$, hence $\pi_i \leqslant \pi_j$.

[7] To see this, minimize $\sum_{k \neq i}^{n} Q_k^2 s_k^2$ subject to $\sum_{k \neq i}^{n} Q_k = Q - Q_i$.

(b) Let $\rho_i \geqslant \rho_j$, $s_i^2 = s_j^2$ and suppose that $\rho_i Q_i < \rho_j Q_j$. Then $Q_j > Q_i$, hence $M > 0$ and by (9″), $\rho_i Q_i > \rho_j Q_j$, a contradiction. Therefore $\rho_i Q_i \not< \rho_j Q_j$ and $\rho_i \pi_i \geqslant \rho_j \pi_j$ because $\gamma > 0$, again by (8b).

(c) Let $\rho_i = \rho_j$, $s_i^2 \geqslant s_j^2$ and suppose that $s_i^2 Q_i > s_j^2 Q_j$. Rearranging terms in the expression for M, one has

$$M = (Q_j s_j^2 - Q_i s_i^2)(Q - Q_j)\, L - (Q_j - Q_i)(L - Q_j Q s_j^2)\, Q_i s_i^2 < 0,$$

because $Q_j \geqslant Q_i$, by (a), and $L > Q_j Q s_j^2$, by Corollary 3.2. Now (9″) implies $Q_i s_i^2 < Q_j s_j^2$, a contradiction. Therefore $s_i^2 Q_i \not> s_j^2 Q_j$ and $s_i^2 \pi_i \leqslant s_j^2 \pi_j$, because $\gamma > 0$ by (8b).

The sharpening of the conclusions when the inequalities in the premises are strict is proved by the same argument. Q.E.D.

Proof of Proposition 5.1

From (8a) and Lemma 3.1 one has

(i) $Q_i(n)\, \rho_i(n)\, s_i^2(n) < 1$,

(ii) $$Q_i(n)\, \rho_i(n)\, s_i^2(n) > 1 - Q_i(n)\, s_i^2(n) \sum_{k \in \mathscr{A}^n} Q_k(n)/\Delta^2(n)$$

$$> 1 - \sum_{k \in \mathscr{A}^n} \frac{1}{\rho_k(n)\, s_k^2(n)} \Big/ \rho_i(n)\, \Delta^2(n) \qquad \text{by (i)}$$

$$\geqslant 1 - \frac{1}{n} \frac{1}{\underline{\rho}^2 \underline{s}^2 \Delta^{*2}}, \qquad \text{by (B.2) and (B.3).}$$

Q.E.D.

Proof of Proposition 5.2

For any n, define

$$A(n) = \frac{1}{n} \sum_{i \in \mathscr{A}^n} \frac{1}{\rho_i(n)}, \qquad B(n) = \frac{1}{n} \sum_{i \in \mathscr{A}^n} \frac{1}{\rho_i(n)\, s_i^2(n)}.$$

Using (B.2) and Proposition 5.1 to substitute for $\bar{Z}(n) = nZ^*$, $\Delta^2(n) = n^2 \Delta^{*2}$ and $Q_i(n) = 1/\rho_i(n)\, s_i^2(n) + o(1/n)$ in (8b) and (8c), one has, by elementary algebra

$$n\gamma(n) = \frac{\sigma^2 \Delta^{*2} + \sigma^2 A(n)\, B(n)}{\Delta^{*2} A(n) + \sigma^2 B(n)\, \Delta^{*2} + \sigma^2 A(n)\, B(n)^2} + o(1/n),$$

$$\pi_0(n) = \frac{\bar{X} A(n)\, \Delta^{*2} + \sigma^2 Z^* A(n)\, B(n)}{\Delta^{*2} A(n) + \sigma^2 B(n)\, \Delta^{*2} + \sigma^2 A(n)\, B(n)^2} + o(1/n).$$

Note that

$$A(n) = \int \frac{1}{\rho}\, d\mu_{\mathscr{E}_n} \qquad \text{and} \qquad B(n) = \int \frac{1}{\rho s^2}\, d\mu_{\mathscr{E}_n},$$

so that (B3) and (B4) imply $\lim_{n\to\infty} A(n) = A^*$ and $\lim_{n\to\infty} B(n) = B^*$. Then one has $\lim_{n\to\infty} \pi_0(n) = \pi_0^*$, $\lim_{n\to\infty} \gamma(n) = \gamma^*$, and $\lim_{n\to\infty} \sum_{i\in\mathscr{A}^n} \pi_i(n) = \lim_{n\to\infty} \sum_{i\in\mathscr{A}^n} Q_i(n)\gamma(n) = \lim_{n\to\infty} \sum_{i\in\mathscr{A}^n} Q_i(n) \lim_{n\to\infty} n\gamma(n) = B^*\gamma^*$, by Lemma 5.1.

Rewriting the equation for $\tilde{p}^n$ as

$$\tilde{p}^n = \pi_0(n) + \sum_{i\in\mathscr{A}^n} \pi_i(n)\, \tilde{X} + \sum_{i\in\mathscr{A}^n} \pi_i(n)\, \tilde{\epsilon}_i^{\,n} - \gamma(n)\, n\, \frac{\tilde{Z}^n}{n},$$

one sees that the proposition follows, if $\sum_{i\in\mathscr{A}^n} \pi_i(n)\, \tilde{\epsilon}_i^{\,n}$ converges to zero in probability. To prove the latter statement, it suffices to note that $\sum_{i\in\mathscr{A}^n} \pi_i(n)\tilde{\epsilon}_i^{\,n}$ has mean zero and variance

$$\begin{aligned}\sum_{i\in\mathscr{A}^n} \pi_i(n)^2\, s_i^2(n) &= (\gamma(n)\, n)^2 \frac{1}{n^2} \sum_{i\in\mathscr{A}^n} Q_i(n)^2\, s_i^2(n) \\ &= (\gamma(n)\, n)^2 \frac{1}{n^2} \sum_{i\in\mathscr{A}^n} \left[\frac{1}{\rho_i(n)^2\, s_i^2(n)} + o\left(\frac{1}{n}\right)\right] \\ &\leqslant (\gamma(n)\, n)^2 \frac{1}{n} \left[\frac{1}{\underline{\rho}^2 \underline{s}^2} + o\left(\frac{1}{n}\right)\right],\end{aligned}$$

which converges to zero as $n \to \infty$. Q.E.D.

Proof of Proposition 5.3

The proposition follows from normal distribution theory (Raiffa and Schlaifer, 1961, p. 250) after noting that the triple $(\tilde{X}, \tilde{y}, \tilde{p}^*)$ is normally distributed with mean $(\overline{X}, \overline{X}, \pi_0^* + \pi^*\overline{X} - \gamma^* Z^*)$ and variance–covariance matrix:

$$\begin{pmatrix} \sigma^2 & \sigma^2 & \pi^*\sigma^2 \\ \sigma^2 & \sigma^2 + s^2 & \pi^*\sigma^2 \\ \pi^*\sigma^2 & \pi^*\sigma^2 & \pi^{*2}\sigma^2 + \gamma^{*2}\Delta^{*2} \end{pmatrix}. \qquad \text{Q.E.D.}$$

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